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THE POISSON EQUATION WITH WENTZELL BOUNDARY CONDITIONS IN THE SQUARE

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The Wentzell boundary value problems with condition for second-order linear elliptic equations were studied by various methods. Over time, condition was understood as a description of a process occurring at the boundary of the domain and influenced by processes within the domain. Since in the mathematical literature the Wentzell boundary conditions has been considered from the two points of view (in classical and neoclassical cases), the purpose of this work is to show the solvability of the Wentzel problem for the Poisson equation in the square in neoclassical one, when we divide the desired function into two components.

Keywords: Laplace operator, Wentzell boundary condition, Fourier series.

Introduction

Let $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N} \setminus \{1\}$, be a smooth Riemannian manifold with boundary $\partial\Omega$ of class C^∞. Let us consider the Poisson equation

$$-\Delta v(x) = f(x),\ x \in \Omega. \tag{1}$$

with the Wentzell boundary conditions

$$\Delta w(x) + \partial_\nu v(x) + \gamma w(x) = 0,\ x \in \partial\Omega, \tag{2}$$

and with the agreement's conditions

$$Tr\ v = w. \tag{3}$$

Apparently there the symbol Δ in equation (1) denotes the Laplace operator and the Laplace-Beltrami operator in equation (2), this will be clear from the context. Moreover, equation (2) will be considered exclusively in 0-form spaces. Here $v : \Omega \to \mathbb{R}$ and $w : \partial\Omega \to \mathbb{R}$ are the functions sought, the parameter $\gamma \in \mathbb{R}$, the symbol ∂_ν denotes the derivative on the external (with respect to the region Ω) normal to the boundary of $\partial\Omega$. From now on, we consider this problem as a system of equations (1) – (3), then the solution of the system will be in the form $u = v + w$. In general, by way of example we can say that there exist of the modificatons of the model (1), (2), which describes the processes in the cell cytoplasm and on the cell membrane and generalizes the model proposed in [1].

The purpose of this work is to show new approach for resolvability of problem (1) with Wentzell boundary conditions. Namely, according to the used agrement condition, describe the solution of the Wentzell problem.The article contains one sections except introduction and the list of references. The solvability of the Wentzell problem in the square by side π is given in the first section.

1. The Solvability of the Wentzell Problem in the Square

Let us consider the solvability of the problem (1)–(3), where as the domain Ω consider the square $\mathrm{P}_\pi = \{(x, y) : (x, y) \in [0, \pi] \times [0, \pi]\}$ with side π. To solve it, it is convenient to review two auxiliary problems of the form (4)–(5)

$$\Delta v(x, y) = f(x, y),\ (x, y) \in \Pi_\pi, \tag{4}$$

$$\partial_\nu v = \varphi(x, y),\ (x, y) \in \partial\Pi_\pi \tag{5}$$

and agreement's equation (6)

$$\Delta w(x, y) + \gamma w(x, y) = -\varphi(x, y),\ (x, y) \in \partial\Pi_\pi \tag{6}$$

having previously done the following replacement on each side of the square

$$\partial_\nu v = \varphi(x, y) = \begin{cases} \varphi_1(y) & (x, y) \in \Pi_\pi^1 = \{(x, y) : x = 0, 0 \le y \le \pi\}, \\ \varphi_2(y) & (x, y) \in \Pi_\pi^3 = \{(x, y) : x = \pi, 0 \le y \le \pi\}, \\ \psi_1(x) & (x, y) \in \Pi_\pi^2 = \{(x, y) : y = 0, 0 \le x \le \pi\}, \\ \psi_2(x) & (x, y) \in \Pi_\pi^4 = \{(x, y) : y = \pi, 0 \le x \le \pi\}. \end{cases}$$

In the first step, let us consider the solution of (4)–(5), decomposing it into three components

$$\begin{aligned} &\Delta v_1(x, y) = f(x, y), (x, y) \in \Pi_\pi, \\ &\begin{cases} \partial_\nu v_1 = 0 & (x, y) \in \Pi_\pi^1 = \{(x, y) : x = 0, 0 \le y \le \pi\}, \\ \partial_\nu v_1 = 0 & (x, y) \in \Pi_\pi^3 = \{(x, y) : x = \pi, 0 \le y \le \pi\}, \\ \partial_\nu v_1 = 0 & (x, y) \in \Pi_\pi^2 = \{(x, y) : y = 0, 0 \le x \le \pi\}, \\ \partial_\nu v_1 = 0 & (x, y) \in \Pi_\pi^4 = \{(x, y) : y = \pi, 0 \le x \le \pi\}. \end{cases} \end{aligned} \tag{7}$$

$$\begin{aligned} &\Delta v_2(x, y) = 0, (x, y) \in \Pi_\pi, \\ &\begin{cases} \partial_\nu v_2 = \varphi_1(y) & (x, y) \in \Pi_\pi^1 = \{(x, y) : x = 0, 0 \le y \le \pi\}, \\ \partial_\nu v_2 = \varphi_2(y) & (x, y) \in \Pi_\pi^3 = \{(x, y) : x = \pi, 0 \le y \le \pi\}, \\ \partial_\nu v_2 = 0 & (x, y) \in \Pi_\pi^2 = \{(x, y) : y = 0, 0 \le x \le \pi\}, \\ \partial_\nu v_2 = 0 & (x, y) \in \Pi_\pi^4 = \{(x, y) : y = \pi, 0 \le x \le \pi\}. \end{cases} \end{aligned} \tag{8}$$

$$\begin{aligned} &\Delta v_3(x, y) = 0, (x, y) \in \Pi_\pi, \\ &\begin{cases} \partial_\nu v_3 = 0 & (x, y) \in \Pi_\pi^1 = \{(x, y) : x = 0, 0 \le y \le \pi\}, \\ \partial_\nu v_3 = 0 & (x, y) \in \Pi_\pi^3 = \{(x, y) : x = \pi, 0 \le y \le \pi\}, \\ \partial_\nu v_3 = \psi_1(x) & (x, y) \in \Pi_\pi^2 = \{(x, y) : y = 0, 0 \le x \le \pi\}, \\ \partial_\nu v_3 = \psi_2(x) & (x, y) \in \Pi_\pi^4 = \{(x, y) : y = \pi, 0 \le x \le \pi\}. \end{cases} \end{aligned} \tag{9}$$

It is easy to see by simple substitution that the function

$$v(x, y) = v_1(x) + v_2(x, y) + v_3(x, y)$$

is a solution of the original problem (4)–(5). Let us proceed to find each function $v_i(x, y)$, $i = 1, 2, 3$ from corresponding problems (7)–(9).

Consider the solution of the problem (7). Let find eigenvalues λ_{np} and eigenfunctions v_{np} of the problem

$$-\Delta v = \lambda v$$

with boundary conditions (7). It is easy to show that the required characteristics are $v_{np} = \cos(nx)\cos(py)$, $\lambda_{np} = n^2 + p^2, n, p \in \{0\} \cup \mathbb{N}$. On the other hand, let us decompose the function f in the right-hand side of equation (7) into a Fourier series of functions v_{np}

$$f(x, y) = \sum_{n,p=1}^{\infty} f_{np} v_{np}(x, y),$$

where

$$f_{np} = \frac{(f, v_{np})}{(v_{np}, v_{np})} = \frac{4}{\pi^2} \int\limits_0^{\pi} dx \int\limits_0^{\pi} (cos(nx)cos(py)f(x, y))dy$$

Thus, by presenting the solution of problem (7) as

$$v_1(x, y) = \sum_{n=1}^{\infty}\sum_{p=1}^{\infty} c_{np} v_{np}(x, y),$$

we have,

$$-\sum_{n=1}^{\infty}\sum_{p=1}^{\infty} \lambda_{np} c_{np} v_{np}(x, y) = \sum_{n=1}^{\infty}\sum_{p=1}^{\infty} f_{np} v_{np}(x, y),$$

which is equal to

$$c_{np} = -\frac{f_{np}}{\lambda_{np}}$$

or

$$v_1(x, y) = -\sum_{n=1}^{\infty}\sum_{p=1}^{\infty} \frac{4\int\limits_0^{\pi} dx \int\limits_0^{\pi} \cos(nx)\cos(py) f(x, y) dy}{\pi^2(n^2 + p^2)} v_{np}(x, y).$$

Let us turn to the solution of problem (8). To do this, solve the auxiliary problem

$$\Delta v = 0, \quad \begin{cases} \partial_\nu v = 0 & (x, y) \in \Pi_\pi^2 = \{(x, y) : y = 0, 0 \le x \le \pi\}, \\ \partial_\nu v = 0 & (x, y) \in \Pi_\pi^4 = \{(x, y) : y = \pi, 0 \le x \le \pi\}. \end{cases} \tag{10}$$

We will look for all possible solutions of this problem by the method of separation of variables

$$v(x, y) = X(x)Y(y).$$

We have,

$$X''(x)Y(y) + X(x)Y''(y) = 0,$$

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = \lambda.$$

As a result, the equations for the unknown functions X and Y are divided by

$$X''(x) = \lambda X(x), \quad -Y''(y) = \lambda Y(y). \tag{11}$$

Combining boundary conditions (10) and equations (11), we obtain the following Sturm – Liouville problem

$$\begin{cases} -Y''(y) = \lambda Y(y), \\ Y'(0) = 0, \\ Y'(\pi) = 0, \end{cases}$$

where the eigenfunctions and eigenvalues of the problem are

$$Y_n(y) = \cos(\sqrt{\lambda_n} y) = \cos(ny),$$

$$\lambda_n = \left(\frac{\pi n}{\pi}\right)^2 = n^2, \ n \in \{0\} \cup \mathbb{N}.$$

Thus, the second equation (11) takes the following form

$$X''(x) = n^2 X(x),$$

and the general solution of the equation is

$$X_n(x) = C_n e^{nx} + \frac{C_{1,n}}{e^{nx}},$$

and all solutions of the homogeneous problem, represented as $v_n(x, y) = X_n(x)Y_n(y)$. We further decompose the functions $\varphi_1(y)$ and $\varphi_2(y)$ into a Fourier series of eigenfunctions of $Y_n(y)$ of the auxiliary problem and proceed to the system. We have,

$$\varphi_1(y) = \sum_{n=1}^{\infty} \varphi_{1,n} Y_n(y), \quad \left(\varphi_2(y) = \sum_{n=1}^{\infty} \varphi_{2,n} Y_n(y)\right) \tag{12}$$

Let us proceed to the solution of problem (9). To do this, solve the auxiliary problem

$$\Delta v = 0, \quad \begin{cases} \partial_\nu v = 0 \quad (x, y) \in \Pi_\pi^1 = \{(x, y) : x = 0, 0 \le y \le \pi\}, \\ \partial_\nu v = 0 \quad (x, y) \in \Pi_\pi^3 = \{(x, y) : x = \pi, 0 \le y \le \pi\}. \end{cases} \tag{13}$$

We will look for all possible solutions of this problem by the method of separation of variables

$$v(x, y) = X(x)Y(y).$$

We have,

$$X''(x)Y(y) + X(x)Y''(y) = 0,$$

$$-\frac{X''(x)}{X(x)} = \frac{Y''(y)}{Y(y)} = \lambda.$$

As a result, the equations for the unknown functions X and Y are divided by

$$-X''(x) = \lambda X(x), \quad Y''(y) = \lambda Y(y). \tag{14}$$

Combining boundary conditions (14) and equations (15), we obtain the following Sturm – Liouville problem

$$\begin{cases} -X''(x) = \lambda X(x), \\ -X'(0) = 0, \\ X'(\pi) = 0, \end{cases}$$

where the eigenfunctions and eigenvalues of the problem have the form

$$X_n(y) = \cos(\sqrt{\lambda_n} x) = \cos(nx),$$

$$\lambda_n = \left(\frac{\pi n}{\pi}\right)^2 = n^2, \; n \in \{0\} \cup \mathbb{N}.$$

Thus, the second equation (15) takes the following form

$$Y''(y) = n^2 Y(y),$$

and the general solution of the equation is

$$Y_n(y) = B_n e^{ny} + \frac{B_{1,n}}{e^{ny}},$$

and all solutions of the homogeneous problem, represented as $v_n(x, y) = X_n(x)Y_n(y)$. We further decompose the functions $\psi_1(x)$ and $\psi_2(x)$ into a Fourier series of eigenfunctions of $X_n(x)$ of the auxiliary problem and proceed to the system. We have,

$$\psi_1(x) = \sum_{n=1}^{\infty} \psi_{1,n} X_n(x), \quad \left(\psi_2(x) = \sum_{n=1}^{\infty} \psi_{2,n} X_n(x)\right) \tag{15}$$

where

$$\psi_{1,n} = \frac{(g_1, X_n)}{(X_n, X_n)} = \frac{2}{\pi} \int_0^{\pi} g_1(x) \cos(nx)\, dx, \quad \left(\psi_{2,n} = \frac{(g_2, X_n)}{(X_n, X_n)} = \frac{2}{\pi} \int_0^{\pi} g_2(x) \cos(nx) dx\right).$$

Since we needed to find a Fourier series solution of the form

$$v_n^3(x, y) = \sum_{n=1}^{\infty} X_n(x) Y_n(y), \tag{16}$$

it was obtained that the series satisfies the equation and the third and fourth boundary conditions of the problem. It remains to choose constants such that the first and second boundary conditions of (9) are satisfied. Substituting series (16) and (17) into the first and second boundary conditions of (9), we obtain the following system

$$\begin{cases} -\sum\limits_{n=1}^{\infty} Y_n'(0) X_n(x) = \sum\limits_{n=1}^{\infty} \psi_{1,n} X_n(x), \\ \\ \sum\limits_{n=1}^{\infty} Y_n'(\pi) X_n(x) = \sum\limits_{n=1}^{\infty} \psi_{2,n} X_n(x), \end{cases}$$

which is equivalent to,

$$\begin{cases} -Y_n'(0) = \psi_{1,n}, \\ Y_n'(\pi) = \psi_{2,n}, \end{cases}$$

where

$$Y_n'(0) = nB_n - B_{1,n}n,$$

$$Y_n'(\pi) = nB_n e^{n\pi} - \frac{B_{1,n}n}{e^{n\pi}},$$

$$\psi_{1,n} \quad \text{and} \quad \psi_{2,n} \quad \text{determined earlier.}$$

Thus, the solution of problem (9) has the form

$$v_3(x,y) = \sum_{n=1}^{\infty} X_n(x)Y_n(y) = \sum_{n=1}^{\infty} \left(B_n(\psi 1, \psi 2)e^{nx} + \frac{B_{1,n}(\psi 1, \psi 2)}{e^{nx}} \right) \cos(nx)$$

And so, the general solution of (4)–(5) in the square is

$$v(x,y) = -\sum_{n,p=1}^{\infty} \frac{f_{np}}{\lambda_{np}} v_{np}(x,y) + \sum_{n=1}^{\infty} \left(C_n(\varphi_1, \varphi_2)e^{nx} + \frac{C_{1,n}(\varphi_1, \varphi_2)}{e^{nx}} \right) \cos(ny) +$$

$$+ \sum_{n=1}^{\infty} \left(B_n(\psi_1, \psi_2)e^{nx} + \frac{B_{1,n}(\psi_1, \psi_2)}{e^{nx}} \right) \cos(nx).$$

Let us move on to the solution of equation (6), we have on each side of the square Π_π the following decomposition into four equations

$$w'(y) + \gamma w(y) = -\varphi_1(y), \ (x,y) \in \Pi_\pi^1 = \{(x,y) : x = 0, 0 \le y \le \pi\},$$

$$w'(y) + \gamma w(y) = -\varphi_2(y), \ (x,y) \in \Pi_\pi^3 = \{(x,y) : x = \pi, 0 \le y \le \pi\},$$

$$w'(x) + \gamma w(x) = -\psi_1(x), \ (x,y) \in \Pi_\pi^2 = \{(x,y) : y = 0, 0 \le x \le \pi\},$$

$$w'(x) + \gamma w(x) = -\psi_2(x), \ (x,y) \in \Pi_\pi^4 = \{(x,y) : y = \pi, 0 \le x \le \pi\}.$$

Since the condition of coupling of the two solutions in the region and on the boundary is fulfilled

$$Tr \ v = w,$$

it is easy to show that for the existence of one-valued solutions of the problem in (1)–(3) we should add additional conditions to the quotients $w_{i,n}$, $i = 1, 2, 3, 4$, by pre-decomposing each of the parts into Fourier series. Thus, the following system is valid

$$\begin{cases} -\dfrac{f_{np}(0,y)}{\lambda_p} + C_p\varphi_1,\varphi_2) + C_{1,p}(\varphi_1,\varphi_2) = -\dfrac{\varphi_{1,p}}{\gamma+p^2},\ (x,y)\in\Pi^1_\pi, \\ B_n(\psi_1,\psi_2) + B_{1,n}(\psi_1,\psi_2) = 0,\ (x,y)\in\Pi^1_\pi, \\ -\dfrac{f_{np}(\pi,y)}{\lambda_p}\cos(\pi n) + C_p(\varphi_1,\varphi_2)e^{\pi p} + \dfrac{C_{1,p}(\varphi_1,\varphi_2)}{e^{\pi p}} = -\dfrac{\varphi_{2,p}}{\gamma+p^2},\ (x,y)\in\Pi^3_\pi, \\ \left(B_n(\psi_1,\psi_2)e^{\pi n} + \dfrac{B_{1,n}(\psi_1,\psi_2)}{e^{\pi n}}\right)\cos(n\pi) = 0,\ (x,y)\in\Pi^1_\pi, \\ -\dfrac{f_{np}(x,\pi)}{\lambda_p}\cos(\pi y) + \left(B_n(\psi_1,\psi_2)e^{\pi n} + \dfrac{B_{1,n}(\psi_1,\psi_2)}{e^{\pi n}}\right) = -\dfrac{\psi_{1,n}}{\gamma+n^2}, m=2k,\ (x,y)\in\Pi^2_\pi, \\ \left(C_n(\varphi_1,\varphi_2)e^{nx} + \dfrac{C_{1,n}(\varphi_1,\varphi_2)}{e^{nx}}\right)\cos(\pi y) = 0,\ (x,y)\in\Pi^2_\pi, \\ -\dfrac{f_{np}(x,0)}{\lambda_p} + \left(B_n(\psi_1,\psi_2) + B_{1,n}(\psi_1,\psi_2)\right) = -\dfrac{\psi_{2,n}}{\gamma+n^2}, m=2k,\ (x,y)\in\Pi^4_\pi, \\ C_n(\varphi_1,\varphi_2) + C_{1,n}(\varphi_1,\varphi_2) = 0,\ (x,y)\in\Pi^2_\pi. \end{cases} \tag{17}$$

Thus, the following takes place

Theorem 1. *For the Poisson equation with Wentzell boundary value problem in the square by side π there exist unique solution, which has the following form*

$$v(x,y) = -\sum_{n,p=1}^{\infty}\frac{f_{np}}{\lambda_{np}}v_{np}(x,y) + \sum_{n=1}^{\infty}\left(C_n(\varphi_1,\varphi_2)e^{nx} + \frac{C_{1,n}(\varphi_1,\varphi_2)}{e^{nx}}\right)\cos(ny) +$$

$$+\sum_{n=1}^{\infty}\left(B_n(\psi_1,\psi_2)e^{nx} + \frac{B_{1,n}(\psi_1,\psi_2)}{e^{nx}}\right)\cos(nx),$$

provided that the coefficients $\varphi_{1,p}, \varphi_{2,p}, \psi_{1,n}, \psi_{2,n}$ satisfy the solution of the system (13).

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УРАВНЕНИЕ ПУАССОНА С ГРАНИЧНЫМИ УСЛОВИЯМИ ВЕНТЦЕЛЯ В КВАДРАТЕ

Н. С. Гончаров

Задачи с граничным условием Вентцеля для линейных эллиптических уравнений второго порядка изучались различными методами. Со временем условие стало пониматься как описание процесса, происходящего на границе области и на который влияют процессы внутри области. Поскольку в математической литературе граничные условия Вентцеля рассматривались с двух точек зрения (в классическом и неоклассическом случаях), целью данной работы является показать разрешимость задачи Вентцеля для уравнения Пуассона в квадрате в неоклассическом случае, когда мы разделяем искомую функцию на две компоненты.

Keywords: уравнение Пуассона; краевое условие Вентцеля, ряд Фурье.

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