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EXPONENTIAL DICHOTOMIES OF STOCHASTIC SOBOLEV TYPE EQUATIONS

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The article is of a review nature and contains results on the study of the stability of Sobolev type stochastic linear equations in terms of stable and unstable invariant spaces and exponential dichotomies. The article considers stochastic analogues of the Barenblatt – Zheltov – Kochina equation for the pressure of a fluid filtering in a fractured porous medium, the Oskolkov linear equation of plane-parallel flows of a viscoelastic fluid, the Dzektser equation describing the evolution of the free surface of a filtering liquid, the Ginzburg – Landau equation, which models the conductivity in a magnetic field. These equations can be considered as special cases of the stochastic Sobolev-type equations, where the stochastic **K**-process acts as the required quantity, and its derivative is understood as the Nelson – Glicklich derivative. The paper presents results on the existence of stable and unstable invariant spaces of stochastic equations that are Barenblatt – Zheltov – Kochina, Oskolkov, Dzektser and Ginzburg – Landau equations. The general scheme of a numerical algorithm for finding stable and unstable solutions to these equations is described, and the results of computational experiments are presented.

Keywords: stochastic Sobolev type equations; invariant spaces; exponential dichotomies.

Introduction

A large number of models emerging recently in natural science and physics can be considered in the form

$$L\dot{u} = Mu, \tag{1}$$

where the operators $L,\ M$ are linear and continuous. Equations of the form (1), where the operator L is an irreversible operator, are usually called Sobolev type equations [19]. At present, the number of papers devoted to the issues of solvability, stability of the equation (1), numerical studies of the equation (1) is constantly growing. Therefore, it is not possible to consider them all in this review.

In this review, we consider several stochastic analogues of such equations. The Barenblatt – Zheltov – Kochina equation

$$(\lambda - \Delta)u_t = \alpha\Delta u \tag{2}$$

simulates the dynamics of the pressure of a fluid filtered in a fractured porous medium [1]. The real parameters α and λ characterize the medium and properties of the fluid, respectively. In [19], in order to study the initial – boundary value problems for the Barenblatt – Zheltov – Kochina equation, equation (7) is reduced to the Cauchy problem for the linear Sobolev type equation in the suitable function spaces. The paper [24] was the first to consider dichotomies of solutions to homogeneous Sobolev type equation (2), where the operator M is relatively spectrally bounded. The paper [13] proves the existence of invariant spaces of equation (2) in spaces of differential forms defined on smooth Riemannian manifolds without boundary. The numerical solution of equation (2) was started in [26]. The paper [25] considers equation (2) on graphs. The paper [10] is devoted to the study of the asymptotic stability of the Barenblatt – Zheltov – Kochina equation in the sense of Lyapunov. Here the Lyapunov function method is applied, and a computational experiment based on the Galerkin method is constructed.

The Oskolkov equation

$$(\lambda - \Delta)\Delta u_t = \nu\Delta^2 u + \frac{\partial(u, \Delta u)}{\partial(x_1, x_2)} \tag{3}$$

is a model of a multipurpose flow of a high-performance, non-essential liquid [9]. Here the coefficients $\alpha,\ \nu \in \mathbb{R}$ characterize the parameters of the fluid. Consider the equation

$$(\lambda - \Delta)\Delta\alpha_t = \nu\Delta^2\alpha. \tag{4}$$

Equations (3), (4) were previously considered in different aspects of [22], [23], [20].

The equation

$$(\lambda - \Delta)u_t = \alpha\Delta u - \beta\Delta^2 u, \tag{5}$$

where $\alpha,\ \beta \in \mathbb{R}_+$ and $\lambda \in \mathbb{R}$, simulates the evolution of the free surface of a filtered fluid [3]. Here the parameters $\alpha,\ \beta,\ \lambda$ characterize the environment. The solvability of the initial-boundary value problem for equation (5) is considered, for example, in [21]. The paper [12] shows the existence of a unique solution to equation (5) in the spaces of differential forms defined on a compact smooth oriented manifold without boundary.

The equation

$$(\lambda - \Delta)\alpha_t = \nu\Delta\alpha - id\Delta^2\alpha \tag{6}$$

describes weakly linear effects in hydrodynamics in a particular case. Here the coefficients $\nu \in \mathbb{R}_+$, $\lambda,\ d \in \mathbb{R}$ describe the parameters of the system [2]. The work [11] proves solvability of equation (7) in the case when the right-hand side contains nonlinearity. The paper [28] considers the question of the stability of solutions and shows the existence of stable and unstable invariant spaces of the linear Ginzburg – Landau equation.

This article presents the author's results on numerical stable and unstable solutions of stochastic analogues of the equations (2)–(6). For this, these equations will be considered in the form of a linear stochastic Sobolev type equation

$$L \overset{\circ}{\eta} = M\eta, \tag{7}$$

where η is a stochastic **K**-process, $\overset{\circ}{\eta}$ is a Nelson – Glicklikh derivative [8]. At present, a large number of works are devoted to the study of the Cauchy and Showalter – Sidorov problems for stochastic Sobolev type equations [4–7], [27] and etc. In [27], the question of the solvability of the Cauchy and Showalter – Sidorov problems for the equation (7) was studied in the case when the operator M is a (L, p)-bounded operator. In [4] the case of the relative sectoriality of the operator L was considered, and in the paper [5] the case of the relative radiality of the operator L was considered. The papers [14, 29, 31, 35] study the solvability and stability of the equation (7) in spaces of differential forms given on a smooth Riemannian manifold without boundary. [15–18] are devoted to the numerical study of Sobolev type stochastic linear equations on manifolds.

In this review article, we present the results of the author [30, 31, 33, 34] which present numerical experiments on the stability of stochastic analogs of the equations (2), (4), (5), (6). The article consists of Introduction, five sections and References. The first section defines the concepts of a random variable, a stochastic process, the Nelson–Glicklikh derivative, random **K**-variables, and **K**-«noise» spaces, establishes the existence of stable and unstable invariant spaces. The next four sections are devoted to applications of the results of Section 1. Namely, results on the existence of stable and unstable invariant spaces of the Barenblatt – Zheltov – Kochina equation are presented in Section 2; of the Oskolkov — in Section 3; of the Dzektser equation — in Section 4; of the Ginzburg – Landau — in Section 5.

Construct the spaces of **K**-variables and **K**-«noises». Let $\mathfrak{U}$ ($\mathfrak{F}$) be a real separable Hilbert space with a basis $\{\varphi_k\}$ ($\{\psi_k\}$) orthonormal with respect to the scalar product $<\cdot,\cdot>_{\mathfrak{U}}$ ($<\cdot,\cdot>_{\mathfrak{F}}$). Choose the sequence $\mathbf{K} = \{\lambda_k\} \subset \mathbb{R}$ such that $\sum\limits_{k=1}^{\infty} \lambda_k^2 < \infty$, and the sequence $\{\xi_k\} \subset \mathbf{L_2}$ ($\{\zeta_k\} \subset \mathbf{L_2}$) uniformly bounded random variables. Next, we construct *random* **K**-*value*

$$\xi = \sum_{k=1}^{\infty} \lambda_k \xi_k \varphi_k \ \left(\zeta = \sum_{k=1}^{\infty} \lambda_k \zeta_k \psi_k \right).$$

The completion of the linear shell with random **K**-values according to the norm

$$\|\xi\|^2_{\mathbf{U_K L_2}} = \sum_{k=1}^{\infty} \lambda_k^2 \mathbf{D}\xi_k \ \left(\|\zeta\|^2_{\mathbf{F_K L_2}} = \sum_{k=1}^{\infty} \lambda_k^2 \mathbf{D}\zeta_k \right)$$

is a Hilbert space. We it denote the symbol $\mathbf{U_K L_2}$ ($\mathbf{F_K L_2}$) and call *space of random* **K**-*values.*

The stochastic process $\eta : (\varepsilon, \tau) \to \mathbf{U_K L}_2$ define by formula

$$\eta(t) = \sum_{k=1}^{\infty} \lambda_k \xi_k(t) \varphi_k, \tag{8}$$

here $\{\xi_k\}$ be some sequence from $\mathbf{CL}_2$ and $\mathcal{J} = (\varepsilon, \tau) \subset \mathbb{R}$. It is called a *stochastic continuous* **K**-*process*, if the number of the right side converges uniformly on any compact set in $\mathcal{J}$ with the norm $\|\cdot\|_{\mathbf{U_K L_2}}$, and the trajectory of the process $\eta = \eta(t)$ almost surely continuously. A continuous stochastic **K**-process $\eta = \eta(t)$ is called a *continuously differentiable process by Nelson – Gliklikh on* $\mathcal{J}$, if the series

$$\overset{o}{\eta}(t) = \sum_{k=1}^{\infty} \lambda_k \overset{o}{\xi}_k (t) \varphi_k \tag{9}$$

converges on any compact in $\mathcal{J}$ according to the norm $\|\cdot\|_{\mathbf{U}}$ and the trajectories of the process $\overset{o}{\eta} = \overset{o}{\eta}(t)$ are almost certainly continuous. The symbol $\mathbf{C}(\mathcal{J}, \mathbf{U_K L}_2)$ denote the space of continuous stochastic $\mathbf{K}$-processes and symbol $\mathbf{C}^l(\mathcal{J}, \mathbf{U_K L}_2)$ denote the space of continuously differentiable up to order $l \in \mathbb{N}$ of the stochastic $\mathbf{K}$-processes.

Let the operators $L, M \in \mathcal{L}(\mathbf{U_K L}_2; \mathbf{F_K L}_2)$, consider the equation (7). Let $\mathcal{J} = \{0\} \cup \mathbb{R}_+$. Stochastic $\mathbf{K}$-process $\eta \in \mathbf{C}^1(\mathcal{J}; \mathbf{U_K L}_2)$ is called a *solution of the equation (7)*, if all its trajectories satisfy the equation (7) for all $t \in \mathcal{J}$. The solution of $\eta = \eta(t)$ of the equation (7) is called a *solution of the Cauchy problem*

$$\eta(0) = \eta_0, \tag{10}$$

if the equality (7) holds for some random $\mathbf{K}$-value $\eta_0 \in \mathbf{U_K L}_2$.

Definition 1. *The set $\mathbf{P_K L}_2 \subset \mathbf{U_K L}_2$ is called a stochastic phase space of equation (7), if*

(i) probably almost every solution path $\eta = \eta(t)$ of the equation (7) lies in $\mathbf{P_K L}_2$, i.e. $\eta(t) \in \mathbf{P_K L}_2, t \in \mathbb{R}$, for almost all trajectories;

(ii) for almost all $\eta_0 \in \mathbf{P_K L}_2$ exists a solution to the problem (7), (10).

Suppose the operator M is (L,p)-bounded, then there is an analytic group of operators

$$U^t = \frac{1}{2\pi i} \int\limits_{\Gamma} (\mu L - M)^{-1} M e^{\mu t} d\mu, \tag{11}$$

where the contour Γ limits the region containing the L-spectrum of the operator M.

Theorem 1. *Suppose that the operator M is (L,p)-bounded, then phase space $\mathfrak{P}$ of the equation (7) is the image* $\text{im}U^{\bullet}$ *of the group (11).*

Definition 2. *The subspace $\mathbf{I} \subset \mathbf{H}^q_{0\mathbf{K}}$ is called an invariant space of the equation (7), if the solution to problem (7), (10) $\eta \in \mathbf{C}^1(\mathbb{R}; \mathbf{I})$ for any $\eta_0 \in \mathbf{I}$.*

Remark 1. For the existence of invariant spaces of equation (7), it is sufficient for the equation (7) to represent the L-spectrum of the operator M in the form of two disjoint parts, and at least one of these parts is closed.

Definition 3. *If the phase space $\mathbf{P} = \mathbf{I}^1 \oplus \mathbf{I}^2$, and there exist constants $N_k \in \mathbb{R}_+, \nu_k \in \mathbb{R}_+,\ k = 1,\ 2$, such that*

$$\begin{array}{ll} \|\eta^1(t)\|_{\mathbf{U}} \leq N_1 e^{-\nu_1(s-t)} \|\eta^1(s)\|_{\mathbf{U}} & \text{for} \quad s \geq t, \\ \|\eta^2(t)\|_{\mathbf{U}} \leq N_2 e^{-\nu_2(t-s)} \|\eta^2(s)\|_{\mathbf{U}} & \text{for} \quad t \geq s, \end{array}$$

where $\eta^k = \eta^k(t) \in \mathbf{I}^k$ for all $t \in \mathbb{R}$, and $\mathbf{I}^k,\ k = 1,\ 2$, is invariant space of equation (7), then solutions $\eta = \eta(t)$ to the equation (7) have exponential dichotomy. The space $\mathbf{I}^1$ ($\mathbf{I}^2$ is called a stable (unstable) invariant space of equation (7).

Theorem 2. [29] *Suppose that the operator M is (L,p)-bounded, and the L-spectrum of the operator M $\sigma^L(M) = \sigma^L_+(M) \oplus \sigma^L_-(M)$, where $\sigma^L_+(M) = \{\mu \in \sigma^L(M) : \text{Re } \mu > 0\} \neq \oslash$, $\sigma^L_-(M) = \{\mu \in \sigma^L(M) : \text{Re } \mu < 0\} \neq \oslash$. Then solutions of the equation (7) have exponential dichotomy.*

Remark 2. Similar results were obtained in the case when the operator M is (L,p)-sectorial operator [31], the operator M is (L,p)-radial operator [33].

Let us describe a general algorithm for finding stable and unstable numerical solutions of linear stochastic equations, which can be considered as an equation (7).

Step 1. Input coefficients of the equation.

Step 2. Subroutine for calculating the eigenvalues of the Laplace operator.

Step 3. Subroutine for computing functions of the basis $\{\varphi_k\}$ in the space $\mathfrak{U}$.

Step 5. Construct the vector $\zeta_0 = \sum\limits_{k=1}^{K} \frac{1}{k^2}\xi_k\varphi_k$, where ξ_k is a random variable $\backsim N(0,1)$ (i.e., ξ_k is a random variable with normal distribution, zero mathematical expectation, and dispersion equal to unity).

Step 6. Checking the invertibility of the operator L. Calculation of the number N_0 under which the operator L is irreversible.

Step 7. Relative spectrum calculation.

Step 8. Calculation of numbers N_u and N_s for which the relative spectrum is positive and negative.

Step 9. If $N_s \neq 0$, then the subroutine for finding a stable solution to the equation. Implementation of the Galerkin method for finding a stable solution to the equation. Derivation of Galerkin coefficients, solutions, solution graph.

Step 10. If $N_u \neq 0$, then the subroutine for finding an unstable solution to the equation. stable solution of the equation. Implementation of the Galerkin method for finding an unstable solution to an equation. Derivation of Galerkin coefficients, solutions, solution graph.

Step 11. If $N_0 \neq 0$, then finding a stationary solution.

Finally, consider the space of random **K**-variables and spaces of **K**-«noises» defined on Riemannian manifolds. Let M be a connected oriented compact Riemannian manifold without boundary having class C^∞ and dimension d. Similarly to the reasoning above, consider *spaces of random* **K**-*variables* defined on the manifold M: $\mathbf{H}^q_{0\mathbf{K}}\mathbf{L}_2$ and $\mathbf{H}^q_{l\mathbf{K}}\mathbf{L}_2$, where $\mathbf{K} = \{\lambda_k\}$ is a monotone sequence such that $\sum\limits_{k=1}^{\infty} \lambda_k^2 < +\infty$. The elements of these spaces are vectors $\alpha = \sum\limits_{k=1}^{\infty} \lambda_k\xi_k\varphi_k$ and $\beta = \sum\limits_{k=1}^{\infty} \lambda_k\xi\psi_k$, respectively, where $\{\varphi_k\}$ and $\{\psi_k\}$ are the operator eigenvectors, which are orthonormal with respect to $< \cdot, \cdot >_0$ and $< \cdot, \cdot >_l$.

1. Exponential Dichotomies in «Noise» Spaces

2. The Stochastic Barenblatt – Zheltov – Kochina Equation

Consider the question on the stability of equation (2) in the spaces $\mathbf{H}^q_{0\mathbf{K}}\mathbf{L}_2$. Denote

$$L = (\lambda + \Delta), \ M = \alpha\Delta, \tag{12}$$

where Δ is the Laplace – Beltrami operator. Consider the stochastic equation with differential forms

$$L \overset{\circ}{\zeta} = M\zeta \tag{13}$$

with the Cauchy condition

$$\chi(0) = \zeta_0. \tag{14}$$

Theorem 3. [30] *The solutions $\eta = \eta(t)$ to problem (13), (14) have exponential dichotomies for any $\alpha \in \mathbb{R}$, $\lambda \in \mathbb{R}_-$, $\eta_0 \in \mathbf{H}^q_{0\mathbf{K}}\mathbf{L}_2$.*

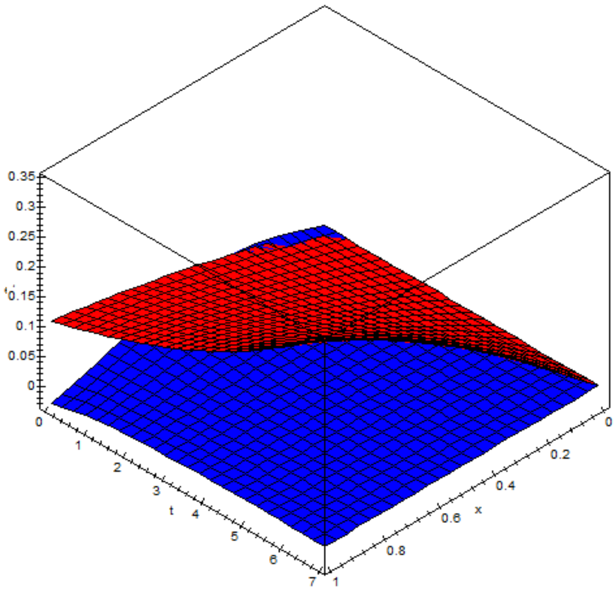

Fig. 1. Exponential dichotomies of the stochastic Barenblatt – Zheltov – Kochina equation for $\lambda = -4$, $\alpha = 0.5$, $t \in [0, 3]$ in the section $y = 0.5$.

(ii) The phase space of problem (13), (14) coincides with a stable invariant space for any $\alpha \in \mathbb{R}_-$ and $\lambda \in \mathbb{R}_+$, and the phase space of problem (13), (14) coincides with an unstable invariant space for any $\alpha, \lambda \in \mathbb{R}_+$.

On the two-dimensional torus $T^2 = [0, 1] \times [0, 1]$, consider the map $\delta : Sq \to U$, where $U \subset T^2$, and Sq is the inside of the square with vertices at the points (0,0), (0,1), (1,0), and (1,1). If $u = u(t)$ is a solution to this differential equation on the torus, then $z = z(t) = \delta(u(t))$ is a solution to this differential equation on the map. Then if the solution to the differential equation is stable on the torus T^2, then the solution is stable on the map Sq. The converse is also true. Due to the smoothness of the solutions, the nature of stability does not change at the places where the maps are «glued». Therefore, we reduce consideration of the question of stability of solutions to equation (13) on the torus T^2 to consideration of the same question on one of the maps Sq.

The solutions to problem (13), (14) have exponential dichotomy for $\lambda = -4$, $\alpha = -0.5$. The solutions

$$\zeta_-(t) = \sum_{l=1}^{n-1} e^{\mu_l t} \left(\sum_{k=1}^{M} \lambda_k \xi_k (\varphi_k, \varphi_l)_{L_2} \varphi_l \right) \tag{15}$$

belong to an unstable invariant space, and the solutions

$$\zeta_+(t) = \sum_{l=n+1}^{K} e^{\mu_l t} \left(\sum_{k=1}^{M} \lambda_k \xi_k (\varphi_k, \varphi_l)_{L_2} \varphi_l \right) \tag{16}$$

belong to a stable invariant space. Fig. 1 shows the graph of the solution for $t \in [0, 7]$ in the section $y = 0.5$.

3. The Stochastic Oskolkov Equation

We will study the stochastic analogue of the equation (4). Define the operators L and M by the formulas

$$L = (\lambda + \Delta)\Delta, \ M = -\nu\Delta^2. \tag{17}$$

Then the stochastic equation (4) can be considered as the equation

$$L \overset{\circ}{\varsigma} = M\varsigma, \tag{18}$$

where operators $L, \ M \in \mathcal{L}(\mathbf{H}_0^q, \mathbf{H}_2^q)$, and the operator M is $(L, 0)$-bounded operator. The phase space of the stochastic Oskolkov equation has the form

$$\mathbf{P} = \begin{cases} \mathbf{H}_0^q, \ \vartheta_l \neq \lambda, \\ \varsigma \in \mathbf{H}_0^q : <\varsigma, \varphi_l>_0 = 0, \ \vartheta_l = \lambda, \end{cases}$$

where $\sigma(\Delta) = \{\vartheta_l\}$.

Theorem 4. [35] *For any $\lambda \in \mathbb{R}_-$, $\nu \in \mathbb{R}$ solutions of the equation (18) have exponential dichotomy.*

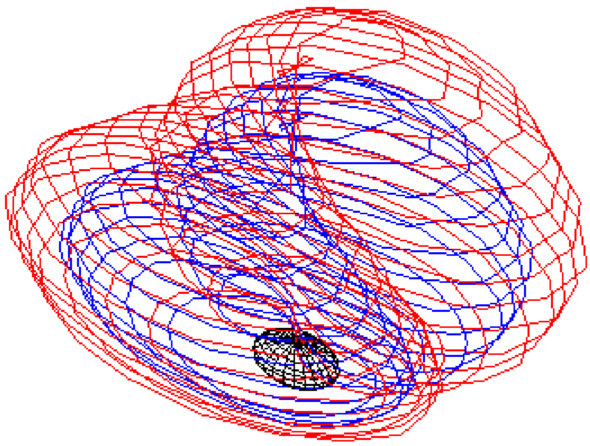

Fig. 2. Stable solutions of the stochastic Oskolkov equation (red color for $t = 0.1$, blue color for $t = 0.5$) for $\lambda = -3.6$, $\nu = 0.5$ on a two-dimensional sphere (black).

Consider the space of 0-form on a single sphere with a center in the initial order. The Laplace – Beltrami operator in the spheral system of coordinates (Θ, φ) is assigned with the formula

$$\Delta_{S^2} = (\sin\varphi)^{-1}\frac{\partial}{\partial\varphi}(\sin\varphi\partial\varphi) + (\sin\varphi)^{-2}\frac{\partial}{\partial\Theta}.$$

If ϑ_l are eigenvalues of the Laplace operator, then

$$Y_l^m(\Theta, \varphi) = \begin{cases} P_l^m(\cos\Theta)\cos m\varphi, \ m = 0, 1, ..., l; \\ P_l^{|m|}(\cos\Theta)\sin|m|\varphi, \ m = -l, -(l+1), ..., -1 \end{cases} \tag{19}$$

are corresponding eigenfunctions, orthonormal with respect to a scalar product. Here

$$P_l(t) = \frac{1}{2^l l!}\frac{d^l}{dt^l}(t^2 - l)^l$$

is a Lagrange polynomial of degree l, and

$$P_l^{|m|}(t) = (1 - t^2)^{\frac{|m|}{2}}\frac{d^{|m|}}{dt^{|m|}}P_l(t) \tag{20}$$

are associated Lagrange polynomials. The scalar product is calculated by the formula

$$(Y_{l1}^{m1}, Y_{l2}^{m2}) = \int\limits_0^{2\pi}\cos m_1\varphi\cos m_2\varphi d\varphi\int\limits_{-1}^{1}P_{l1}^{m_1}(t)P_{l2}^{m_2}(t)dt. \tag{21}$$

Find a stable solution

$$\varsigma_1(t) = \sum_{l=M_1}^{K} e^{\mu_l t}\left(\sum_{k=1}^{K}(\vartheta_k)^{-1}\xi_k\sum_{m_1=1}^{k}\sum_{m_2=1}^{l}(Y_l^{m_1}, Y_k^{m_2})Y_l^{m_1}\right) \tag{22}$$

and an unstable solution

$$\varsigma_2(t) = \sum_{l=1}^{M2} e^{\mu_l t}\left(\sum_{k=1}^{K}(\vartheta_k)^{-1}\xi_k\sum_{m_1=1}^{k}\sum_{m_2=1}^{l}(Y_l^{m_1}, Y_k^{m_2})Y_l^{m_1}\right) \tag{23}$$

to the equation (18). Fig. 2 shows a graph of a stable solution of the equation (18) on a sphere.

4. The Stochastic Dzektser Equation

In order to study the existence of stable and unstable invariant spaces of equation (5) in the spaces $\mathbf{H}_0^q$, consider

$$L = (\lambda + \Delta),\ M = -\alpha\Delta + \beta\Delta^2, \tag{24}$$

where Δ is the Laplace – Beltrami operator.

Theorem 5. [31] *Let $\lambda \neq \frac{\alpha}{\beta}$ and $\alpha,\ \beta,\ \lambda \in \mathbb{R}_+$. Then*

(i) if $\frac{\alpha}{\beta} > \vartheta_1$, then the solutions to equation (7) have an exponential dichotomy;

(ii) if $\frac{\alpha}{\beta} < \vartheta_1$, then there exists only a stable invariant space of equation (4).

Consider the 3-torus $T^3 = [0, 2\pi] \times [0, 2\pi] \times [0, 2\pi]$. As a map, consider the cube with the side equal to 2π. If the solutions to equation (1) are stable on the map of the manifold, then the solutions are also stable on the manifold. The converse is also true. In view of the foregoing, we consider a computational experiment on the map. The «gluing» conditions are satisfied due to the choice of the functions $\varphi(x, y, z)$. Fig. 3 shows the dichotomies of solutions of the stochastic equation (5) in the section $x,\ y,\ z = 3.1$ at t from 0 to 1.22.

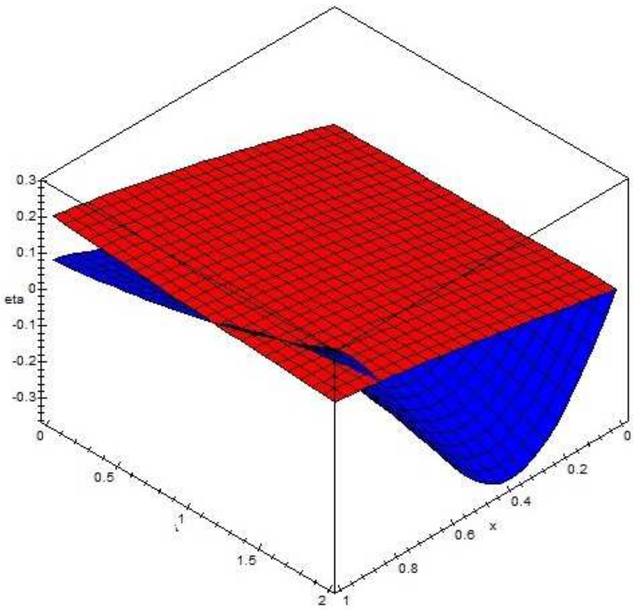

Fig. 3. Exponential dichotomies of the stochastic Dzektser for $\lambda = 0.2$, $\alpha = 8$, $\beta = 2$, $x,\ y,\ z = 3.1$, $t = 0..3$

5. The Stochastic Ginzburg – Landau Equation

In the spaces $\mathbf{H}_0^q,\ q = 0, 1, 2$, consider equation (6) as the stochastic linear Sobolev-type equation

$$L \overset{\circ}{\phi} = M\phi, \tag{25}$$

where the operators $L,\ M : \mathbf{H}_0^q \to \mathbf{H}_2^q$ are defined by the following formulas:

$$L = \lambda + \Delta,\ M = -\nu\Delta - id\Delta^2.$$

For any $\lambda,\ d \in \mathbb{R}$ and $\nu \in \mathbb{R}_+$, the operator M is strongly $(L, 0)$-radial. Due to the fact that the relative spectrum has the form

$$\sigma^L(M) = \left\{ \mu \in \mathbb{C} : \mu_k = \frac{-\nu\vartheta_k - id\vartheta_k^2}{\lambda + \vartheta_k} \right\} \tag{26}$$

the following theorem is true.

Theorem 6. [33] *(i) Let $\lambda,\ \nu \in \mathbb{R}_+$ and $d \in \mathbb{R}$. Then the solutions to equation (25) are exponentially stable.*

(ii) Let $\lambda \in \mathbb{R}_-$, $\nu \in \mathbb{R}_+$ and $d \in \mathbb{R}$. Then for $-\lambda > \lambda_1$ the solutions to equation (25) have exponential dichotomy, and for $-\lambda < \lambda_1$ the solutions to equation (25) are exponentially stable.

The torus $T^2 = [0,\pi] \times [0,\pi]$ can be represented as a direct product $T^2 = S^1 \oplus S^1$, where S^1 is a circle of the radius π. Choose a square $Sq = \{(x,y) : 0 \le x \le \pi, 0 \le y \le \pi\}$, which is one of the maps of the torus T^2. The two-dimensional torus T^2 can be represented as «gluing» opposite sides of the square Sq.

Let's apply the algorithm described in paragraph 1, construct graphs of a stable

$$\eta_1(t) = \sum_{l=1}^{M_1} e^{\mu_l t} \left(\int_0^\pi \int_0^\pi \eta_0 \varphi_l dx dy \right) \varphi_l$$

and unstable

$$\eta_2(t) = \sum_{l=M_2}^{K} e^{\mu_l t} \left(\int_0^\pi \int_0^\pi \eta_0 \varphi_l dx dy \right) \varphi_l$$

solutions of the stochastic equation (6), where $M_1 = \max\{l : \lambda_l < -\lambda\}$ and $M_2 = \min\{l : \lambda_l > -\lambda\}$.

Fig. 4 shows a graph of the real part of the stable solution Re $\eta_1(t)$ to equation (25) for $\lambda = 4.2$, $\nu = 5$, $d = 2$ at times $t = 9,\ 9.1,\ 9.263$. Fig. 5 shows the exponentially dichotomous behavior of the real part of the solution to equation (25) for $\lambda = -4.2$, $\nu = 0.2$, $d = 2$ in the section $x = \frac{\pi}{2}$, $y = \frac{\pi}{2}$.

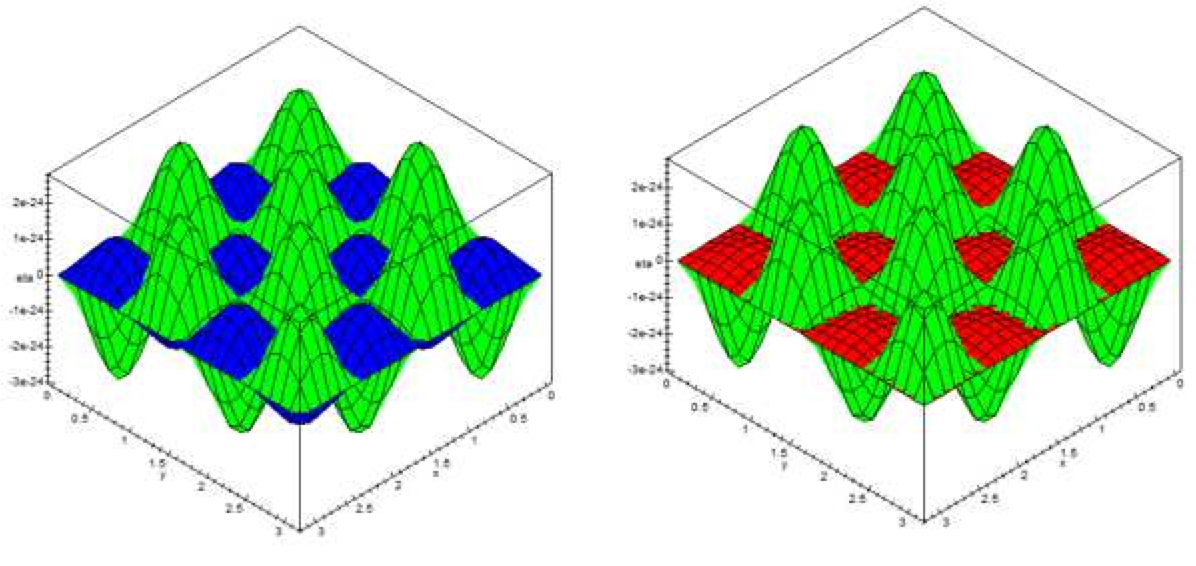

Fig. 4. Stable solutions of the stochastic Ginzburg – Landau equation for $t = 9$ (green), $t = 9.1$ (blue) and $t = 9.263$ (red)

Conclusion

In the future, it is planned to extend the results of [38] to semilinear Sobolev type equations. It is also planned to carry out numerical experiments for the semilinear stochastic equations of Hoff and Benjamin – Bonn – Mahoney, analytical studies of which were presented in [36, 37].

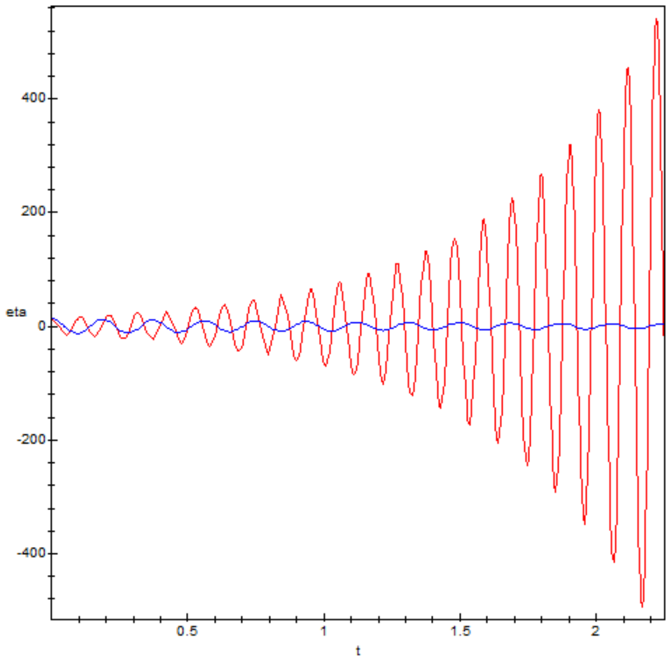

Fig. 5. Exponential dichotomies of the stochastic Ginzburg – Landau equation for $\lambda = -4.2$, $\nu = 0.2$, $d = 2$, $x = \frac{\pi}{2}$, $y = \frac{\pi}{2}$

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ЭКСПОНЕНЦИАЛЬНЫЕ ДИХОТОМИИ СТОХАСТИЧЕСКИХ УРАВНЕНИЙ СОБОЛЕВСКОГО ТИПА

О. Г. Китаева

Статья носит обзорный характер. Она содержит результаты по исследованию устойчивости стохастических линейных уравнений соболевского типа в терминах устойчивого и неустойчивого инвариантных пространств и экспоненциальных дихотомий. Рассмотрены стохастические аналоги уравнения Баренблатта – Желтова – Кочиной давления жидкости, фильтрующейся в трещиновато-пористой среде, линейного уравнения Осколкова плоскопараллельных течений вязкоупругой жидкости, уравнения Дзекцера, описывающего эволюцию свободной поверхности фильтрующейся жидкости, уравнения Гинзбурга – Ландау, моделирующего проводимость в магнитном поле. Данные уравнения можно рассматривать как частные случаи стохастического уравнения соболевского типа, где в качестве искомой величины выступает стохастический **K**-процесс, а под его производной понимается производная Нельсона – Гликлиха. В статье представлены результаты о существовании устойчивых и неустойчивых инвариантных пространств стохастических уравнений Баренблатта – Желтова – Кочиной, Осколкова, Дзекцера и Гинзбурга – Ландау. Описана общая схема численного алгоритма для нахождения устойчивого и неустойчивого решений этих уравнений, приведены результаты вычислительных экспериментов.

Ключевые слова: стохастические уравнения соболевского типа; инвариантные пространства; экспоненциальные дихотомии.

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