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AN OPERATOR APPROACH TO PERIODIC SOLUTIONS OF DIFFERENTIAL EQUATIONS ON LIE GROUPS

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The machinery of integral operators with parallel translation is elaborated such that for a T-periodic ordinary differential equation (i.e., a vector field) on a Lie group with continuous right-hand side the fixed points of those operators are T-periodic solutions. It is shown that under some natural conditions the second iteration of such operator is completely continuous.

Keywords: Lie groups; ordinary differential equations; integral operators with parallel translation; periodic solutions.

Dedicated to anniversary of Professor A.L. Shestakov

Introduction

Up to the moment a lot of processes in technics and engineering are described mathematically in terms of differential equations on non-linear manifolds. And very often the important problem of finding periodical solutions of those equations arises. This problem is especially complicated when the right-hand side of the equation is only continuous, i.e., there is no uniqueness of solution theorem for the Cauchy problem. In the case of equations in linear spaces the method of integral operators is applied for investigating the periodic solutions. Recall that an ordinary differential equation in a vector space can be turned into equivalent Volterra type integral equation. For example, one can turn the Cauchy problem $\dot{x} = f\big(t, x(t)\big)$, $x(0) = x_0$ in $\mathbb{R}^n$ into the integral equation $x(t) = x_0 + \int_0^t f\big(\tau, x(\tau)\big)\, d\tau$.

Unfortunately, the classical integral operators on manifolds are not covariant, i.e. depend on the choice of a chart. Previously the so-called integral operators with parallel translation were constructed (see details, e.g., in [1, 2]) that were covariant, allowed one to deal with equations with continuous right-hand sides but were not applicable to investigation of periodec solutions. In this paper we construct a new sort of those operators on Lie groups such that their fixed points are periodic solutions of differential equations with continuous periodic righ-hand sides.

The paper contains a short survey of the theory of integral operators with parallel translation, the description of the class of such operators applicable in the problem of periodic solutions and investigation of some their properties.

1. Integral operators with parallel translation

Everywhere below we deal with all objects given on a finite interval $[0, T], T > 0$. Let M be a complete Riemannian manifold, $m_0 \in M$ and $v\colon [0, T] \to T_{m_0}M$ be a continuous curve in the tangent space $T_{m_0}M$. Everywhere below we deal with the Levi-Civita connection on M.

Theorem 1. *There exists a unique C^1-curve $m\colon [0,T] \to M$ such that $m(0) = m_0$ and the tangent vector $m'(t)$ is parallel along this curve to the vector $v(t) \in T_{m_0}M$ for every $t \in [0,T]$.*

The existence of the curve $m(t)$ from Theorem 1 follows from some classical constructions. Let $m(t)$ be a C^1-smooth curve in M, $t \in [0,T]$, $m(0) = m_0$. Denote by Γ the operator of parallel translation of vector fields along $m(\cdot)$ to $T_{m_0}M$. Recall that the curve $C(m(t)) = \int_0^t \Gamma m'(s)ds$ is known as *Cartan's development* of $m(t)$ at $T_{m_0}M$. Note the well-known fact that Cartan's development is convertible and it is obvious that the curve $m(t)$ from Theorem 1 is expressed via Cartan's development as $C^{-1}(\int_0^t v(s)ds)$.

We denote the operator that sends $v(t)$ to $m(t)$ in Theorem 1 by the symbol $\mathcal{S}$. It is easy to show that $\mathcal{S}$ is continuous.

Since the parallel translation preserves the norm of the vector, the following statement is valid.

Theorem 2. *Let $\mathcal{U}_K$ be the ball of the radius K centered at the origin of the space of continuous curves $C^0([0,T], T_{m_0}M)$. Then, at every point $t \in [0,T]$, the inequality $\|m'(t)\| \le K$ holds for all curves $m(\cdot)$ from the set $\mathcal{S}\mathcal{U}_K$.*

Lemma 1. (Compactness lemma). *Let $\Xi \subset C^0([0,T], TM)$ be such that $\pi\Xi \subset C^1([0,T], M)$, where $\pi\colon TM \to M$ is the natural projection. If Ξ is relatively compact in $C^0([0,T], TM)$, then so is $\Gamma\Xi$.*

The proof of Lemma 1 can be found, e.g., in [2, Lemma 3.51].

Let Ω_K be the set of curves from $C^1([0,T], M)$ satisfying the inequality $\|m'(t)\| \le K$, where $K > 0$ is a real number, at every point $t \in [0,T]$ and such that the set $\{m(0) \mid m(\cdot) \in \Omega_K\}$ is relatively compact in M.

Theorem 3. *The set of curves $\Gamma(\Omega_K)$ is relatively compact in $C^0([0,T], TM)$.*

Proof. Since Ω_K is compact in $C^0([0,T], M)$ and the field $X(t,m)$ is continuous, the set of curves $\{X(t, m(t)) \mid m(\cdot) \in \Omega_K\}$ is compact in $C^0([0,T], TM)$. Then the aqssertion follows from Lemma 1. □

Let a continuous vector field $X(t,m)$ be given on M. Consider the set $C^1_{m_0}([0,T], M) \subset C^1([0,T], M)$ consisting of curves with initial value $m(0) = m_0$. Introduce the composition operator

$$\mathcal{S} \circ \Gamma X(t, m(t))\colon C^1_{m_0}([0,T], M) \to C^1_{m_0}([0,T], M).$$

One can easily see that this operator is continuous since the parallel translation continuously depends on the curves, along which it is carried out.

Theorem 4. *The fixed point of $\mathcal{S} \circ \Gamma$ is precisely the solution of equation $m'(t) = X(t, m(t))$ with the initial condition $m(0) = m_0$.*

Indeed, if $m(t)$ is a fixed point, $m'(t)$ is parallel along $m(t)$ to $\Gamma X(t, m(t))$. But by construction $\Gamma X(t, m(t))$ is parallel to $X(t, m(t))$. Hence $m'(t) = X(t, m(t))$.

2. The case of Lie groups

Now let M be a Lie group being a finite-dimensional manifold.

Remark 1. We denote the elements of the Lie group M as points of manifold M, i.e., by the symbol m, $m(\cdot)$ or $m(t)$ are curves on M. But for simplicity of presentation, the element considered as a diffeomorphism in M, is denoted by the symbol g. In particular g_{m_0,m_1} denotes the unique diffeomorphsm that sends m_0 to m_1. $Tg_{m_0,m_1} : T_{m_0}M \to T_{m_1}M$ is its tangent mapping.

Introduce an arbitrary complete Riemannian metric $\langle \cdot, \cdot \rangle$ on M (not necessarily left or right invariant). The corresponding norms in the tangent spaces are denoted by $\| \cdot \|$.

Consider the Banach manifold $C^1([0,T],M)$ of C^1-smooth curves in M. According to Remark 1, for $m(t) \in C^1([0,T],M)$ denote by $g_{m(0),m(t)}$ the element of Lie group (i.e., the operator) that sends $m(0)$ to $m(t)$, and by $Tg_{m(0),m(t)} : T_{m(0)}M \to T_{m(t)}M$ the tangent map of this operator. For $m(t) \in C^1([0,T],M)$ introduce the operator B_s by the formula

$$B_s(m(\cdot)) = \mathcal{S} \circ Tg_{m(0),\mathcal{S}\circ\Gamma\mathcal{X}(t,m(t))(s))}\Gamma X(t,m(t)) \tag{1}$$

that sends the vectors $\Gamma X(t,m(t))$ at $m(0)$ to the points at time instant s of the curves from $\mathcal{S} \circ \Gamma X(t,m(t))$. One can easily see that B_s is continuous.

Let also the vector field $X(t,m)$ be T-periodic, i.e., for every $m \in M$ the equality $X(t,m) = X(t+T,m)$ holds. In this case we will mainly deal with the operator $B_T(m(\cdot)) = \mathcal{S} \circ Tg_{m(0),\mathcal{S}\circ\Gamma X(t,m(t))(T))}\Gamma X(t,m(t))$.

Theorem 5. *Fixed points of operator B_T and only they are T-periodic solutions of the equation $m'(t) = X(t,m(t))$.*

Proof. If $m(t)$ is a T-periodic solution of the equation $m'(t) = X(t,m(t))$. Then $g_{m(0),\mathcal{S}\circ\Gamma X(m(\cdot)(T))}m(0) = m(0)$ and so for $X(t,m(t))$

$$Tg_{m(0),m(T)}\mathcal{S} \circ Tg_{m(0),\mathcal{S}\circ\Gamma X(t,m(t)))}\Gamma X(t,m(t)) = \Gamma X(t,m(t)).$$

Then $B_T(m(\cdot)) = \mathcal{S} \circ \Gamma X(t,m(t))$. Recall that $\Gamma X(t,m(t))$ is parallel along $m(\cdot)$ to $X(t,m(t))$. On the other hand, $\frac{d}{dt}\mathcal{S} \circ \Gamma X(t,m(t))$ is parallel along $m(\cdot)$ to $\Gamma X(t,m(t))$ and so

$$\frac{d}{dt}\mathcal{S} \circ \Gamma X(t,m(t)) = X(t,m(t)).$$

Thus $m(t)$ is a fixed point of $B_T(m(\cdot)$.

Now let $m(t)$ be an arbitrary curve in $C^1([0,T].M)$. If $\mathcal{S} \circ \Gamma X(t,m(t))(T) = m(0)$, the above arguments are valid and so $m(t)$ is both a fixed point of B_T and a T-periodic solution. If $\mathcal{S} \circ \Gamma X(t,m(t))(T) \neq m(0)$, $m(t)$ is neither a fixed point of B_T nor a periodic solution. □

3. Properties of operator B_s

Recall the following notion.

Definition 1. *A map from the topological space Y to the topological space Z is called proper, if the preimage of every relatively compact set in Z is relatively compact in Y. In* *particular, a function $\varphi : M \to \mathbb{R}$ is called proper if the preimage of every bounded subset of $\mathbb{R}$ is relatively compact in M.*

Let $\Xi \subset M$ be a compact set. Denote by $\mathfrak{C} \subset C^1([0,T], M)$ the set of curves $\{m(t)|m(0) \in \Xi, t \in [0,T]\}$. Since all curves from $\mathfrak{C}$ are given on the closed interval $[0,T]$ and M is complete, all the curves from $\mathfrak{C}$ lie in another compact set Ξ_1.

Theorem 6. *Let for any compact set $\Xi \subset M$*

$$\sup_{m\in\Xi, t\in[0,T]} \|X(t,m)\| < \sup_{m\in\Xi} \varphi(m) \tag{2}$$

where $\varphi : M \to \mathbb{R}$ is a certain proper function. Then for all $m(\cdot) \in \mathfrak{C} \subset C^1([0,T], M)$, all curves $\mathcal{S} \circ \Gamma X(t, m(t))$ are well-defined on $[0,T]$ and belong to another compact set $\Xi_2 \subset M$.

Proof. From (2) it follows that the norms of all $X(t, m(t))$ for $m(\cdot) \in \mathfrak{C}$ are uniformly bounded by $\sup\limits_{m\in\Xi_1} \varphi(m)$. Since the parallel translation preserves the norms, all norms of the corresponding curves $\Gamma X(t, m(t))$ are uniformly bounded by the same constant. Thus all the C^1-curves $\mathcal{S} \circ \Gamma X(t,m)$ have bounded lengths. Since the metric is complete, those curves lie in a compact set Ξ_2. $\square$

Theorem 7. *The set of curves $B_s\mathfrak{C} \subset C^1([0,T], M)$ is compact in $C^1([0,T], M)$.*

Proof. Since the set Ξ_2 is compact and the operators $g_{m(0),m(T)}$ and $g_{m(0),\mathcal{S}\circ\Gamma X(m(t)(T))}$ are smooth by the definition of the Lie group, the norms of operators

$$Tg_{m(0),m(T)}\mathcal{S} \circ Tg_{m(0),\mathcal{S}\circ\Gamma X(t,m(t)))}$$

are also uniformly bounded. Then the assertion follows from Theorem 6 and Theorem 3. $\square$

Thus, unlike the classical integral operators in Euclidean spaces, only the second iteration of operator B_s is completely continuous.

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ОПЕРАТОРНЫЙ ПОДХОД К ПЕРИОДИЧЕСКИМ РЕШЕНИЯМ ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ НА ГРУППАХ ЛИ

Ю. Е. Гликлих

Разработан аппарат интегральных операторов с параллельным переносом такой, что для T-периодического обыкновенного дифференциального уравнения (т.е., векторного поля) с непрерывной правой частью на группе Ли неподвижные точки таких операторов являются T-периодическими решениями. Показано, что при некоторых естественных условиях вторая итерация такого оператора вполне непрерывна.

Ключевые слова: группы Ли; обыкновенные дифференциальные уравнения; интегральные операторы с параллельным переносом; периодические решения.

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