MSC 15B99 + 05C50 **DOI: 10.14529/jcem220304**

OPERATIONS ON GRAPH FUNCTIONS AND SPECTRAL PROPERTIES OF COMPOSITIONS OF REFLECTIONS

E. V. Kolmykova, Voronezh State University, Voronezh, Russian Federation, kolmykova_kate@bk.ru.

The article deals with operators acting on spaces of graph functions. Graph-theoretic methods are used to find the properties of the introduced operators. These properties show that the introduced operators are discrete analogues of differentiation and integration. The values of operators on some important graph functions are found. A method of using operators to study graph functions and methods of expressing some functions through others are developed. The characteristic polynomials of the Coxeter transformation is considered. Its coefficients can be expressed in terms of simple graph functions. With the help of the developed methodology a method of finding such expressions is proposed. The results of the article can be used to find spectral characteristics of compositions of reflections. These methods are simple and convenient to use.

Keywords: graph; tree; reflection; Coxeter transformation.

Introduction

The binding energy of a physical system is the difference between the energy of the state in which the components of the system are removed from each other and the total energy of the bound state of the system:

$$\Delta E = \sum_{i=1}^{N} E_i - E,$$

where E_i is total energy of i-th component in an unbound state, and E is the total energy of the bound system. For example suppose there is some tree-like molecule and some two atoms in it which are connected by a covalent bond. The energy of this bond is defined as follows. Two molecules obtained from the original molecule by breaking this bond are considered. The binding energy is the difference between the sum of the total energies of these two molecules and the total energy of the original molecule.

In this article we introduce the following construction. We take an arbitrary function defined on a set of trees, an arbitrary tree and an arbitrary edge in it. The sum of the function values on each of the two trees obtained by removing the taken edge is subtracted from the value of the function on the tree. We call this number the increment of the function on the edge of the tree. The operation of an increment considered by us as some linear operator Δ acting on functional spaces resembles the differentiation operator, and the inverse multivalued transformation I has properties similar to those of an indefinite integral. We obtain a convenient model that can be used to determine the binding energy between two atoms of tree-like molecules. However we apply this model in another area. Also, we analyze the constructed model and apply it to obtaining some spectral characteristics of compositions of reflections.

1. Graphs and Graph Functions

Consider finite undirected graphs without loops and multiple edges. Formal definition: a graph is a pair (V_0, V_1) of sets, where V_0 is a finite nonempty set (its elements are called vertices), V_1 is some set of two-element subsets of the set V_0 (elements of the set V_1 are called edges). The concept of a forest is a graph without cycles, and a tree is a connected graph without cycles is defined naturally ([1], Chapter IV, Supplement). We are interested in some operations on graphs and on graph functions.

Deleting an edge in a graph is an operation in which all vertices of the graph and all edges except one edge are preserved. Formal definition: let l be some edge of the graph $G = (V_0, V_1)$; the graph obtained by removing the edge l, is called the graph $(V_0, V_1 \backslash \{l\})$, which is denoted by $G \backslash l$. For example, if an edge is removed from a tree, a forest consisting of two trees is obtained.

A graph function is a function whose argument (or arguments) is a graph (or graphs, their subgraphs, vertices, edges, etc.).

For $n = 0, 1, 2, 3, ...$ in [2] on the set of all graphs, the functions θ_n were introduced as follows: $\theta_n(G)$ is the sum of the n-th degrees of the degrees of the vertices of the graph G (the degree of a vertex is the number of edges containing this vertex). For example, $\theta_0(G)$ is the number of vertices of the graph G, and $\theta_1(G)$ is twice the number of its edges.

We use constrictions of various graph functions defined on the set of all forests or on some sets of trees (without specifying this specifically, if it is clear which constriction is meant).

2. Construction of Model

First, let us consider one construction in which graph functions, graphs and their edges are used. The set of all trees is denoted by $\mathcal{T}$. And by $\mathcal{T}_*$ we denote the set of all trees, except for a tree consisting of one vertex.

Let us take some function $f_0 : \mathcal{T} \to \mathbb{R}$, some tree $T_0 \in \mathcal{T}_*$ and some edge l_0 of the tree T_0. Remove the edge l_0 from the tree T_0, we get a forest consisting of two trees. Let us denote one of them T_0^-, and the other T_0^+ (relatively speaking, the left and right trees). Consider the number

$$f_0(T_0) - (f_0(T_0^-) + f_0(T_0^+)).$$

It does not depend on which of the trees is chosen by the left and which by the right. This number is denoted by $[f_0, T_0, l_0]$ and is called the increment of the function f_0 on the edge l_0 of the tree T_0.

Example 1. Consider the graph shown in Figure 1. And consider horizontally edge. Figure 2 shows the graph obtained by removing this edge. For the tree H shown in Figure 1, we have $\theta_2(H) = 1^2 + 1^2 + 1^2 + 3^2 + 3^2 + 2^2 + 4^2 + 1^2 + 1^2 + 1^2 = 44$. For the trees shown in Figure 2 the values of the function θ_2 are equal to 16 and 20 respectively.

If $f_0 = \theta_2$, $T_0 = H$, and l_0 is the horizontal edge, then:

$$[f_0, T_0, l_0] = 44 - (16 + 20) = 8.$$

Example 2. Let M be some set of tree-like molecules. Each such molecule corresponds to a certain tree. Let the set M be closed with respect to the breaking of bonds (i.e., for any molecule and any bond, if you remove the bond, then both resulting molecules belong

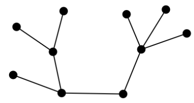

Fig. 1

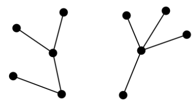

Fig. 2

to the set M). Consider the function $E : \mathcal{T} \to \mathbb{R}$, defined as follows. If T is a tree that does not correspond to any of the molecules under consideration, then $E(T) = 0$. If T is a tree corresponding to some molecule, then by $E(T)$ we denote the total binding energy of the electronic structure of this molecule. Note that this is a rather crude model, since we do not take into account the energy caused by vibrations, rotational movements, etc. If the graph of a molecule is some tree T_0, and the bond between two atoms corresponds to some edge l_0, then the energy of this bond is $-[E, T_0, l_0]$.

Let us introduce the linear spaces L_1 and L_2. The space L_1 consists of all possible real functions $f(X)$, where the variable X runs through the set $\mathcal{T}$. The space L_2 consists of all possible real functions $g(X, l)$, where the variable X runs through the set $\mathcal{T}_*$, and the variable l runs through the set of edges of the tree X.

Define the operator $\Delta : L_1 \to L_2$. For $f \in L_1$, we define the function $\Delta(f) \in L_2$ as follows:

$$\Delta(f)(X, l) = [f, X, l].$$

Example 3. Let us define $\gamma_1 \in L_2$ as follows: $\gamma_1(T, l)$ is the number of edges of the tree T which are incidented to the edge l (two edges are called incident if they have a common vertex). Let us show $\Delta(\theta_2) = 2\gamma_1 + 2$. Let T be an arbitrary tree of $\mathcal{T}_*$, l be its arbitrary edge, the ends of this edge denote u and v. Remove the edge l. The tree containing the vertex u is denoted by T^-, the other is denoted by T^+. The degree of the vertex u (or v)

in the tree T^- (or T^+) we denote by p (or q). Let a_1 (or a_2) be the sum of the squares of the degrees of the vertices of the tree T^- (or T^+), which are different from u (or v). Then $\theta_2(T) = a_1 + (p+1)^2 + (q+1)^2 + a_2$, $\theta_2(T^-) = a_1 + p^2$, $\theta_2(T^+) = q^2 + a_2$. And

$$\Delta(\theta_2)(T, l) = [\theta_2, T, l] = \theta_2(T) - (\theta_2(T^-) + \theta_2(T^+)) = 2(p+q) + 2 = 2\gamma_1(T, l) + 2.$$

Example 4. Let $M_1, M_2, ...$ be arbitrary sets of tree-like molecules closed with respect to bond breaking. Let us define the functions $E_{M_i} : \mathcal{T} \to \mathbb{R}$ similarly as in Example 4. Different sets M_i define different functions from the space L_1, since the considered sets of molecules may differ in the atoms of which elements the molecules consist of. If the graph of a molecule from the set M_i is some tree T, and the bond between two atoms corresponds to some edge l, then the binding energy is $E_{\text{bin}} = -\Delta(E_{M_i})(T, l)$. Thus, the operator Δ models a general scheme for determining the total binding energy of molecules.

3. Properties of the Model

The operator Δ is linear. Sometimes we write Δf instead of $\Delta(f)$. The operator Δ acts on functions defined on the set $\mathcal{T}$. If the function f is defined on some wider set, then by Δf we denote the value of the operator Δ on the restriction of the function f on the set $\mathcal{T}$. By C we denote the set of all real constant functions defined on $\mathcal{T}$. The product $C\theta_0$ is the set of all products $f\theta_0$, where $f \in C$. The operator $\Delta : L_1 \to L_2$ is not injective. Inverse mapping (defined on the image of the operator Δ)is multi-valued one, we denote that by I, its domain of definition $D(I)$ is $\Delta(L_1)$. Let us formulate the properties of mappings Δ and I (they resemble of some properties of differentiation and integration from classical analysis).

Theorem 1. *The general solution of the equation $\Delta y = 0$ is $y = C\theta_0$.*

Proof.

1) Let f be a solution of the equation $\Delta y = 0$, then

$$f(T) = f(T^-) + f(T^+), \quad \forall T, l. \tag{1}$$

Let us number all edges of the graph $T : l_1, l_2, ..., l_r$. We apply the statement (1) to the tree T and the edge l_1. Then, to the one of the obtained trees that contains the edge l_2, we similarly apply the statement (1). And so we do r once. At the rth step we get $f(T) = f(A_1) + f(A_1) + ... + f(A_1)$, where A_1 is a one-vertex tree, and the number of terms is equal to the number of vertices of the tree T. In other words: $f(T) = f(A_1)\theta_0(T)$, hence $f \in C\theta_0$.

2) Let $f \in C\theta_0$, then $f = c \cdot \theta_0$ for some $c \in \mathbb{R}$, so $\Delta(f) = \Delta(c \cdot \theta_0) = c\Delta(\theta_0) = 0$.

□

Theorem 2. *If $\Delta F = f$, then $I(f) = F + C\theta_0$.*

Proof.

If A is a linear operator acting from one linear space to another, then the general solution of the equation $Ax = b$ is $x_0 + \ker A$, where x_0 is some particular solution ([3], Chapter 2, §3, Theorem 1). In our case, $A = \Delta$ and (by Theorem 1) $\ker A = C\theta_0$.

□

Theorem 3. *1) If $f, g \in D(I)$ then $I(f+g) = I(f) + I(g)$.*
2) If $f \in D(I)$ and $\alpha \in \mathbb{R}\backslash\{0\}$ then $I(\alpha f) = \alpha I(f)$.

Proof.
This statement is a special case of the following one: let A be a linear operator acting from one linear space to another, A^{-1} be the inverse (possibly multi-valued) map defined on the image of the operator A. Then $A^{-1}(b+c) = A^{-1}(b) + A^{-1}(c)$ and $A^{-1}(\alpha b) = \alpha A^{-1}(b)$ for any nonzero scalar α. Really, $\alpha A^{-1}(b) = \alpha\{x|Ax = b\} = \{\alpha x|Ax = b\} = \{\alpha x|\alpha Ax = \alpha b\} = \{\alpha x|A(\alpha x) = \alpha b\} = \{z|Az = \alpha b\} = A^{-1}(\alpha b)$.

Further: $A^{-1}(b) + A^{-1}(c) = \{x|Ax = b\} + \{y|Ay = c\} = \{x + y|Ax = b \wedge Ay = c\} \subseteq \{x + y|Ax + Ay = b + c\} = \{x + y|A(x + y) = b + c\} = \{z|Az = b + c\} = A^{-1}(b + c)$.

Let us prove the inclusion $A^{-1}(b+c) \subseteq A^{-1}(b)+A^{-1}(c)$. Take arbitrarily $x \in A^{-1}(b+c)$ and $x_1 \in A^{-1}(b)$, put $x_2 = x - x_1$. Then $Ax_2 = A(x - x_1) = Ax - Ax_1 = (b + c) - b = c$, i.e. $x_2 \in A^{-1}(c)$. So, $x = x_1 + x_2 \in A^{-1}(b) + A^{-1}(c)$.

□

An edge and a vertex of a graph are called incident to each other if the vertex is one of the ends of the edge.

Let $e = \{u, v\}$ be an edge of the graph. Via e_u (resp. e_v) denote the number of edges incident to the vertex u (resp. v), except for the edge e. The numbers e_u and e_v are called the degrees of the edge e.

On the set of all graphs we define the function τ as follows: $\tau(G)$ is the sum of the products of the degrees of the vertices of the edges of the graph G. Let us define $\rho, \xi \in L_2$ as follows: $\rho(T, l)$ is the product of the degrees of the edge l, and $\xi(T, l)$ is the sum of the degrees of the vertices of the forest $T\backslash l$ adjacent to at least one end of the edge l.

For $n \in \mathbb{N}_0$, we define $\gamma_n \in L_2$ as follows: $\gamma_n(T, l)$ is the sum of the nth degrees of the degrees of the edge l.

Lemma 1. *The following equalities are true:*

$$\Delta(\theta_n) = \sum_{i=0}^{n-1} C_n^i \gamma_i, \quad n \in \mathbb{N},$$

$$\Delta(\tau) = \xi + \rho + \gamma_1 + 1.$$

Proof.
Let T be an arbitrary tree, l be its arbitrary edge, the ends of this edge are denoted by u and v. Remove the edge l. The tree containing the vertex u is denoted by T^-, the other is denoted by T^+. Via p (resp. q) we denote the degree of the vertex u (resp.v) in the tree T^- (resp. T^+).

Let a_1 (resp. a_2) be the sum of n-th degrees of the degrees of the vertices of the tree T^- (resp. T^+) exept by u (resp. v). Then $\theta_n(T) = a_1 + (p+1)^n + (q+1)^n + a_2$, $\theta_n(T^-) = a_1 + p^n$, $\theta_n(T^+) = q^n + a_2$. Then $\Delta(\theta_n)(T, l) = [\theta_n, T, l] = \theta_n(T) - (\theta_n(T^-) + \theta_n(T^+)) = (p + 1)^n - p^n + (q + 1)^n - q^n = \sum\limits_{i=1}^{n} C_n^i (p^{n-i} + q^{n-i}) = \sum\limits_{i=1}^{n} C_n^i \gamma_{n-i} = \sum\limits_{i=0}^{n-1} C_n^i \gamma_i$.

Let b_1 (resp. b_2) be the sum of the products of the degrees of the vertices of the edges of the tree T^- (resp. T^+) which are incident to the vertex u (resp. v). Let s_u (resp. s_v) be the sum of the degrees of the vertices of the tree T^- (resp. T^+) adjacent to the vertex u (resp. v). Then $\tau(T) = b_1 + s_u(p+1) + (p+1)(q+1) + s_v(q+1) + b_2$, $\tau(T^-) = b_1 + s_u p$, $\tau(T^+) = s_v q + b_2$. Then $\Delta(\tau)(T,l) = [\tau, T, l] = \tau(T) - (\tau(T^-) + \tau(T^+)) = s_u + (p+1)(q+1) + s_v = (s_u + s_v) + pq + (p+q) + 1 = \xi(T,l) + \rho(T,l) + \gamma_1(T,l) + 1$.

□

Example 5. 1) We have: $\Delta(\theta_3) = 3\gamma_2 + 3\gamma_1 + \gamma_0 = 3\gamma_2 + 3\gamma_1 + 2$, $\Delta(\theta_2) = 2\gamma_1 + \gamma_0 = 2\gamma_1 + 2$, $\Delta(\theta_1) = \gamma_0 = 2$. From Theorem 1 follows $\Delta(\theta_0) = 0$.

2) Using the previous lemma and theorems 1–3 we find $I(0) = C\theta_0$, $I(1) = I(0,5\gamma_0) = 0,5\theta_1 + C\theta_0$, $I(\gamma_1) = 0,5\theta_2 - 0,5\theta_1 + C\theta_0$, $I(\gamma_2) = \frac{1}{3}\theta_3 - \frac{1}{2}\theta_2 + \frac{1}{6}\theta_1 + C\theta_0$ $I(\xi + \rho) = \tau - I(\gamma_1) - I(1) = \tau - (0,5\theta_2 - 0,5\theta_1) - 0,5\theta_1 + C\theta_0 = \tau - 0,5\theta_2 + C\theta_0$

4. Reflections, Coxeter Transformations and Trees

We use the constructed model to study the spectral characteristics of some compositions of reflections.

Let G be some graph with n vertices. Its vertices denote by $1, 2, ..., n$. The set of all vertices adjacent to the vertex i we denote by $S_1(i)$. For $i \in \{1, 2, ..., n\}$ and $x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$ we denote

$$s_i(x) = -x_i + \sum_{j \in S_1(i)} x_j,$$

$$\sigma_i(x) = (x_1, x_2, ..., x_{i-1}, s_i(x), x_{i+1}, ..., x_n).$$

Thus the linear operators $\sigma_i : \mathbb{R}^n \to \mathbb{R}^n$ are defined. These operators are reflections with respect to some hyperplanes ([1], Chapter V).

The composition of operators

$$\sigma_n \sigma_{n-1} ... \sigma_1$$

is called the Coxeter transformation.

The monograph [4] is devoted to the Coxeter transformation. The most important problem associated with Coxeter transformations is to find their spectral properties.

Since the vertices of the graph G can be denoted by the numbers $1, 2, ..., n$ in several ways, several Coxeter transformations are associated with the graph G. It is shown in [5] and [6] that if a graph has cycles then the spectral properties of various Coxeter transformations associated with such a graph can significantly differ among themselves. Therefore in this case the graph itself is not a convenient enough tool for spectral investigation of the Coxeter transformation.

The situation strongly changes when there are no cycles in the graph. It is proved in [7] that if a graph is a forest then all Coxeter transformations associated with it are conjugated to each other and therefore have the same characteristic polynomial. In [8] it is proved that the Jordan form of the Coxeter transformation associated with a tree is uniquely determined by the spectrum of this transformation. Therefore in the case of a tree all spectral properties of the Coxeter transformation (including the sizes of the Jordan cells and the features of the iterative process) are uniquely determined by its characteristic polynomial.

The characteristic polynomial of the forest G is denoted by $\mathcal{H}_G(\lambda)$. The coefficient for λ^i in the polynomial $\mathcal{H}_G(\lambda)$ we denote by $h_i(G)$. This sets the functions h_i defined on the set $\mathcal{F}$ of all forests.

The functions h_i can be represented in terms of some other functions defined on the set $\mathcal{F}$ of all forests. In [2] such expressions are obtained for h_0, h_1, h_2 and h_3. The proof for these statements is quite cumbersome. In section 6 we show how shorter proofs can be obtained using the constructed model.

5. Properties of Characteristic Polynomial of the Coxeter Transformation

The n-vertex graph-chain has $V_0 = \{v_1, v_2, ..., v_n\}$, $V_1 = \{\{v_1, v_2\}, \{v_2, v_3\}, ..., \{v_{n-1}, v_n\}\}$ for $n \geq 2$; and $V_0 = \{v_1\}$, $V_1 = \emptyset$ when $n = 1$. We denote this graph by A_n. It is well known that $\mathcal{H}_{A_n}(\lambda) = \lambda^n + \lambda^{n-1} + ... + \lambda + 1$, i.e. $h_i(A_n) = 1$ for $i \leq n$; $h_i(A_n) = 0$ for $i > n$.

If l is an edge of the graph G then the graph obtained from G by removing the vertices of the edge l we denote by $G\backslash[l]$. Formal definition: if $G = (V_0, V_1)$ and $l = \{u, v\}$ then $G\backslash[l] = (V_0', V_1')$, where $V_0' = V_0\backslash l$, and V_1' is obtained from V_1 by removing all edges containing the vertex u or v. The following result is proved in [8].

Proposition 1. *(The splitting formula). If $T \in \mathcal{T}\backslash\{A_1, A_2\}$ and l is edge of T then:*

$$\mathcal{H}_T(\lambda) = \mathcal{H}_{T\backslash l}(\lambda) - \lambda\mathcal{H}_{T\backslash[l]}(\lambda).$$

The namber of trees of forest F is denoted by $k(F)$. Thus we inrtoduce the map k defined on set of all forests.

If F is an arbitrary forest consisting of trees $T_1, ..., T_{k(F)}$ then

$$\mathcal{H}_F(\lambda) = \prod \mathcal{H}_{T_i}(\lambda). \tag{2}$$

The following result is proved in [2].

Proposition 2. *The equalities:* $h_0 = 1$, $h_1 = k$, $h_2 = 2(\theta_0 - 1) - \dfrac{\theta_2}{2} + \dfrac{k^2 - 5k + 4}{2}$, $h_3 = \dfrac{5}{2}\theta_2 - \dfrac{10}{3}(\theta_0 - 1) - \dfrac{1}{3}\theta_3 - \tau + (k-1)(2(\theta_0 - 1) - \dfrac{\theta_2}{2}) + \dfrac{(k-1)(k^2 - 14k + 32)}{6}$ *are true.*

Let us rewrite this proposition by using forula $\theta_0 = 0.5\theta_1 + k$.

Proposition 3. *The equalities:* $h_0 = 1$, $h_1 = k$, $h_2 = \theta_1 - \dfrac{\theta_2}{2} + \dfrac{k(k-1)}{2}$, $h_3 = -\dfrac{5}{3}\theta_1 + \dfrac{5}{2}\theta_2 - \dfrac{1}{3}\theta_3 - \tau + (k-1)(\dfrac{\theta_1}{2} - \dfrac{\theta_2}{2}) + \dfrac{k(k-1)(k-2)}{6}$ *are true.*

6. Another Way to Get the Formula for h_3

1) Let us take an arbitrary tree $T \notin \{A_1, A_2\}$ and its arbitrary edge l. Using the proposition 1 we obtain: $\mathcal{H}_T(\lambda) = \mathcal{H}_{T^-}(\lambda)\mathcal{H}_{T^+}(\lambda) - \lambda\mathcal{H}_{T\backslash[l]}(\lambda)$.

In more detail, given that $h_1(T^-) = h_1(T^+) = 1$, we get:

$$\mathcal{H}_T(\lambda) = (...+h_3(T^-)\lambda^3 + h_2(T^-)\lambda^2 + \lambda + 1)(...+h_3(T^+)\lambda^3 + h_2(T^+)\lambda^2 + \lambda + 1) - \lambda\mathcal{H}_{T\backslash[l]}(\lambda).$$

Comparing the coefficients for λ^3 we get:

$$h_3(T) = h_3(T^-) + h_2(T^-) + h_2(T^+) + h_3(T^+) - h_2(T\backslash[l]),$$

$$\Delta h_3(T,l) = h_2(T^-) + h_2(T^+) - h_2(T\backslash[l]),$$

$$\Delta h_3(T,l) + \Delta h_2(T,l) = h_2(T) - h_2(T\backslash[l]).$$

Using the formula for h_2 from proposition 3 and the equality $k(T\backslash[l]) = \gamma_1(T,l)$ we get:

$$\Delta h_3(T,l) + \Delta h_2(T,l) = \theta_1(T) - \frac{\theta_2(T)}{2} - \theta_1(T\backslash[l]) + \frac{\theta_2(T\backslash[l])}{2} - \frac{\gamma_1(T,l)(\gamma_1(T,l)-1)}{2},$$

$$\Delta h_3(T,l) + \Delta h_2(T,l) = (\theta_1(T) - \theta_1(T\backslash[l])) - \frac{\theta_2(T) - \theta_2(T\backslash[l])}{2} - \frac{\gamma_1(T,l)(\gamma_1(T,l)-1)}{2},$$

$$\Delta h_3(T,l) + \Delta h_2(T,l) = 2(\gamma_1(T,l)+1) - \frac{2\xi(T,l) - \gamma_1(T,l) + \gamma_2(T,l) + 2\gamma_1(T,l) + 2}{2} -$$

$$- \frac{\gamma_2(T,l) + 2\rho(T,l) - \gamma_1(T,l)}{2}.$$

This equality is true for any tree $T \notin \{A_1, A_2\}$. Direct substitution checks that it is also true for $T = A_2$. So it is true for any T of $\mathcal{T}_*$. We get the equality of functions:

$$\Delta h_3 + \Delta h_2 = 2(\gamma_1 + 1) - \frac{2\xi - \gamma_1 + \gamma_2 + 2\gamma_1 + 2}{2} - \frac{\gamma_2 + 2\rho - \gamma_1}{2}.$$

Hence $\Delta h_3 + \Delta h_2 = 2\gamma_1 + 1 - \xi - \rho - \gamma_2$. Hence $h_3 + h_2 \in I(2\gamma_1 + 1 - \xi - \rho - \gamma_2)$, $h_3 \in -h_2 + I(2\gamma_1 + 1 - \xi - \rho - \gamma_2)$. Using Theorem 2, Theorem 3, equalities from Example 5 and the expression for h_2 from Proposition 3 we obtain

$$h_3 \in -\frac{5}{3}\theta_1 + \frac{5}{2}\theta_2 - \frac{1}{3}\theta_3 - \tau + C\theta_0.$$

Hence we get, because $h_3(A_1) = 0$, that

$$h_3 = -\frac{5}{3}\theta_1 + \frac{5}{2}\theta_2 - \frac{1}{3}\theta_3 - \tau.$$

2) Let F be an arbitrary forest consisting of trees $T_1, ..., T_{k(F)}$. Comparing the coefficient at λ^3 in equality (2) we get:

$$h_3(F) = \sum h_3(T_i) + (k(F)-1)\sum h_2(T_i) + C^3_{k(F)} =$$

$$= \sum\left(-\frac{5}{3}\theta_1(T_i) + \frac{5}{2}\theta_2(T_i) - \frac{1}{3}\theta_3(T_i) - \tau(T_i)\right) + (k(F)-1)\sum\left(\theta_1(T_i) - \frac{\theta_2(T_i)}{2}\right) + C^3_{k(F)} =$$

$$= -\frac{5}{3}\theta_1(F) + \frac{5}{2}\theta_2(F) - \frac{1}{3}\theta_3(F) - \tau(F) + (k(F)-1)\left(\theta_1(F) - \frac{\theta_2(F)}{2}\right) + C^3_{k(F)}.$$

References

1. Bourbaki N. *Lie Groups and Lie Algebras*. Berlin, Heidelberg, New York, Springer-Verlag, 2002.
2. Kolmykov V.A., Men'shikh V.V, Subbotin V.F., Sumin M.V. Four Coefficients of the Characteristic Polynomial of the Coxeter Transformation. *Mathematical Notes*, 2003, vol. 73, no. 5–6, pp. 742–746.
3. Winberg E.B. [*Course of Algebra*]. Moscow, Factorial Press, 2001. (in Russian)
4. Stekolshchik R. *Notes on Coxeter Transformations and the McKay Correspondence*. Berlin, Heidelberg, Springer-Verlag, 2008.
5. Men'shikh V.V., Subbotin V.F. Properties of the Coxeter Transformation for Affine Dynkin Cycle. *Russian Mathematics*, 2014, vol. 58, no. 9, pp. 36–40.
6. Men'shikh V.V. [Conjugacy of Coxeter Products in Coxeter Groups]. *Chernozemniy Almanah Naychnih Issledovaniy*, 2005, vol. 1, no. 1, pp. 69–74. (in Russian)
7. Bernstein I.N., Gelfand I.M., Ponomarev V.A. Coxeter Functors and Gabriel's Theorem. *Russian Mathematics Surveys*, 1973, vol. 28, no. 2, pp. 17–32.
8. Subbotin V.F., Stekolshchik R.B. The Jordan Form of the Coxeter Transformation and Applications to Representations of Finite Graphs. *Functional Analysis and Its Applications*, 1978, vol. 12, no. 1, pp. 67—68.
9. Subbotin, V.F. [Study of Polynomials Associated with Graph Representations] / V.F. Subbotin, M.V. Sumin. – Moscow, 1982. Dep. at VINITI. no. 6480-B82. – P. 1–37. (in Russian)

Ekaterina V. Kolmykova, Lecturer, Department of Functional Analysis and Operator Equations, Voronezh State University (Voronezh, Russian Federation), kolmykova_kate@bk.ru.

Received May 30, 2022.

УДК 519.17 + 512.817.73 **DOI: 10.14529/jcem220304**

ОПЕРАЦИИ НАД ГРАФОВЫМИ ФУНКЦИЯМИ И СПЕКТРАЛЬНЫЕ СВОЙСТВА КОМПОЗИЦИЙ ОТРАЖЕНИЙ

Е. В. Колмыкова

В статье вводятся операторы, действующие на пространствах графовых функций. С помощью теоретико-графовых методов находятся свойства введенных операторов. Эти свойства показывают, что введенные операторы являются дискретными аналогами дифференцирования и интегрирования. Найдены значения операторов на некоторых важных графовых функциях. Разработана методика использования операторов для исследования графовых функций и способов выражения одних функций через другие. Рассматриваются характеристические многочлены преобразований Кокстера. Их коэффициенты могут быть выражены через простые графовые функции. С помощью разработанной методики предложен способ нахождения таких выражений. Результаты статьи можно использовать для нахождения спектральных характеристик композиций отражений. Эти способы являются простыми и удобными для применения.

Ключевые слова: граф; дерево; отражение; преобразование Кокстера.

Литература

1. Бурбаки, Н. Группы и алгебры Ли / Н. Бурбаки. – Москва: Мир, 1972.
2. Колмыков, В.А. Четыре коэффициента характеристического многочлена преобразования Кокстера / В.А. Колмыков, В.В. Меньших, В.Ф. Субботин, М.В. Сумин // Математические заметки. – 2003. – Т. 73, № 5. – С. 788–791.
3. Винберг, Э.Б. Курс Алгебры / Э.Б. Винберг. – Москва: Факториал Пресс, 2001.
4. Stekolshchik, R. Notes on Coxeter Transformations and the McKay Correspondence / R. Stekolshchik. – Berlin, Heidelberg: Springer-Verlag, 2008.
5. Меньших, В.В. Свойства преобразований Кокстера для аффинного цикла Дынкина / В.В. Меньших, В.Ф. Субботин // Известия высших учебных заведений. Математика. – 2014. – Т. 58, № 9. – С. 43–48.
6. Меньших, В.В. Сопряженность кокстеровских произведений в группах Кокстера / В.В. Меньших // Черноземный альманах научных исследований. – 2005. – Т. 1, № 1. – С. 69–74.
7. Бернштейн, И.Н. Функторы Кокстера и теорема Габриеля / И.Н. Бернштейн, И.М. Гельфанд, В.А. Пономарев // Успехи математический наук. – 1973. – Т. 28, № 2(170). – С. 19–33.
8. Субботин, В.Ф. Жорданова форма преобразования Кокстера и приложения к представлениям конечных графов. / В.Ф. Субботин, Р.Б. Стекольщик // Функциональный анализ и приложения. – 1978. – Т. 12, № 1. – С. 84–85.
9. Субботин, В.Ф. Исследование многочленов, связанных с представлениями графов / В.Ф. Субботин, М.В. Сумин. – Москва, 1982. Деп. в ВИНИТИ. № 6480-В82. – С. 1–37.

Колмыкова Екатерина Владиславовна, преподаватель, кафедра функционального анализа и операторных уравнений, Воронежский государственный университет (г. Воронеж, Российская Федерация), kolmykova_ kate@bk.ru.

Поступила в редакцию 30 мая 2022 г.