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ON OBSERVATION WHILE SOLVING THE PROBLEM OF OPTIMAL DYNAMIC MEASUREMENTS

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The article presents the rationale of a new algorithm for solving the problem of optimal dynamic measurements. The authors named the algorithm the spline method with simple averaging. The algorithm is based on the application of the Kotelnikov theorem. The discussed algorithm is one of the numerical methods used in the theory of optimal dynamic measurements, which allow to find the input signal from a known output signal (or observation) and a known transfer function of the measuring device. In all formulations of the problem, it is assumed that the inertia of the measuring device is taken into account, and the differences are due to the inclusion of interferences of various natures in the mathematical model. Consideration of interference as «white noise» led to the development of analytical and numerical methods for solving the problem under discussion. In recent years, one of the areas of numerical research is the work with the observed signal. The article provides brief necessary theoretical information, an overview of numerical methods for using digital filters to process observation results with subsequent application of the spline method, and uses the results of an experiment to show the advantages of the spline method with simple averaging in the work with observation.

Keywords: optimal dynamic measurements; spline method; Leontief type system; Kotelnikov theorem.

Introduction

The theory of optimal dynamic measurements originates from a mathematical model of restoring a dynamically distorted signal from a known observed output signal and parameters of a measuring device (MD), which is based on the problem of optimal control for a Leontief type system [1]. The measuring device is modeled by a Leontief type system (or a description system)

$$\begin{cases} L\dot{x} = Ax + Bu, \\ y = Cx, \end{cases} \tag{1}$$

where L and A are matrices that characterize the structure of the MD, in some cases it is possible that $\det L = 0$ [2]; $x(t)$ and $\dot{x}(t)$ are vector-functions of the state of the MD and the velocity of the state change, respectively; $y(t)$ is a vector-function of observation; C is a rectangular matrix characterizing the interrelation between the system state and observation; $u(t)$ is a vector-function of measurements; B is a matrix characterizing interrelation between the system state and measurement. If L is not degenerate then system (1) can be reduced to

$$\begin{cases} \dot{x} = Mx + Fu, \\ y = Cx, \end{cases}$$

where $M = L^{-1}A$, $F = L^{-1}B$.

The initial Showalter – Sidorov condition

$$\left[(\alpha L - A)^{-1} L\right]^{p+1} (x(0) - x_0) = 0 \tag{2}$$

reflects initial state of the MD for some $x_0 \in R^n$, $\alpha \in \rho^L(M)$. The initial Showalter – Sidorov condition is equivalent to the initial Cauchy condition $x(0) = x_0$ in the case of $\det L \neq 0$.

The unknown input signal is found as a solution to the optimal control problem in which we minimize the penalty functional

$$J(v) = \min_{u \in U_\partial} J(x(u), u)$$

of the form

$$J(u) = J(x(u)) = \sum_{k=0}^{1} \int_0^\tau \left\| Cx^{(k)}(t) - y_0^{(k)}(t) \right\|^2 dt. \tag{3}$$

The form of functional (3) determines the main idea of the mathematical model of optimal dynamic measurements that is minimizing the discrepancy between the output signal $y(t) = Cx(t)$ modelled by system (1) and the observed output signal $y_0(t)$ (or observation) according to the readings of MD and their derivatives [3]. The function $v(t)$, at which the minimum of the penalty functional is reached, is called the optimal dynamic measurement.

Assuming that the input signal is distorted by interferences of the «white noise» type, it is necessary to consider the stochastic model of dynamic measurements, which is presented in Section 1 of the article. In Section 2, we give a brief overview of the proposed approaches to «purification» of observation [4] with the transition to a deterministic model of optimal dynamic measurements. In Section 3, we discuss the advantages of using the Kotelnikov sampling theorem for observations, and present the results of computational experiments.

1. Stochastic Model of Optimal Dynamic Measurements

Let $\Omega \equiv (\Omega, \mathrm{A}, P)$ be a complete probability space, $\mathbb{R}$ be a set of real numbers endowed with the Boreal σ-algebra. The measurable mapping $\xi : \Omega \to \mathbb{R}$ is called a random variable. The set of random variables with $E\xi = 0$ and finite variance forms a Hilbert space $\mathbf{L_2}$ with an inner product $< \xi_1, \xi_2 >= E(\xi_1 \xi_2)$. Let $I \subset \mathbb{R}$ be some interval. The mapping $\eta : I \times \Omega \to \mathbb{R}$ of the form $\eta = \eta(t, \omega)$ is called an *(one-dimensional) stochastic process*, therefore the value of the mapping $\eta = \eta(t, \cdot)$ is a random variable for every fixed $t \in I$, i.e. $\eta = \eta(t, \cdot) \in \mathbf{L}_2$ and the value of a stochastic process $\eta = \eta(\cdot, \omega)$ is called a *(sample) trajectory* for every fixed $\omega \in \Omega$. The random process η is called *continuous*, if almost surely all its trajectories are continuous. Denote by $C\mathbf{L}_2$ the space of continuous random processes. A continuous random process, which independent random variables are Gaussian, is called *Gaussian*. Denote by $\overset{o}{\eta}{}^{(\ell)}$ the ℓ-th Nelson – Gliklikh derivative of the stochastic process η [5]. The set of continuous stochastic processes having continuous Nelson – Gliklikh derivatives up to order $k \in \mathbb{N}$ at each point of the set I forms a space, which is denoted by $C^k\mathbf{L}_2$.

Consider the stochastic model of the MD

$$\begin{cases} L\overset{\circ}{\xi} = A\xi + B(u+\phi), \\ \eta = C\xi + \nu, \end{cases} \tag{4}$$

$$\left[(\alpha L - A)^{-1} L\right]^{p+1} (\xi(0) - \xi_0) = 0. \tag{5}$$

Here the matrices L, A, B, C have the same sense as in (1). Random processes ϕ and ν determine noises in the circuits and at the output of the MD, respectively.

Similarly to the deterministic case, when investigating the problem on restoration of a dynamically distorted signal by random interference in the circuits and at the output of the MD, we consider the control problem

$$J(v) = \min_{u \in U_\partial} J(u), \tag{6}$$

where the functional

$$J(u) = J(\eta(u)) = \sum_{k=0}^{1} \int_0^\tau E \left\| \overset{\circ}{\eta}{}^{(k)}(t) - \eta_0^{(k)}(t) \right\|^2 dt \tag{7}$$

reflects the closeness of the real observation $\eta_0(t)$ and the virtual observation $\eta(t)$ obtained on the basis of a mathematical model of the MD.

The minimum point $v(t)$ of functional on the set U_∂ that is a solution to optimal control problem (4) – (7) is called an optimal dynamic measurement. In practice, there is only indirect information about $v(t)$.

2. Digital Filters and Observation

One of the developed directions in the theory of optimal measurements is the application of various methods to filtering the observation in order to obtain a smoothed observation function $\overline{y}_0(t)$ with a subsequent transition from stochastic model of optimal dynamic measurements (4) – (7) to the deterministic model

$$\begin{cases} L\dot{\overline{x}} = A\overline{x} + B\overline{u}, \\ \overline{y} = C\overline{x}, \end{cases} \tag{8}$$

$$\left[(\alpha L - A)^{-1} L\right]^{p+1} (\overline{x}(0) - x_0) = 0, \tag{9}$$

$$J(\overline{v}) = \min_{\overline{u} \in U_\partial} J(\overline{x}(\overline{u}), \overline{u}), \tag{10}$$

$$J(\overline{u}) = J(\overline{x}(\overline{u})) = \sum_{k=0}^{1} \int_0^\tau \left\| C\overline{x}^{(k)}(t) - \overline{y}_0^{(k)}(t) \right\|^2 dt. \tag{11}$$

Note that the solution $\overline{v}$ to problem (8) – (11) is an approximate solution to problem (4) – (7).

To obtain a smoothed observation, the work [6] uses an algorithm for constructing a smoothed one-dimensional observation signal under the condition that the signal shape is convex upwards and has a single maximum point. To accept the assumption of similarity of an observation and a smoothed observation function, we test a statistical hypothesis of normal distribution of the parameters for the cross sections of the process η_0. In addition, in combination with the numerical algorithm described in [7], this approach allows to take into account the condition of degradation of the MD.

To obtain a smoothed observation, the works [8] and [9] use a digital moving average filter and the Savitsky – Golay digital filter, respectively. In both cases, for each experiment, it is necessary to select the parameters of digital filters that are the value and shift of the time window, data weights, which is a disadvantage of such methods. The advantage of these methods is their simplicity and the insignificance of information about the numerical characteristics of the noise. The work [10] uses an one-dimensional Kalman filter to obtain a smoothed observation under the assumption that «white noise» takes place only at the output of the MD. Note that its application requires information about the noise variance.

Note that all numerical algorithms for solving problem (8) – (11) use the approaches described in detail in [11, 12].

3. Kotelnikov Sampling Theorem and Observation

In [13], it is proposed to use Kotelnikov sampling theorem [14] to filter an observation. Its application allows to choose the discretization time interval Δ, which determines the number of samples $K = \frac{\Delta}{\delta}$, where $\delta = t_{i+1} - t_i$ is the time interval, through which the observation values are taken from the MD. Each of the samples has one discretization time interval, but different initial times t_{0+j-1}, where $j = 1, 3, ..., K$ is the number of the sample. Then, for each sample, the optimal dynamic dimension $\overline{v_j}(t), j = 1, \ldots, K$ is found using the spline method. This allows to determine $\overline{v_j}(t_i), j = 1, \ldots, K$ for all t_i. Based on them, for each nodal point t_i, a simple arithmetic mean value $V_i, i = 1, \ldots, n$ is determined, which is taken as the result of solving the problem.

In [13], the results of the computational experiment are given in part, and the algorithm is presented without any rationale of its steps. When comparing the results of different approaches to filtering observations, the work [4] also does not provide a rationale of the advantages of this approach compared to standard digital filters. Let us pay attention to these aspects in this section of the article.

Note the significant aspects and assumptions:

1) the characteristics of the MD are such that the discretization frequency allows to restore the input signal $u(t)$ without loss in its frequency range;

2) the technical capabilities of the MD are such that the frequency of fixing observations δ is less than the discretization frequency Δ;

3) the sum of the deterministic signal u and the «white noise» ϕ is a random process, therefore, the values of $u + \phi$ are independent at each moment of time, which allows us to consider the obtained K samples as independent ones;

4) the array of data on n measurements is constructed within one experiment, which is considered to be one implementation or trajectory, hence the deterministic part of the input signal u is unchanged.

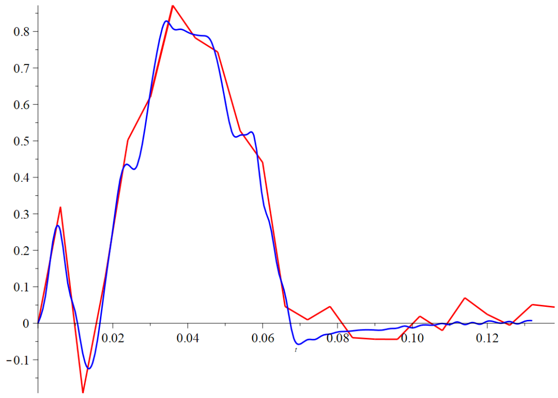

Fig. 1. Input signal reconstructed from the results of the 1-st sample of observations; blue color – $v(t)$, red color – $\overline{v_1}(t)$.

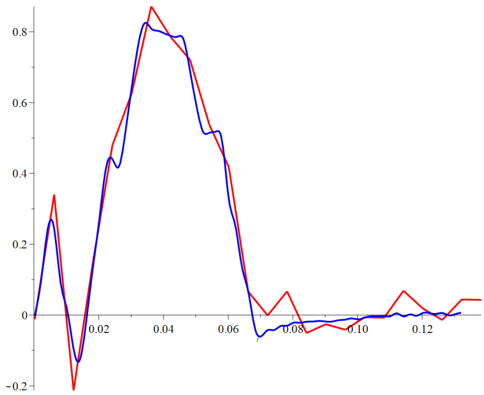

Fig. 2. Input signal reconstructed from the results of the 2-nd sample of observations; blue color – $v(t)$, red color – $\overline{v_2}(t)$.

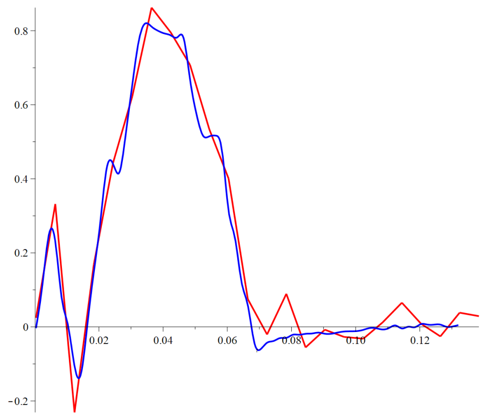

Fig. 3. Input signal reconstructed from the results of the 3-rd sample of observations; blue color – $v(t)$, red color – $\overline{v_3}(t)$.

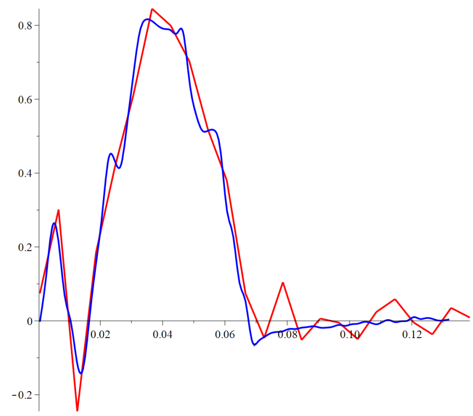

Fig. 4. Input signal reconstructed from the results of the 4-th sample of observations; blue color – $v(t)$, red color – $\overline{v_4}(t)$.

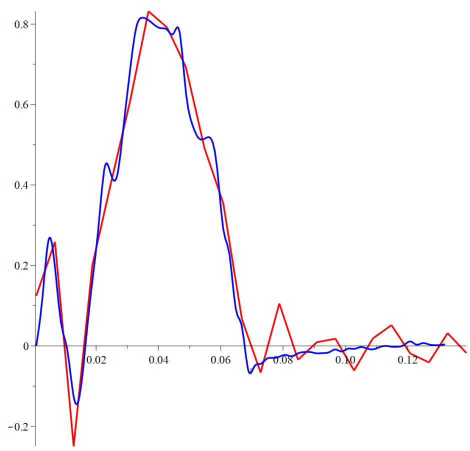

Fig. 5. Input signal reconstructed from the results of the 5-th sample of observations; blue color – $v(t)$, red color – $\overline{v_5}(t)$.

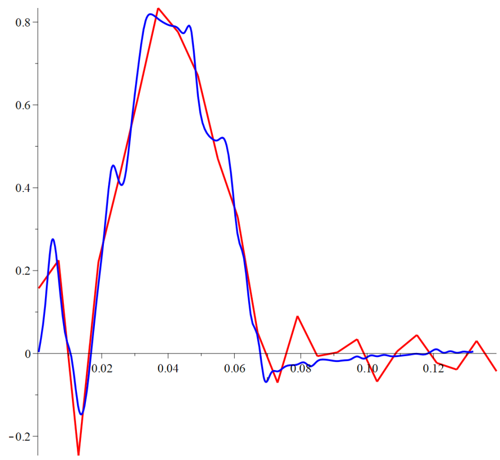

Fig. 6. Input signal reconstructed from the results of the 6-th sample of observations; blue color – $v(t)$, red color – $\overline{v_6}(t)$.

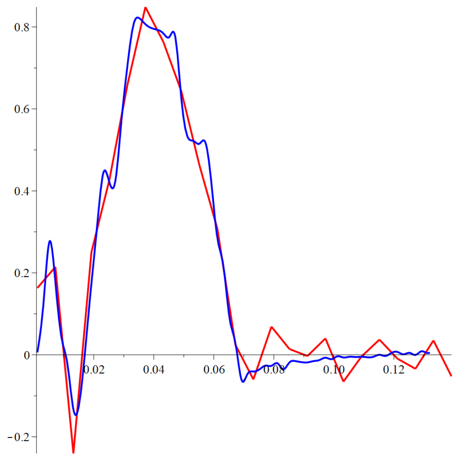

Fig. 7. Input signal reconstructed from the results of the 7-th sample of observations; blue color – $v(t)$, red color – $\overline{v_7}(t)$.

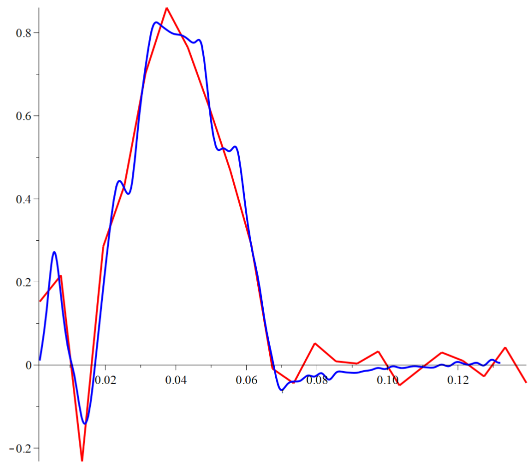

Fig. 8. Input signal reconstructed from the results of the 8-th sample of observations; blue color – $v(t)$, red color – $\overline{v_8}(t)$.

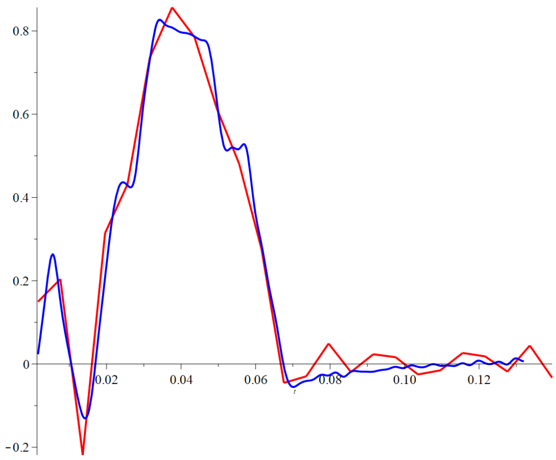

Fig. 9. Input signal reconstructed from the results of the 9-th sample of observations; blue color – $v(t)$, red color – $\overline{v_9}(t)$.

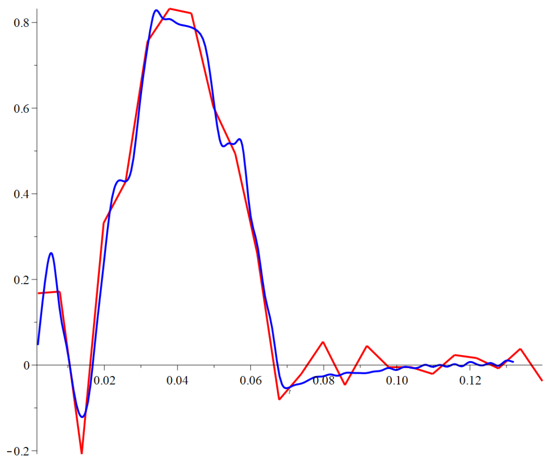

Fig. 10. Input signal reconstructed from the results of the 10-th sample of observations; blue color – $v(t)$, red color – $\overline{v_{10}}(t)$.

Figs. 1 – 10 show the results of a computational experiment, in which $n = 700$, $K = 10$.

In the figures, the blue color denotes the input signal taken from the control sensor and accepted as a «true» input signal $v(t)$. The red color denotes the approximate input signals $\overline{v_j}(t)$ obtained by one sample as a result of the numerical algorithm.

Note that during dynamic measurements, the value of the entire time interval T is small, and the value of δ is extremely small, therefore, following item 4, based on the results of each sample, we expect to obtain one reconstructed signal. But the results of K samples show that the resulting approximate signals have not only different values, but also different shapes in the time neighborhood, where the amplitude highest signal values are achieved.

At the same time, it follows from item 3 that the independence of the samples makes it possible to consider them to be «conditionally different» realizations of one signal and apply to them the statistical methods of processing. That is why the correct application of the Kotelnikov sampling theorem to the observed signal allows to obtain a more accurate approximate optimal dynamic measurement than the approximate optimal measurement obtained after filtering the observed signal with standard digital filters.

Fig. 11 shows the result of simple averaging.

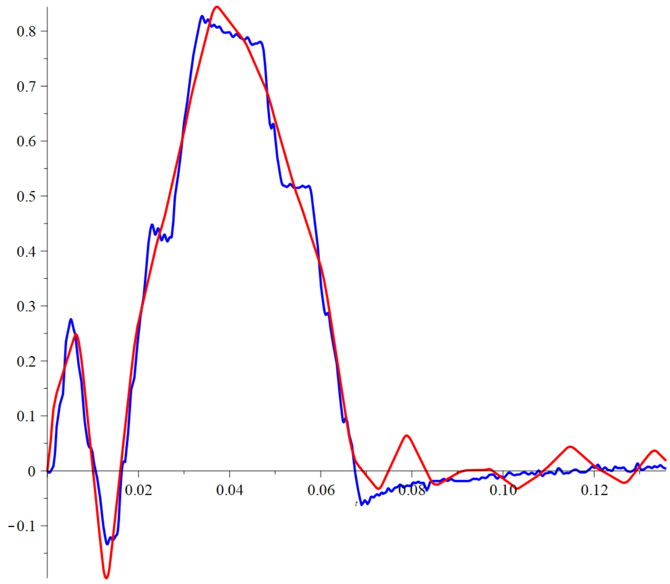

Fig. 11. Input signal after averaging; blue color – $v(t)$, red color – $V(t)$.

In addition, the average approximate measurement allows to calculate the variance from the data of K samples. Taking into account that the resulting value includes not only the noise variance, but also the error of the methods and calculations, the resulting value can be used as an estimate of the noise variance.

Therefore, the algorithm proposed in [13] makes it possible to restore a dynamically distorted signal with unknown noise parameters under the assumption of known frequency characteristics of the useful signal and the sensor.

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О НАБЛЮДЕНИИ ПРИ РЕШЕНИИ ЗАДАЧИ ОПТИМАЛЬНЫХ ДИНАМИЧЕСКИХ ИЗМЕРЕНИЙ

А. В. Келлер

В статье представлено обоснование нового алгоритма решения задачи оптимальных динамических измерений, названного авторами сплайн методом с простым усреднением, в его основе лежит применение теоремы Котельникова. Обсуждаемый алгоритм является одним из используемых в теории оптимальных динамических измерений численных методов, позволяющих по известному выходному сигналу (или наблюдению) и известной передаточной функции измерительного устройства находить входной сигнал. Во всех постановках задачи предполагается учет инерционности измерительного устройства, а различия обусловлены включением в математическую модель различных по природе помех. Рассмотрение помехи в качестве «белого шума» привело к развитию аналитических и численных методов решения обсуждаемой задачи. В последние годы одним из направлений численных исследований стала работа с наблюдаемым сигналом. В статье приведены кратко необходимые теоретические сведения, обзор численных методов по использованию цифровых фильтров для обработки результатов наблюдения с последующим применением сплайн метода, показаны преимущества подхода в работе с наблюдением сплайн метода с простым усреднением на основе результатов одного эксперимента.

Ключевые слова: оптимальные динамические измерения, сплайн метод, система леонтьевского типа, теорема Котельникова.

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