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NUMERICAL SOLUTIONS FOR NONCLASSICAL EQUATIONS IN THE SPACE OF DIFFERENTIAL FORMS

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The article contains an overview of results obtained by the author in specially assigned spaces, namely the spaces of differential forms with stochastic coefficients defined on some Riemannian manifold without boundary. This work presents graphs of trajectories of numerical solutions to the Cauchy problem for the Barenblatt–Zheltov–Kochina equation and the Showalter–Sidorov problem for the Dzektser and Ginzburg–Landau equations. Since the equations are studied in a space of differential forms, the operators themselves are understood in a special form, in particular, instead of the Laplace operator, we take its generalization that is the Laplace–Beltrami operator. Graphs of coefficients of differential forms obtained during the computational experiments are given for different values of the parameters of the initial equations.

Keywords: Sobolev type equation; differential forms; Riemannian manifold; Laplace–Beltrami operator; numerical solution.

Introduction

Consider the following equations:

– the Barenblatt–Zheltov–Kochina equation [1]

$$(\lambda - \Delta)u_t = \alpha \Delta u, \tag{1}$$

which is a model of dynamics of a fluid filtering in a fractured-porous environment;

– the Dzektser equation [2]

$$(1 - \kappa\Delta)\varphi_t = \alpha\Delta\varphi - \beta\Delta^2\varphi, \tag{2}$$

which is a model of flow of a viscous-elastic incompressible zero-order Kelvin–Voigt fluid in the first approximation;

– the Ginzburg – Landau equation

$$(\lambda - \Delta)u_t = \alpha\Delta u + id\Delta^2 u \tag{3}$$

from the phenomenological theory of superconductivity.

In the functional spaces $\mathfrak{U}$, $\mathfrak{F}$ chosen by us, (1)–(3) are reduced [4] to the linear equation of Sobolev type

$$L\dot{u} = Mu \tag{4}$$

with the irreversible operator L.

Consider the Cauchy problem [5]

$$u(0) = u_0 \tag{5}$$

for equations (1), (2), and the Showalter–Sidorov problem [6]

$$P(u(0) - u_0) = 0 \tag{6}$$

for equations (2), (3).

The papers [8]–[10] propose a transition of (4) to the stochastic Sobolev type equations

$$L \overset{\circ}{\eta} = M\eta \tag{7}$$

with the condition

$$\eta(0) = \eta_0 \tag{8}$$

or

$$P(\eta(0) - \eta_0) = 0 \tag{9}$$

in spaces of Wiener stochastic processes in the case of an abstract (L, p)-bounded operator M, (L, p)-sectorial operator M and (L, p)-radial operator M, respectively. Since Wiener processes are continuous, but nondifferentiable in the usual sense at each point, we use the Nelson–Gliklikh derivative [7]. In this article, we study numerical solutions to all three equations (the Barenblatt – Zheltov – Kochina equation [13], the Dzektser equation [16] and the Ginzburg – Landau equation [18]) in spaces of differential forms defined on a torus in the form (7).

1. Structure of Differentiable «Noises» Spaces

Consider the complete probability space $\Omega = (\Omega, \Sigma, P)$ with the probability measure P associated with the sigma-algebra Σ of subsets of the space Ω. If $\mathbb{R}$ is the set of real numbers endowed with a sigma algebra, then the mapping $\xi : \Sigma \mapsto \mathbb{R}$ is called a random variable. The set of random variables ξ, the mathematical expectation of which is equal to zero, i.e. $M\xi = 0$, while variance is finite, i.e. $D\xi < \infty$, form the Hilbert space $\mathbf{L_2}$ with the scalar product $(\xi_1, \xi_2) = M\xi_1\xi_2$ and with the norm denoted by $\|\xi\|_{\mathbf{L_2}}$. If we take the subalgebra Σ_0 of the sigma-algebra Σ, then we obtain the subspace of random variables $\mathbf{L_2^0} \subset \mathbf{L_2}$ measurable with respect to Σ_0.

A measurable mapping $\eta = \eta(t, \omega) : J \times \Sigma \mapsto \mathbb{R}$, where $J = (a, b) \subset \mathbb{R}$, is called a stochastic process, a random variable $\eta(\cdot, \omega), \omega \in \Omega$ is said to be a section of the stochastic process, and a function $\eta(t, \cdot), t \in J$ is said to be a trajectory of the stochastic process. The stochastic process $\eta = \eta(t, \omega)$ is called continuous, if the trajectories $\eta = \eta(t, \omega_0)$ are continuous functions almost sure (i.e. for a.a. (almost all) $\omega_0 \in \Sigma$). The set $\eta = \eta(t, \omega)$ of continuous stochastic processes forms a Banach space $\mathbf{CL_2}$.

By the Nelson – Gliklikh derivative of the stochastic process $\eta \in \mathbf{CL_2}$ at the point $t \in J$ we mean the random variable

$$\overset{\circ}{\eta} = \frac{1}{2}\left(\lim_{\Delta t \mapsto 0+} M_t^\eta \left(\frac{\eta(t + \Delta t, \cdot) - \eta(t, \cdot)}{\Delta t}\right) + \lim_{\Delta t \mapsto 0+} M_t^\eta \left(\frac{\eta(t - \Delta t, \cdot) - \eta(t, \cdot)}{\Delta t}\right)\right), \tag{10}$$

if the limit exists in the sense of a uniform metric on $t \in J$. Here M_t^η is the expectation on a subalgebra of the sigma-algebra Σ that is generated by the random variable $\eta = \eta(t, \omega)$. If there exist the Nelson – Gliklikh derivatives $\overset{\circ}{\eta}(\cdot, \omega)$ of the stochastic process η at almost all points of the interval J, then we say that there exists the Nelson – Gliklikh derivative

$\overset{\circ}{\eta}(\cdot,\omega)$ almost sure on J. The set of continuous stochastic processes with continuous Nelson – Gliklikh derivatives $\overset{\circ}{\eta}$ form the Banach space $\mathbf{C^1L_2}$. Further, by induction, we obtain the Banach spaces $\mathbf{C^lL_2}$, $\mathbf{l} \in \mathbb{N}$ of the stochastic processes having continuous Nelson – Gliklikh derivatives on J up to the order $\mathbf{l} \in \mathbb{N}$ inclusively with the norms of the form $\|\eta\|_{\mathbf{C^lL_2}} = \sup\limits_{t\in J}(\sum\limits_{k=0}^{\mathbf{l}} D\overset{\circ}{\eta}^{(k)}(t,\omega))^{\frac{1}{2}}$, where $\overset{\circ}{\eta}^{(0)}(t,\omega) = \eta(t,\omega)$.

2. Resolving Groups or Semigroups of Operators

Let $\mathfrak{U}$ and $\mathfrak{F}$ be real separable Hilbert spaces. Denote by $\mathcal{L}(\mathfrak{U};\mathfrak{F})$ the space of linear bounded operators, and by $Cl(\mathfrak{U};\mathfrak{F})$ the space of linear closed and densely defined operators. Let us construct the Hilbert spaces $\mathbf{U_KL_2}$ and $\mathbf{F_KL_2}$, where $\mathbf{K} = \{\lambda_k\} \subset \mathbb{R}$ is a monotone sequence of numbers such that $\sum\limits_{k=1}^{\infty} \lambda_k^2 < +\infty$.

The operator M is called *spectrally bounded with respect to the operator* L (or, shortly, (L,σ)*-bounded*), if $\exists r > 0 \quad \forall \mu \in \mathbb{C} \quad (|\mu| > r) \Rightarrow (\mu \in \rho^L(M))$.

In complex plane $\mathbb{C}$, for the (L,σ)-bounded operator M, we choose a closed circuit of the form $\gamma = \{\mu \in \mathbb{C} : |\mu| = R > r\}$. Then the integrals

$$P = \frac{1}{2\pi i}\int\limits_{\gamma} R_\mu^L(M)d\mu, \quad Q = \frac{1}{2\pi i}\int\limits_{\gamma} L_\mu^L(M)d\mu$$

make sense as the integrals of analytic functions on a closed circuit. Moreover, the operators $P : \mathfrak{U} \to \mathfrak{U}$ and $Q : \mathfrak{F} \to \mathfrak{F}$ are projectors [4]. Denote by L_k, M_k the restrictions of the operators L, M on the subspace $\mathfrak{U}^k$, $k = 0, 1$.

Theorem 1. [4] *Let the operator M be (L,σ)-bounded. Then*

(i) $L_k, M_k \in \mathcal{L}(\mathfrak{U}^k;\mathfrak{F}^k)$, $k = 0, 1$;

(ii) *there exist operators* $L_1^{-1} \in \mathcal{L}(\mathfrak{F}^1;\mathfrak{U}^1)$, $M_0^{-1} \in \mathcal{L}(\mathfrak{F}^0;\mathfrak{U}^0)$.

If the operator M is (L,σ)-bounded, then by virtue of Theorem 1, there exist the operators $H = M_0^{-1}L_0 \in \mathcal{L}(\mathfrak{U}^0)$ and $S = L_1^{-1}M_1 \in \mathcal{L}(\mathfrak{U}^1)$.

Definition 1. The (L,σ)-bounded operator M is called

(i) $(L,0)$*-bounded*, if $H = \mathbb{O}$;

(ii) (L,p)*-bounded*, if $H^p \neq \mathbb{O}$ and $H^{p+1} = \mathbb{O}$ for some $p \in \mathbb{N}$.

Theorem 2. [11] *Let the operator M be (L,p)-bounded, $p \in \{0\} \cup \mathbb{N}$. Then there exists an analytical group of the operators on the space $\mathbf{U_K L}_2$ ($\mathbf{F_KL_2}$).*

Definition 2. The operator $M \in Cl(\mathbf{U_KL_2};\mathbf{F_KL_2})$ is said to be p-sectorial with respect to the operator $L \in \mathcal{L}(\mathbf{U_KL_2};\mathbf{F_KL_2})$ (for shortness, (L,p)-sectorial), $p \in \{0\} \cup \mathbb{N}$, if

(i) there exist constants $\alpha \in \mathcal{R}$ and $\Theta \in (\frac{\pi}{2},\pi)$ such that the sector $S_{\alpha,\Theta}^L(M) = \{\mu \in \mathbb{C} : |\arg(\mu-\alpha)| < \Theta, \mu \neq \alpha\} \subset \rho^L(M)$;

(ii) there exists a constant $K_1 > 0$ such that $\max\{\|R_{(\mu,p)}^L(M)\|_{\mathfrak{U}},\ \|L_{(\mu,p)}^L(M)\|_{\mathfrak{F}}\} < \dfrac{K_1}{\prod\limits_{q=0}^{p}|\mu_q-\alpha|}$ for any $\mu_0,\mu_1,...,\mu_q \in S_{\alpha,\Theta}^L(M)$.

Here $\rho^L(M) = \{\mu \in \mathbb{C} : (\mu L - M)^{-1} \in \mathcal{L}(\mathfrak{F};\mathfrak{U})\}$ is the L-resolvent set, and $\sigma^L(M) =$

$\mathbb{C} \setminus \rho^L(M)$ is the L-spectrum of the operator M. For $\mu_q \in \rho^L(M), q = 0, 1, ..., p$, the operator functions $R^L_\mu(M) = (\mu L - M)^{-1}L$ and $L^L_\mu(M) = L(\mu L - M)^{-1}$ are called the right L-resolvent and the left L-resolvent of the operator M, and $R^L_{(\mu,p)}(M) = \prod\limits_{q=0} p(\mu_q L - M)^{-1}L$ and $L^L_{(\mu,p)}(M) = \prod\limits_{q=0} pL(\mu L - M)^{-1}$ are called the right (L,p)-resolvent and the left (L,p)-resolvent of the operator.

The stochastic Sobolev type equation

$$L \overset{\circ}{\eta} = M\eta \tag{11}$$

can be reduced to two equations of the form

$$A \overset{\circ}{\nu} = B\nu.$$

Let us formulate

Lemma 1. *The following statements are true:*
i) the operator $A \in \mathcal{L}(\mathfrak{U}; \mathfrak{F})$ *exactly if* $A \in \mathcal{L}(\mathbf{U_K L_2}; \mathbf{F_K L_2})$*;*
ii) the operator $B \in Cl(\mathfrak{U}; \mathfrak{F})$ *exactly if* $B \in Cl(\mathbf{U_K L_2}; \mathbf{F_K L_2})$.

Theorem 3. *[15] Let the operator M be (L,p)-sectorial, then there exists an analytical semigroup of the operators on the space* $\mathbf{U_K L}_2$ *(*$\mathbf{F_K L}_2$*).*

Definition 3. The operator $M \in Cl(\mathbf{U_K L_2}; \mathbf{F_K L_2})$ is said to be p-radial with respect to the operator $L \in \mathcal{L}(\mathbf{U_K L_2}; \mathbf{F_K L_2})$ (for shortness, (L,p)-radial), $p \in \{0\} \cup \mathbb{N}$, if
i) there exists a constant $\alpha \in \mathbb{R}$ such that $[\alpha, +\infty) \subset \rho^L(M)$;
ii) there exists a constant $K_1 > 0$ such that $\forall \mu_q \in [\alpha, +\infty), q = 0, 1, ..., p, \forall n \in \mathbb{N}$ $\max\{\|R^L_{(\mu,p)}(M)\|_{\mathfrak{U}},\ \|L^L_{(\mu,p)}(M)\|_{\mathfrak{F}}\} < \dfrac{K_1}{\prod\limits_{q=0}^{p} (\mu_q - \alpha)^n}$.

Theorem 4. *[11] Let the operator M be (L,p)-radial. Then there exists a C_0-semigroup of the operators on the space* $\mathbf{U_K L}_2$ *(*$\mathbf{F_K L}_2$*).*

The set $\ker V^* = \{\nu \in \mathbf{U_K L_2}(\mathbf{F_K L_2}) : V^t\nu = 0\}$ is called the kernel, the set $\mathrm{im}V^* = \{\nu \in \mathbf{U_K L_2}(\mathbf{F_K L_2}) \lim\limits_{t \mapsto 0+} V^t\nu = \nu_0\}$ is said to be the image of the analytical semigroup $V^t : t \geq 0$. Denote $\mathfrak{U}^0 = \{\mathbf{U^0_K L_2}\}$ $(\mathfrak{F}^0 = \{\mathbf{F^0_K L_2}\})$, which form a closure of kernels of semigroups in the norm of the space $\mathfrak{U} = \mathbf{U_K L_2}$ $(\mathfrak{F} = \mathbf{F_K L_2})$. Also, denote $\mathfrak{U}^1 = \{\mathbf{U_K L_2}\}$ $(\mathfrak{F}^1 = \{\mathbf{F^1_K L_2}\})$, which form a closure $\mathrm{im}R^L_{(\mu,p)}(M)$ $(\mathrm{im}L^L_{(\mu,p)}(M))$ in the norm of the space $\mathfrak{U} = \mathbf{U_K L_2}$ $(\mathfrak{F} = \mathbf{F_K L_2})$. The spaces $\mathbf{U_K L}_2$ and $\mathbf{F_K L}_2$ are splitted into the direct sum

$$\mathbf{U_K L}_2 = \mathbf{U^0_K L}_2 \oplus \mathbf{U^1_K L}_2, \mathbf{F_K L}_2 = \mathbf{F^0_K L}_2 \oplus \mathbf{F^1_K L}_2. \tag{12}$$

The following theorem takes place.

Theorem 5. *If the operator M is (L,p)-radial and there exist splittings (12), then* $\mathrm{im}U^* = \mathbf{U^1_K L_2}$ *and* $\mathrm{im}F^* = \mathbf{F^1_K L_2}$.

Previously, the Showalter-Sidorov problem

$$P(\eta(0)-\eta_0)=0 \tag{13}$$

was investigated [11] in the spaces $\mathfrak{U}=\mathbf{U_K L_2}$ ($\mathfrak{F}=\mathbf{F_K L_2}$), where there exist representations of the form

$$\eta(t,\cdot)=\sum_{k=0}^{+\infty}\lambda_k\xi_k(t,\cdot)\varphi_k. \tag{14}$$

Theorem 6. *Suppose that the operator M is (L,p)-radial and there exist splittings (12), then $\forall\eta_0\in\mathfrak{U}^1\subset\mathfrak{U}$ there exists the unique solution to problem (11), (13).*

3. Differential Forms and Computational Experiments

Consider a two-dimensional torus obtained by the direct product of two segments $\mathbb{T}=[0,\pi]\times[0,2\pi]$. The torus is a 2-dimensional smooth compact oriented Riemannian manifold without boundary. Using theory presented in Sections 1 and 2, we construct spaces of smooth differential q-forms with stochastic processes as the coefficient:

$$\omega(t,\omega,x_1,x_2)=\sum_{|i_1,\dots,i_q|=q}\chi_{i_1,\dots,i_q}(t,\omega,x_1,x_2)dx_{i_1}\wedge\dots\wedge dx_{i_2}, \tag{15}$$

where $|i_1,\dots,i_q|$ is a multi-index, and, according to (14), the coefficients have the form

$$\chi_{i_1,i_2,\dots,i_q}(t,\omega,x_1,x_2)=\sum_{k=1}^{\infty}\lambda_k\xi_{k,i_1,\dots,i_q}(t)\varphi_k.$$

As $\mathfrak{U}$, we consider the spaces of differential q-forms defined on a smooth compact oriented Riemannian manifold without boundary and orthogonal to harmonic q-forms. Such spaces take place on the basis of the Hodge–Kodaira theory in the deterministic case for the Cauchy problem for Ginzburg–Landau equation (3). We consider the Showalter – Sidorov problem

$$P(\eta(0)-\eta_0)=0 \tag{16}$$

for the stochastic version of the Ginzburg – Landau equation

$$(\lambda+\Delta)\overset{\circ}{\eta}=\alpha\Delta\eta+id\Delta^2\eta, \tag{17}$$

and the signs differ from (3) since instead of the Laplace operator we use its generalization (up to a sign) to spaces of differential forms, namely, the Laplace – Beltrami operator. Denote the operators

$$L=(\lambda+\Delta),\ M=\alpha\Delta+id\Delta^2$$

and arrive at (7).

For this problem, the work [17] proves (L,p)-radiality of the operator M and constructs the relative spectrum

$$\mu_t=\frac{\alpha\lambda_k+id\lambda_k^2}{\lambda+\lambda_k},$$

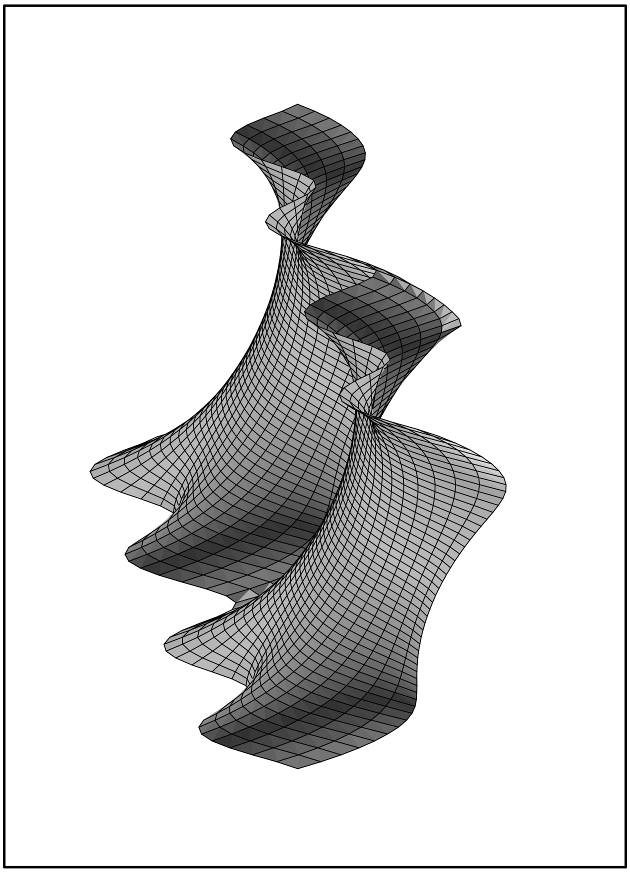

Fig. 1. Solution to (16), (17) for $\alpha = -0.5, d = 0.5, \lambda = 4, q = 0$

where $\{\lambda_k\}$ is the sequence of eigenvalues of the Laplace–Beltrami operator on the torus numbered in increasing order taking into account the multiplicity, and $\{\varphi_k\}$ is the sequence of eigenfunctions, respectively.

Introduce a grid on the torus and construct a difference analogue of the trajectories of the Ginzburg–Landau stochastic equation, and implement the Petrov–Galerkin method in the Maple system.

Here we implement the following algorithm.

Step 1. Enter the parameters of the Ginzburg – Landau equation ($\alpha, d \in \mathbb{R}, \lambda \neq 0$).

Step 2. Construct a grid on the two-dimensional torus $\mathbb{T}$.

Step 3. Calculate eigenvalues and construct eigenfunctions.

Step 4. Represent solutions in the form of expansion in terms of eigenfunctions.

Step 5. Obtain a numerical solution to the problem for a random value that belongs to the probability space Ω.

Step 6. Obtain a graphical representation of the solution and display the solution on the screen.

Fig. 1 shows the unique coefficient for solution to the homogeneous Ginzburg – Landau equation for 0-forms (2-forms) at $\alpha = -0.5, d = 0.5, \lambda = 4$.

Figs. 2 and 3 show the coefficients at dx and dy, respectively, for the solution to the homogeneous Ginzburg – Landau equation for $\alpha = 1, d = 2, \lambda = 0$.

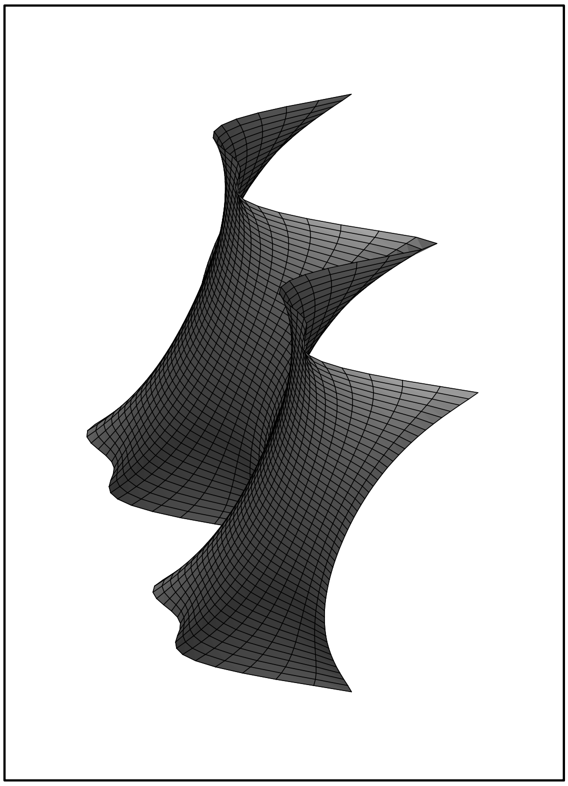

Fig. 2. Coefficient at dx of the solution to (16), (17) for $\alpha = 2, d = 2, \lambda = 0, q = 1$

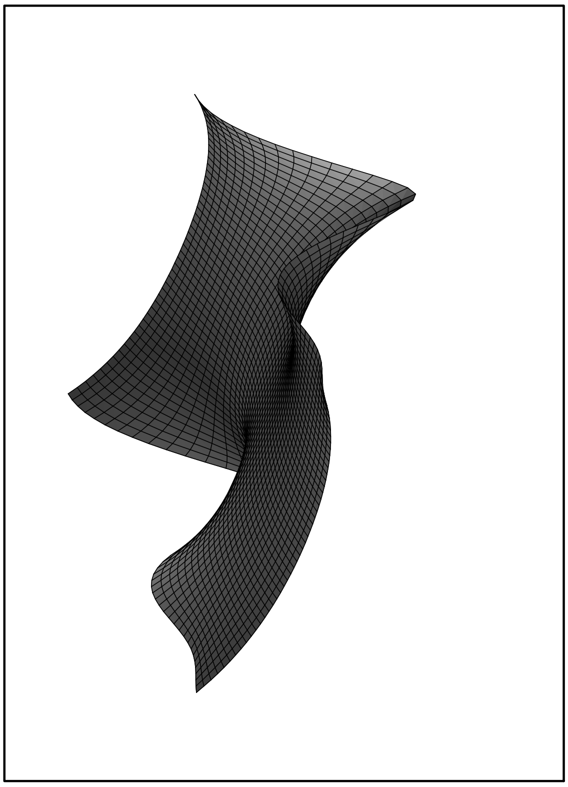

Fig. 3. Coefficient at dy of the solution to (16), (17) for $\alpha = 2, d = 2, \lambda = 0, q = 1$

Then we take the stochastic version of the Dzektser equation

$$(\lambda+\Delta)\overset{\circ}{\eta}=-\alpha\Delta\eta-\beta\Delta^2\eta. \tag{18}$$

Denote the operators

$$L=(\lambda+\Delta),\ M=-\alpha\Delta, N=-\beta\Delta^2$$

and arrive at

$$L\overset{\circ}{\eta}=M\eta+N(\theta), \tag{19}$$

with $\Theta=\eta$.

For this problem, the paper [14] proves $(L,0)$-sectoriality of the operator M and constructs the relative spectrum

$$\mu_t=\frac{-\alpha\lambda_k-\beta\lambda_k^2}{\lambda+\lambda_k}.$$

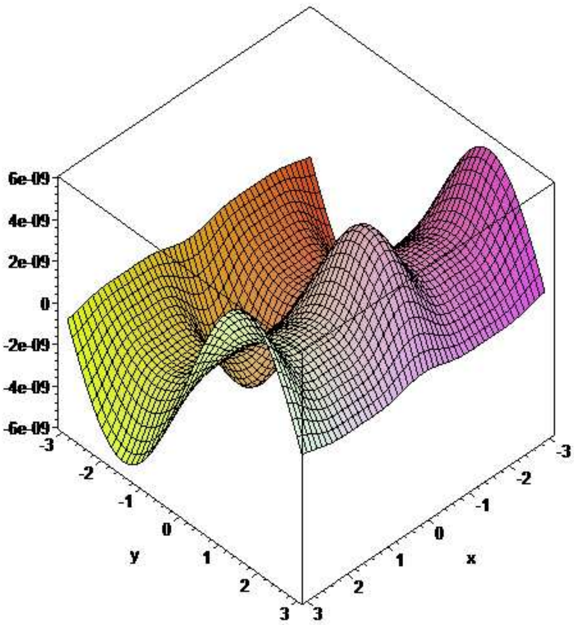

Fig. 4. Coefficient of the solution to the Showalter–Sidorov problem for the Dzektser equation with $\alpha=-2,\beta=-2,\lambda=0.1,q=0$ at the moment $t=1$

Fig. 4 shows the unique coefficient $q=0(q=2)$ for the solution to homogeneous Dzektser equation (18) with Cauchy condition (8) for the values $\alpha=-2,\beta=-2,\lambda=0.1$.

At last we take the stochastic version of the Barenblatt – Zheltov – Kochina (BZK) equation

$$(\lambda+\Delta)\overset{\circ}{\eta}=\alpha\Delta\eta+f. \tag{20}$$

Denote the operators

$$L=(\lambda+\Delta),\quad M=\alpha\Delta$$

and arrive at

$$L\overset{\circ}{\eta}=M\eta+f. \tag{21}$$

For this problem, the paper [11] proves $(L, 0)$-boundedness of the operator M and constructs the relative spectrum
$$\mu_t = \frac{\alpha\lambda_k}{\lambda + \lambda_k}.$$

The algorithm of solution of the Cauchy problem for the Barenblatt–Zheltov–Kochina equation is presented by the block diagram given in Fig. 5.

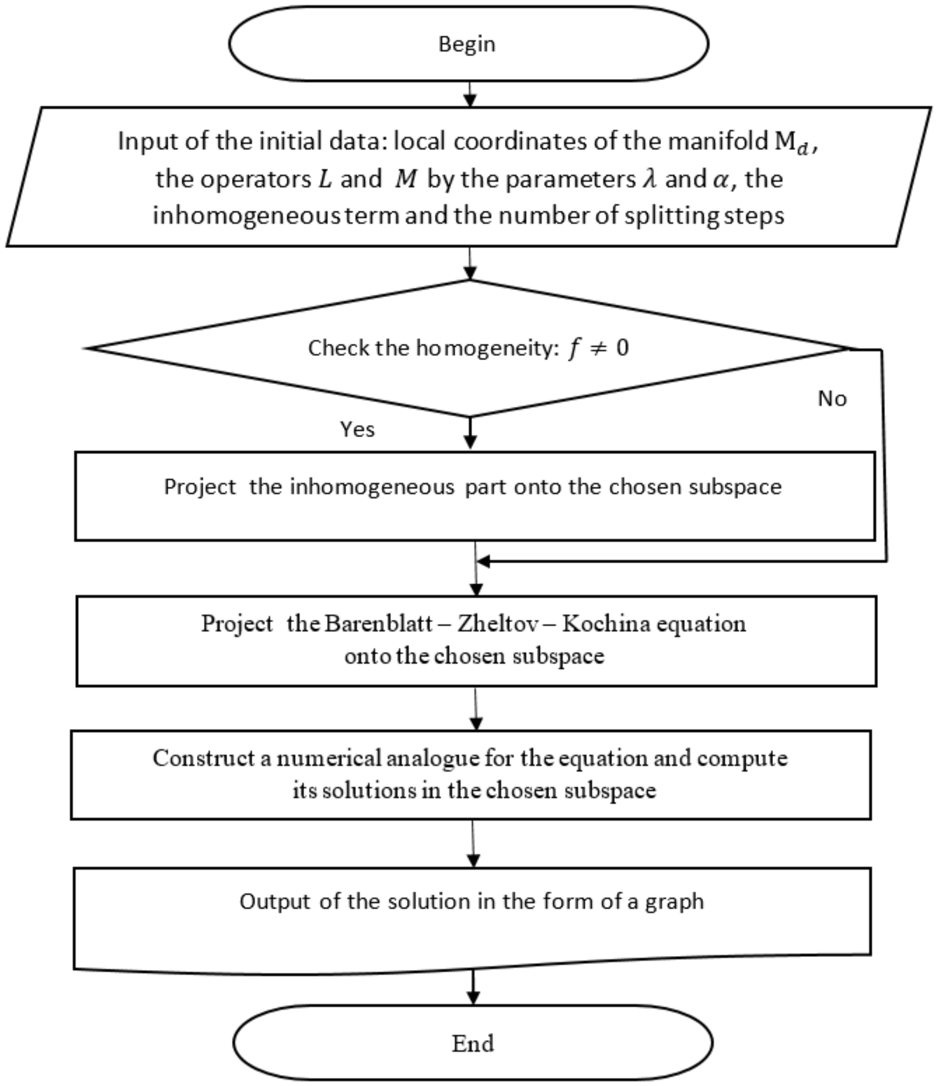

Fig. 5. Block diagram of the algorithm for BZK equation

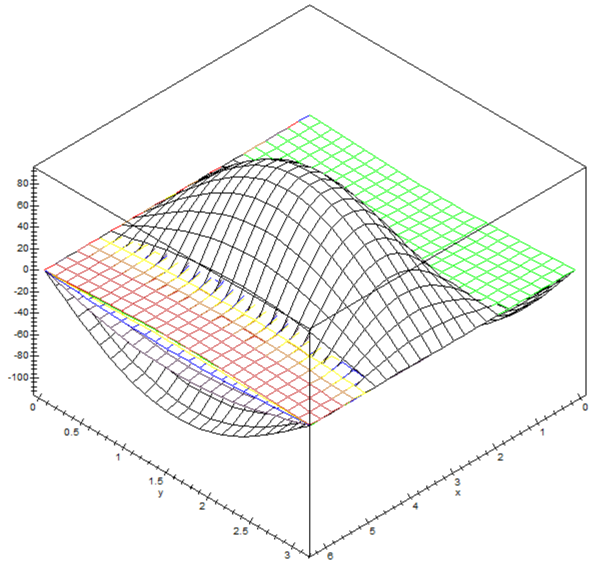

Fig. 6. The graph of the Cauchy problem solution for BZK with $\lambda = 7, \alpha = 0.5, f = 0$

The graphs show the solutions at the time moments $t_k, k = 1, ..., 8$, by the corresponding colors: pink, green, blue, black, yellow, brown, red, black.

Fig. 7 shows the graphs of the solution to the homogeneous Cauchy problem with $\lambda = 7, \alpha = 0.5$ at the first eight time moments. Fig. 8 shows a similar graph with an inhomogeneous term $f = 10\sin(5t)$. Further, we present graphs of the solution with $\lambda = 1, \alpha = 2$ for the homogeneous equation and inhomogeneous equation with $f = 5\sin(2t)$ in Fig. 9 and in Fig. 10, respectively.

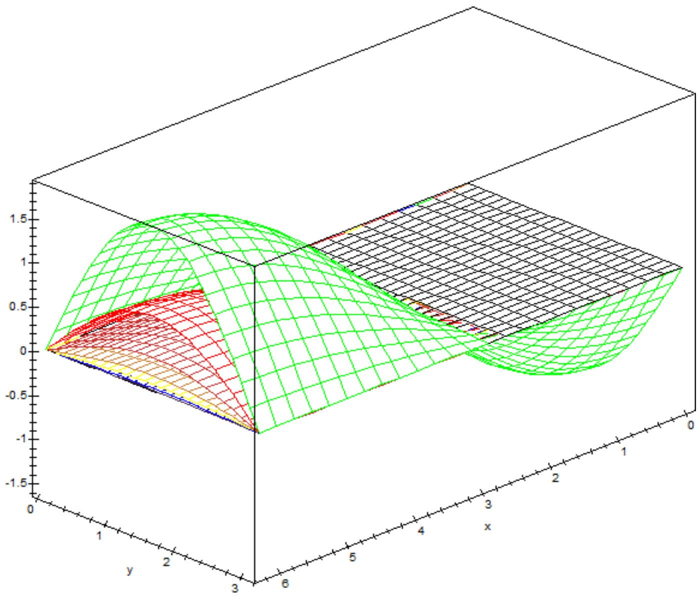

Fig. 7. The graph of the Cauchy problem solution for BZK with $\lambda = 7, \alpha = 0.5, f = 10\sin(5t)$

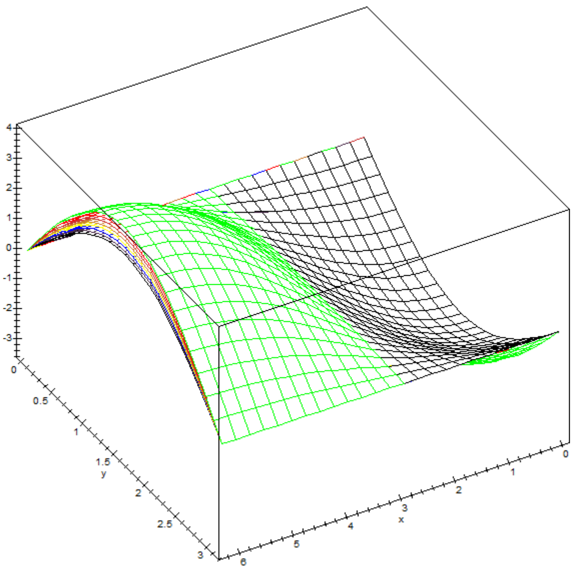

Fig. 8. The graph of the Cauchy problem solution for BZK with $\lambda = 1, \alpha = 2, f = 0$

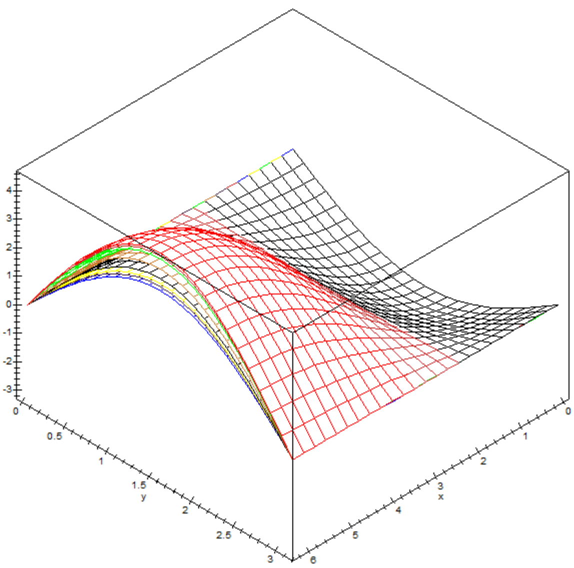

Fig. 9. The graph of the Cauchy problem solution for BZK with $\lambda = 1, \alpha = 2, f = 5\sin(2t)$

Conclusion

As a result of studying the numerical solutions to the Sobolev type equation, we obtain the graphs of the solution for three model cases on the torus. In addition to the presented results, there exist the papers on these equations on the sphere and for the linear and semilinear Hoff equation [20], but they are not included in this overview.

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ЧИСЛЕННЫЕ РЕШЕНИЯ НЕКЛАССИЧЕСКИХ УРАВНЕНИЙ В ПРОСТРАНСТВАХ ДИФФЕРЕНЦИАЛЬНЫХ ФОРМ

Д. Е. Шафранов

Статья содержит обзор результатов, полученных автором в специально заданных пространствах, а именно пространствах дифференциальных форм со стохастическими коэффициентами, определенных на некотором римановом многообразии без края. В данной работе представлены графики траекторий численных решений задачи Коши для уравнения Баренблатта–Желтова–Кочиной и задачи Шоуолтера–Сидорова для уравнений Дзекцера и Гинзбурга–Ландау. Поскольку уравнения изучаются в пространстве дифференциальных форм, сами операторы понимаются в специальной форме, в частности, вместо оператора Лапласа берется его обобщение – оператор Лапласа–Бельтрами. Графики коэффициентов дифференциальных форм полученные при проведении вычислительных экспериментов приведены для различных значений параметров исходных уравнений.

Ключевые слова: уравнение соболевского типа; дифференциальные формы; риманово многообразие; оператор Лапласа - Бельтрами; численное решение.

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