SHORT NOTES

MSC 37C75 DOI: 10.14529/jcem220305

STABILITY ANALYSIS OF PERTURBED SYSTEMS FOR INVESTIGATION OF LIMITED BOUNDEDNESS OF THEIR SOLUTIONS

I. A. Yeletskikh[1]**,** yeletskikh.irina@yandex.ru,
K. S. Yeletskikh[1]**,** kostan.yeletsky@gmail.com
[1]Bunin Yelets State University, Yelets, Russian Federation

Stability theory play an important role in systems theory and engineering sciences. The stability of equilibrium points is usually considered within the framework of the theory of stability developed by the Russian mathematician and mechanic A. M. Lyapunov (1857–1918), who laid its foundations and gave it a name. At present, it has become very routine view at stability as stability with respect to a perturbation of the input signal. The research is based on the space-state approach for modeling nonlinear dynamic systems and the alternative ≪input-output≫ approach. The concept of stability in terms of ≪input-output≫ of a nonlinear system is based on the method of Lyapunov functions and its generalization to the case of nonlinear dynamic systems. The interpretation of the problem of the accumulation of perturbations is reduced to the problem of finding the norm of the operator, which makes it possible to expand the range of models under research depending on the space in which the input and output signals act.

Keywords: dynamical system; stability of origin; interconnected and slowly changing systems; equilibrium point; exponential stability; causality; amplification factor.

Introduction

The theory of stability play an important role in the analysis of the stability properties of perturbed systems. We can formulate several problems that arise in the study of dynamical systems. Recall that an equilibrium point is stable if all solutions starting near this point remain in its vicinity; otherwise this point is unstable. An equilibrium point is asymptotically stable if all solutions starting near it tend to this equilibrium point as time tends to infinity. Consider the system

$$x'' = f\left(t, x\right) + g\left(t, x\right) \tag{1}$$

where $f : [0, \infty) \times D \to \mathbb{R}^n$ and $g : [0, \infty) \times D \to \mathbb{R}^n$ are piecewise continuous in t and locally functions are Lipschitz in x on $[0, \infty) \times D$ and $D \subset \mathbb{R}^n$ is an open region containing the origin $x = 0$. We will consider this system as a perturbation of the nominal system

$$x'' = f\left(t,\ x\right). \tag{2}$$

The presence of the perturbation term $g\left(t, x\right)$ may be due to errors in the definition of the model, changes in parameters over time, or other uncertainties and perturbations that always take place in real situations. Let us assume that the original system (2) has a uniformly asymptotically stable equilibrium point at the origin. In order to determine the stability properties of the perturbed system, it is necessary to use the Lyapunov function of the nominal system to analyze the perturbed system. It is this technique that was used in the analysis of the linearization method. The novelty in this case lies in the fact that the perturbation term can have a more general form compared to the perturbation term arising in the investigation of a linearized system. The conclusion that can be reached in the course of the analysis of the system essentially depends on the answer to the question about the equality of the perturbation term at the origin of coordinates to zero. If $g(t,0) = 0$, then the perturbed system (2) has an equilibrium point at the origin. If $g(t,0) \neq 0$, then the origin is not an equilibrium point of the perturbed system. In this case, it is possible to investigate the perturbed system for the limiting boundedness of its solutions ([1] p. 339).

1. Perturbation Vanishing at the Origin

Consider the case when $g(t,0) = 0$. Assume that $x = 0$ is an exponentially stable equilibrium point of the nominal system (2) and let $V(t,\ x)$ be a Lyapunov function satisfying the relations:

$$c_1\|x\|^2 \leq V(t,\ x) \leq c_2\|x\|^2, \tag{3}$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \leq -c_3\|x\|^2, \tag{4}$$

$$\left\|\frac{\partial V}{\partial x}\right\| \leq c_4\|x\| \tag{5}$$

for all $(t,x) \in [0,\infty) \times D$ and some positive constants $c_1,\ c_2,\ c_3$, and c_4. The existence of a Lyapunov function satisfying (3) – (5) is guaranteed by the theorem ([1] p. 162).

Example 1. Consider a second-order system

$$\begin{cases} x_1' = x_2, \\ x_2' = -4x_1 - 2x_2 + \beta x_2^3 \end{cases}$$

with unknown constant $\beta \geq 0$. Consider this system as perturbed with

$$f(x) = Ax = \begin{bmatrix} 0 & 1 \\ -4 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \text{and} \quad g(x) = \begin{bmatrix} 0 \\ \beta x_2^3 \end{bmatrix}.$$

The eigenvalues of the matrix A are $-1 \pm j\sqrt{3}$. And, therefore, A is Hurwitz. The solution of the Lyapunov equation

$$PA + A^T P = -I$$

has a form

$$P = \begin{bmatrix} \frac{3}{2} & \frac{1}{8} \\ \\ \frac{1}{8} & \frac{5}{16} \end{bmatrix}.$$

The Lyapunov function $V(x) = x^T P x$ satisfies inequalities (3) – (4) with constants $c_3 = 1$ and $c_4 = 2 \times 1.513 = 3.026$. The perturbation term $g(x)$ satisfies $\|g(x)\|_2 = \beta|x_2|^3 \leq \beta k_2^2|x_2| \leq \beta k_2^2\|x\|_2$ for all $|x_2| \leq k_2$. Using $V(\ x)$ as the Lyapunov function for the perturbed system, we obtain

$$V'(x) \leq -\|x\|_2^2 + 3.026\beta k_2^2\|x\|_2^2.$$

Hence $V'(x)$ will be negative definite if

$$\beta < \frac{1}{3.026k_2^2}\ .$$

In order to get the bound k_2, we must let $\Omega_c = \{x \in \mathbb{R}^2 / V(x) \leq c\}$. For any positive constant c, the set Ω_c is closed and bounded. The boundary of the set Ω_c is the Lyapunov surface

$$V(x) = \frac{3}{2}x_1^2 + \frac{1}{4}x_1x_2 + \frac{5}{16}x_2^2 = c.$$

The largest value of $|x_2|$ on the surface $V(x) = c$ can be determined by differentiating the surface equation with respect to x_1. This leads to the following result

$$3x_1 + \frac{1}{4}x_2 = 0.$$

Then the extremum of x_2 is reached at the point of intersection of the line $x_1 = -\frac{x_2}{12}$ with a Lyapunov surface. It is easy to show that the maximum value of x_2^2 on the Lyapunov surface is $\frac{96c}{29}$. Thus, all points inside Ω_c satisfy the bound

$$|x_2| \leq k_2, \text{ where } k_2^2 = \frac{96c}{29}\ .$$

Therefore if

$$\beta < \frac{29}{3.026 \times 96\ c} \approx \frac{0.1}{c}$$

the derivative $V'(x)$ will be negative definite on Ω_c and we can conclude that the origin $x = 0$ is exponentially stable and Ω_c as an estimate of the attraction region. The inequality $\beta < \frac{0.1}{c}$ shows the interrelation between the estimate of the attraction region and the upper bound β. The smaller the upper bound β, the larger the estimate of the region of attraction. The presence of a compromise interrelation between these two quantities seems to be a natural fact. The change of variables

$$z_1 = \sqrt{\frac{3\beta}{2}}x_2\ ,$$

$$z_2 = \sqrt{\frac{3\beta}{8}}\ \left(4x_1 + 2x_2 - \beta x_2^3\right) = -\sqrt{\frac{3\beta}{8}}x_2'\ , \quad \tau = 2t$$

transforms the equation of state to the following form

$$\frac{dz_1}{d\tau} = -z_2, \quad \frac{dz_2}{d\tau} = z_1 + \left(z_1^2 - 1\right) z_2.$$

In [1] (p. 344) was shown that the area of attraction of this system is limited and surrounded by an unstable limit cycle. The region of attraction, represented in x-coordinates, increases with decreasing β and decreases with increasing β. Example 1 can serve as an illustration of the regularity that any two-dimensional vector satisfying the inequality

$$\|g(t,x)\|_2 \leq \beta k_2^2 \|x\|_2.$$

This class of perturbations seems to be more general in comparison with the perturbation that takes place in the particular example under consideration and has a structural feature – the first component of the perturbation vector g is equal to zero. The analysis carried out shows that it is possible to consider perturbations of a general form, when all components of the vector g can change. The stability analysis of the perturbed system becomes more complicated if the origin of the nominal system (2) is uniformly asymptotically stable, but not exponential stability.

2. Perturbations That do not Vanishing at the Origin

Consider a more general case when the condition $g\left(t,0\right) = 0$ is not satisfied and, consequently, the origin $x = 0$ is not an equilibrium point of the perturbed system (1). In this case, one cannot investigate the stability of the origin of coordinates as an equilibrium point of the system, and one cannot expect that the solution of the perturbed system will tend to the origin of coordinates at $t \to \infty$.We can hope the solution $x(t)$ to be limitingly small if the perturbations $g(t,x)$ is small in some sense. Let's start with the case when the origin of the nominal system (2) is exponentially stable and $V\left(t,\ x\right)$ is a Lyapunov function of the nominal system satisfying (3) – (5) on $[0,\infty)\times D$, where $\left\{x\in\mathbb{R}^n/\|x\|<r\right\}$.

Example 2. Consider a second-order system

$$\begin{cases} x_1' = x_2, \\ x_2' = -4x_1 - 2x_2 + \beta x_2^3 + d\left(t\right), \end{cases}$$

where $\beta \geq 0$ is an unknown constant and $d\left(t\right)$ is a uniformly bounded perturbation that satisfies the inequality $\left|d\left(t\right)\right| \leq \delta$ for all $t \geq 0$. This system coincides with the one we considered in Example 1, except that the new system has an additional perturbation term $d\left(t\right)$. This system can be considered as a perturbation of the nominal linear system of the Lyapunov function, which has the form $V\left(x\right) = x^T P x$, where P is the same matrix.

This function $V(x)$ can be considered as a Lyapunov function for the perturbed system, but the analysis of the influence of two perturbation terms βx_2^3 and $d\left(t\right)$ should be carried out separately, since the first term is equal to zero at the origin, and the second is not. Calculating the derivative $V(x)$ along the trajectories of the perturbed system, we obtain [2]

$$V'\left(t,x\right) = -\|x\|_2^2 + 2\beta\ x_2^2\left(\frac{1}{8}x_1x_2 + \frac{5}{16}x_2^2\right) + 2d\left(t\right)\left(\frac{1}{8}x_1 + \frac{5}{16}x_2\right)$$

$$\leq -\|x\|_2^2 + \frac{3}{4}\beta\ k_2^2\|x\|_2^2 + \frac{\sqrt{29}\delta}{8}\|x\|_2.$$

In obtaining this estimate, we used the inequality

$$|2x_1 + 5x_2| \leq \|x\|_2\sqrt{29},$$

and also the fact that k_2 is an upper bound on $\left|x_2\right|$. $\left|x_2\right|^2$ is bounded on Ω_c by the value $\dfrac{96c}{29}$. For the case of exponential stability, in the analysis of which some singularity was used, which had no analogue in the analysis of a more general situation of uniform asymptotic stability, taking into account $t \to \infty$ [3]. The assumption that all uniformly bounded perturbations are satisfied, the solutions of the perturbed system will be uniformly bounded. On the other hand, in the case of uniform asymptotic stability, an analysis of the properties of stability shows that when uniformly bounded disturbances arise in a nominal system with a uniformly asymptotically stable equilibrium point at the origin, its solutions will remain bounded regardless of the magnitude of the acting disturbances. We cannot prove this fact, which does not mean that the system cannot have this property. However, it turns out that this statement is indeed false. One can give examples of systems for which the origin of coordinates is a globally uniformly asymptotically stable equilibrium point, but bounded perturbations lead to the fact that the solutions of the perturbed system go to infinity.

Conclusion

In conclusion, it should be noted that the complexity of the analysis of a nonlinear dynamical system increases rapidly with an increase in its order. This serves as a motivation to look for ways to simplify the analysis. If the model can be represented as interconnected lower order subsystems, we can divide the stability analysis into two steps. At the first step, perform the decomposition of the system into isolated subsystems of a smaller order and for each of them we conduct an appropriate study of the stability properties. At the second step, we use the results obtained at the first step and information about the interconnections between these subsystems in order to determine the stability properties of the entire interconnected system. This method is shown in the article on the given examples and used to find the Lyapunov functions for interconnected systems [4].

References

1. Khalil H.K. *Nonlinear Systems.* New Jersey, Prentice Hall, 2002.
2. Yeletskikh I.A. Application of the Method of Lyapunov to Investigate on the Stability of Linear Stationary Systems // *CONTINUUM. Mathematics. Informatics. Education*, 2019, vol. 14, no. 2, pp. 27–33. (in Russian)
3. La Salle J. *Stability by Liapunov's Direct Method with Applications.* New York, London, Academic Press, 1961.
4. Lyapunov A.M. *The General Problem of the Stability of Motion.* Boca Raton, CRC Press, 1992.

Irina A. Yeletskikh, PhD (Math), Associate professor, Institute of Mathematics, Natural Science and Technics, Bunin Yelets State University (Yelets, Russian Federation), yeletskikh.irina@yandex.ru.

Konstantin S. Yeletskikh, PhD (Math), Senior Lecturer, Institute of Mathematics, Natural Science and Technics, Bunin Yelets State University (Yelets, Russian Federation), kostan.yeletsky@gmail.com.

Received August 18, 2022.

УДК 517.925 DOI: 10.14529/jcem220305

АНАЛИЗ СВОЙСТВ УСТОЙЧИВОСТИ ВОЗМУЩЕННЫХ СИСТЕМ НА ПРЕДМЕТ ПРЕДЕЛЬНОЙ ОГРАНИЧЕННОСТИ ИХ РЕШЕНИЙ

И. А. Елецких, К. С. Елецких

Теория устойчивости играет ключевую роль в теории систем и инженерных науках. Устойчивость точек равновесия обычно рассматривается в рамках теории устойчивости, разработанной русским математиком и механиком А. М. Ляпуновым (1857–1918), заложившим ее основы и давшим ей имя. В настоящее время стала очень распространенной точка зрения на устойчивость, как устойчивость по отношению к возмущению входного сигнала. В основу исследования положен подход пространства-состояния для моделирования нелинейных динамических систем и альтернативный подход «вход-выход». В основу концепции устойчивости в терминах «вход-выход» нелинейной системы, положен метод функций Ляпунова и его обобщение на случай нелинейных динамических систем. Трактовка задачи о накоплении возмущений сводится к задаче отыскания нормы оператора, что позволяет расширить круг исследуемых моделей в зависимости от пространства, в котором действуют входные и выходные сигналы.

Ключевые слова: динамическая система; устойчивость начала координат; взаимосвязанные и медленно меняющиеся системы; точка равновесия; экспоненциальная устойчивость; казуальность; коэффициент усиления.

Литература

1. Khalil, H.K. Nonlinear Systems / H.K. Khalil. – New Jersey: Prentice Hall, 2002.
2. Елецких, И.А. Приложение метода Ляпунова к исследованию на устойчивость линейных стационарных систем / И.А. Елецких // CONTINUUM. Математика. Информатика. Образование. – 2019. – Т. 14, № 2. – Р. 27–33.
3. La Salle, J. Stability by Liapunov's Direct Method with Applications / J. La Salle, S. Lefschetz. – New York, London: Academic Press, 1961.
4. Ляпунов, А.М. Общая задача об устойчивости движения / А.М. Ляпунов. – М., Л.: ГИТТЛ, 1950. – 472 с.

Елецких Ирина Адольфовна, кандидат физико-математических наук, доцент, кафедра «Математики и методики ее преподавания», Елецкий государственный университет им. И.А. Бунина (г. Елец, Российская Федерация), yeletskikh.irina@yandex.ru.

Елецких Константин Сергеевич, кандидат физико-математических наук, старший преподаватель, кафедра «Математики и методики ее преподавания», Елецкий государственный университет им. И.А. Бунина (г. Елец, Российская Федерация), kostan.yeletsky@gmail.com.

Поступила в редакцию 18 августа 2022 г.