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MODELS OF VISCOUS FLUIDS GENERATED BY MARTINGALES ON THE GROUPS OF DIFFEOMORPHISMS

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We study two martingales on the group of Sobolev diffeomorphisms of the flat n-dimensional torus, they both are described by systems of two special equations with mean derivatives. The first one describes a solution of the Burgers equation on the torus that also satisfies an analog of continuity equation. The second martingale describes a certain non-Newtonian fluid on the torus that satisfies some special analogs of the Burgers equation and the continuity equation.

Keywords: mean derivatives; flat torus; groups of diffeomoirphisms; viscous hydrodynamics.

Introduction

The paper is devoted to the Lagrangiaan approach to hydrodynamics initiated by the well-known works by V.I. Arnold [1] and then by D. Ebin and J. Marsden [2]. The main difference between those papers and our approach is that we replace the covariant derivatives of weak Riemannian metric on the group of diffeomorphisms, that was used in [2], by the second order backward mean derivatives of stochastic processes. The concept of mean derivatives was introduced by E. Nelson (see, e.g., [3]) for the needs of the so called stochastic mechanics. The use of these derivatives allows us to generalise the geometric method used in [2] for investigation of ideal fluids, to the case of viscous fluids (see details, e.g., in [4]). In spite of the fact that the construction is based on the Stochastic Analysis, the results in particular are obtained for deterministic (not random) fluids.

We consider the hydrodynamics only on the flat n-dimensional torus and we essentially use the properties of the torus in our constructions. Note that the investigation of fluid motion on the torus is a well-known problem in the hydrodynamics. Recall that the flat n-dimensional torus is a quotient space of $\mathbb{R}^n$ under factorisation with respect to the integral lattice where the Riemannian metric on the torus is inherited from the Euclidean inner product on $\mathbb{R}^n$.

We study two martingales on the group of Sobolev diffeomorphisms, they both are described by systems of two special equations with mean derivatives. The first martingale describes a solution of the Burgers equation on the torus that also satisfies an analog of continuity equation. The second one describes a certain non-Newtonian fluid on the torus that satisfies some special analogs of the Burgers equation and the continuity equation.

1. Mean Derivatives

For simplicity of presentation, we describe the theory of mean derivatives for processes in $\mathbb{R}^n$. However, due to the fact that the geometry on the flat torus is inherited from the Euclidean geometry on $\mathbb{R}^n$, this presentation is unchanged applied to the torus.

Consider a random process $\xi(t)$ in $\mathbb{R}^n$ (where we specify the σ-algebra of Borel sets), $t \in [0,T]$, defined on some probability space $(\Omega, \mathcal{F}, \mathsf{P})$ and such that $\xi(t)$ belongs to the space $L_1(\Omega, \mathbb{R}^n)$ for every t.

Denote by $\mathcal{N}_t^\xi$ the σ-subalgebra «presence» in $\mathcal{F}$ generated by preimages of Borel sets from $\mathbb{R}^n$ under the mapping $\xi(t) : \Omega \to \mathbb{R}^n$. $\mathcal{N}_t^\xi$ is assumed to be complete, i.e. containing all zero probability sets. The «past» σ-subalgebra $\mathcal{P}_t^\xi$ is the minimal complete σ-subalgebra such that all $\xi(s)$ for $0 \le s \le t$ are measurable with respect to it. For convenience, we denote by E_t^ξ the conditional mathematical expectation $E(\cdot|\mathcal{N}_t^\xi)$ relative to the «presence» $\mathcal{N}_t^\xi$ of $\xi(t)$.

Following E. Nelson, we introduce the concepts of forward, backward and symmetric mean derivative.

The forward mean derivative $D\xi(t)$ of the process $\xi(t)$ at time t is an L_1 random element of the form

$$D\xi(t) = \lim_{\triangle t \to +0} E_t^\xi \left(\frac{\xi(t+\triangle t) - \xi(t)}{\triangle t} \right), \tag{1}$$

where the limit is assumed to exist in $L_1(\Omega, \mathcal{F}, \mathsf{P})$, and the symbol $\triangle t \to +0$ means that $\triangle t$ tends to zero 0 and $\triangle t > 0$.

The backward mean derivative $D_*\xi(t)$ of the process $\xi(t)$ at the time instant t is an L_1-random element

$$D_*\xi(t) = \lim_{\Delta t \to +0} E_t^\xi \left(\frac{\xi(t) - \xi(t-\Delta t)}{\Delta t} \right) \tag{2}$$

where (as in (1)) the limit is assumed to exist in $L_1(\Omega, \mathcal{F}, \mathsf{P})$, and the symbol $\Delta t \to +0$ means that $\Delta t \to 0$ and $\Delta t > 0$.

The symmetric mean derivative D_S is given by the formula $\frac{1}{2}(D + D_*)$. The derivative $D_S\xi(t)$ is called the current velocity of $\xi(t)$.

Everywhere below by $w(t)$ we denote a certain Wiener process. We mainly need to work with D_* and D_S. That's why let's take a closer look at their properties. It follows from the properties of the conditional expectation that $D_*\xi(t)$ can be represented as a superposition of $\xi(t)$ and a measurable Borel vector field (regression)

$$a(t,x) = \lim_{\Delta t \to +0} E \left(\frac{\xi(t) - \xi(t-\Delta t)}{\Delta t} | \xi(t) = x \right) \tag{3}$$

on $\mathbb{R}^n$. This means that $D_*\xi(t) = a(t, \xi(t))$.

We introduce the symbol v^ξ for the regression of current velocity.

Let $Z(t,x)$ be a C^2-smooth vector field on $\mathbb{R}^n$, and $\xi(t)$ be a stochastic process in $\mathbb{R}^n$.

The L_1-limit of the form

$$D_*Z(t,\xi(t)) = \lim_{\Delta t \to +0} E_t^\xi \left(\frac{Z(t,\xi(t)) - Z(t-\Delta t, \xi(t-\Delta t))}{\Delta t} \right) \tag{4}$$

is called backward mean derivative of Z along $\xi(\cdot)$ at the time instant t.

Of course, $D_*Z(t,\xi(t))$ can be represented as the superposition of $\xi(t)$ with a certain Borel measurable vector field (regression). This vector field we will denote by the symbol $\overline{D_*Z}$.

For a process with diffusion term $\sigma w(t)$, i.e., with the diffusion coefficient $\sigma^2 I$ in $\mathbb{R}^n$ the following formula holds

$$\overline{D_*Z} = \frac{\partial}{\partial t}Z + (Y_*^0 \cdot \nabla)Z - \frac{\sigma^2}{2}\Delta Z, \tag{5}$$

where $\nabla = (\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n})$, Δ is Laplacian, dot denotes the scalar product in $\mathbb{R}^n$ and the vector field $Y_*^0(t,x)$ is the regression for backward mean derivative of $\xi(t)$. One can easily derive (5) using the backward Ito formula.

Definition 1. *The second order mean derivative $D_*D_*\xi(t)$ is the first order derivative D_* of the regression (vector field) $\overline{D_*\xi}$.*

Lemma 1. *[4, Lemmas 8.23 and 8.25] For $t \in (0,T]$ we obtain $D_*w(t) = \frac{w(t)}{t}$ and $D_*\frac{w(t)}{t} = 0$.*

Corollary 1. *$D_*D_*\sigma w(t) = D_*\overline{D_*\sigma w(t)} = 0$.*

In the case when ξ has the diffusion coefficient $\sigma^2 I$, denote the regression of $D_*\xi$ by the symbol Y. Then according to the formula (5) we get

$$D_*D_*\xi = \left(\frac{-\sigma^2}{2}\Delta + Y \cdot \nabla + \frac{\partial}{\partial t}\right)Y, \tag{6}$$

where the right-hand side of formula (6) is the same as the left-hand side of the Burgers equations with viscosity $\frac{\sigma^2}{2}$.

Below, besides the process with diffusion term $\sigma w(t)$ we will deal with the process with diffusion term $\int\limits_0^t B(t)dw(t)$ where $B(t)$ is a smooth in t deterministic non-degenerate and (possibly) non-autonomous linear operator. In this case formula (5) is transformed into the formula

$$\overline{D_*Z} = \frac{\partial}{\partial t}Z + (Y_*^0 \cdot \nabla)Z - \frac{1}{2}\mathfrak{B}(t)Z, \tag{7}$$

where $\mathfrak{B}(t)$ is the second order differential operator with the matrix of coefficients in the form $B(t) \cdot B^*(t)$. Thus formula (6) is transformed into

$$D_*D_*\xi = \left(\frac{1}{2}\mathfrak{B}(t) + Y \cdot \nabla + \frac{\partial}{\partial t}\right)Y. \tag{8}$$

We interpret the right-hand-side of (8) as an analog of the Burgers equation for a certain non-Newtonian fluid.

Note that $\int\limits_0^t B(t)dw(t)$ is a martingale with respect to the «past» of $w(t)$ and so $D\int\limits_0^t B(t)dw(t) = 0$.

Lemma 2. *For $t \in (0,T]$ we obtain $D_* \int\limits_0^t B(t)dw(t) = B(t)\frac{w(t)}{t}$ and the regression of $(t)\frac{w(t)}{t}$ equals $B(t)\frac{x}{t}$.*

*P*roof. To calculate the backward mean derivative in this case, it is enough to consider the summand in the integral sum, that determins the integral, of the form , $B(t-\Delta t)(w(t)-w(t-\Delta t))$. By the definition of backward mean derivative we obtain

$$D_* \int\limits_0^t B(s)dw(s) = \lim_{\Delta t \to +0} E_t^w \frac{B(t-\Delta t)(w(t)-w(t-\Delta t))}{\Delta t}.$$

Since the operator $B(t)$ is not random, we can translate it outside the conditional expectation, and since it is smooth (and so continuous) $\lim\limits_{\Delta t \to 0} B(t-\Delta t) = B(t)$. Thus $D_* \int\limits_0^t B(s)dw(s) = B(t)D_*w(t)$. But by Lemma 1 $D_*w(t) = \frac{w(t)}{t}$.

Since $B(t)$ is not random, the «presence» of $B(t)\frac{w}{t}$ coincides with the «presence» of $w(t)$. Thus the regression of $D_* \int\limits_0^t B(t)dw(t)$ equals $B(t)\frac{x}{t}$. $\square$

Remark 1. The latter formula allows one to calculate $D_*D_* \int\limits_0^t B(t)dw(t)$. Denote the regression of this second derivative by $\mathcal{B}(t,x)$. We needn't its exact form and for simplicity of presentation we leave the calculation for the reader as an exercise.

Let the process $\xi(t)$ have a diffusion coefficient with the matrix (α^{ij}), that is a smooth symmetric $(2,0)$ non-degenerate tensor field. This means that the smooth non-degenerate symmetric $(0,2)$ tensor field with the converse matrix (α_{ij}) is well defined, and it can be considered a new time-depended Riemannian metric on $\mathbb{R}^n$. Below we denote by Div_t the divergence with respect to the Riemannian metric (α_{ij}) restricted to the level surface $t = \text{const}$.

Note that in the case of the diffusion coefficient (α^{ij}), in formulae (6) and (8) the expression $Y \cdot \nabla$ is replaced by the covariant derivative of the Levi-Civita connection of the Riemannian metric (α_{ij}) with respect to Y.

Denote by the symbol $\rho^\xi(t,x)$ the probabilistic density of element $\xi(t)$ with respect to the volume form $dt \wedge \Lambda$ on $\mathbb{R} \times \mathbb{R}^n$, where Λ is the Euclidean form volume on $\mathbb{R}^n$, i.e., for any bounded continuous function $f\colon [0,T] \times \mathbb{R}^n \to \mathbb{R}$ the formula

$$\int\limits_0^T E(f(t,\xi(t)))\, dt = \int\limits_0^T \left(\int\limits_\Omega f(t,\xi(t))d\mathsf{P} \right) dt = \int\limits_{[0,T]\times\mathbb{R}^n} f(t,x)\rho^\xi(t,x)dt \wedge \Lambda$$

takes place.

Let $\xi(t)$ be a diffusion process with diffusion coefficient (α^{ij}) as above and density ρ^ξ. For its current velocity $v^\xi(t,x)$ and density $\rho^\xi(t,x)$ the following relation of the type of continuity equation is satisfied

$$\frac{\partial \rho^\xi(t,x)}{\partial t} = -Div_t(\rho^\xi(t,x)v^\xi(t,x)), \tag{9}$$

(see the proof of formula (9) in [5, Lemma 3]). For a process with the unit diffusion coefficient Div_t is the usual divergence div and so (9) takes the form

$$\frac{\partial \rho^{\xi}(t,x)}{\partial t} = -\mathrm{div}(\rho^{\xi}(t,x).v^{\xi}(t,x)), \tag{10}$$

that is ordinary continuity equation.

All construction of this Section are translated from $\mathbb{R}^n$ to the flat n-dimensional torus $\mathcal{T}^n$ by the factorisation with respect to the integral lattice.

2. The Sobolev Groups of Diffeomorphisms

Let $\mathcal{T}^n$ be a flat n-dimensional torus and $\mathcal{D}^s(\mathcal{T}^n)$ be its Sobolev group of diffeomorphisms of the class $H^s(s > n/2+1)$. Recall that for $s > n/2+1$ the mappings from H^s are C^1 smooth. $\mathcal{D}^s(\mathcal{T}^n)$ is a Hilbert manifold and group with respect to superposition with unity $e = id$. Tangent space $T_e\mathcal{D}^s(\mathcal{T}^n)$ is the space of all H^s-vector fields on $\mathcal{T}^n$.

In any tangent space $T_f\mathcal{D}^s(\mathcal{T}^n)$ one can define L^2-scalar product by the formula

$$(X,Y) = \int\limits_{\mathcal{T}^n} \langle X(m), Y(m)\rangle_{f(m)}\mu(dm) \tag{11}$$

The family of these scalar products forms the so-called weak Riemannian metric on $\mathcal{D}^s(\mathcal{T}^n)$. In particular, in $T_e\mathcal{D}^s(\mathcal{T}^n)$ (11) becomes

$$(X,Y)_e = \int\limits_{\mathcal{T}^n} \langle X(m), Y(m)\rangle_{m}\mu(dm). \tag{12}$$

Right shift $R_f : \mathcal{D}^s(T^n) \to \mathcal{D}^s(T^n)$, where $R_f(\Theta) = \Theta \circ f$ for $\Theta, f \in \mathcal{D}^s(T^n)$ is a C^∞-smooth mapping. The tangent mapping to the right shift is $TR_f(X) = X \circ f$ for $X \in T\mathcal{D}^s(T^n)$. On the other hand, left shift $L_f : \mathcal{D}^s(T^n) \to \mathcal{D}^s(T^n)$, where $L_f(\Theta) = f \circ \Theta$ for $\Theta, f \in \mathcal{D}^s(T^n)$, is only continuous.

For the vector $X \in T_e\mathcal{D}^s(T^n)$, introduce the right-invariant vector field $\bar{X}$ on $\mathcal{D}^s(T^n)$ by the formula $\bar{X}_g = X \circ g$ for any $g \in \mathcal{D}^s(T^n)$.

Recall that $T\mathcal{T}^n = \mathcal{T}^n \times \mathbb{R}^n$. We introduce the operators

$$B : T\mathcal{T}^n \to \mathbb{R}^n,$$

projections to the second factor in $\mathcal{T}^n \times \mathbb{R}^n$, and

$$A(m) : \mathbb{R}^n \to T_m\mathcal{T}^n,$$

the linear isomorphism of $\mathbb{R}^n$ inverse to B onto the tangent space to $\mathcal{T}^n$ at $m \in \mathcal{T}^n$.

Consider the operator $\bar{A} : \mathcal{D}^s(\mathcal{T}^n) \times \mathbb{R}^n \to T\mathcal{D}^s(\mathcal{T}^n)$ such that $\bar{A}_e$ is the same as A introduced above and for each $g \in \mathcal{D}^s(\mathcal{T}^n)$ the mapping $\bar{A}_g : \mathbb{R}^n \to T_g\mathcal{D}^s(\mathcal{T}^n)$ is constructed from $\bar{A}_e$ by right shift, i.e. for $X \in \mathbb{R}^n$:

$$\bar{A}_g(X) = TR_g \circ A_e(X) = (A \circ g)(X).$$

Every right-invariant vector field $\bar{A}(X)$ is C^∞-smooth on $\mathcal{D}^s(\mathcal{T}^n)$ for every $X \in \mathbb{R}^n$. This follows from the results of [2].

For any point $m \in T^n$ denote by $exp_m : T_m T^n \to T^n$ the mapping that sends the vector $X \in T_m T^n$ to the point $m+X$ modulo the factorization with respect to the integer lattice on T^n. The family of such mappings generates a mapping $\overline{exp} : T_e\mathcal{D}^s(T^n) \to \mathcal{D}^s(T^n)$ that sends the vector $X \in T_e\mathcal{D}^s(T^n)$ into $e + X \in \mathcal{D}^s(T^n)$, where $e + X$ is the diffeomorphism T^n of the form: $(e + X)(m) = m + X(m)$.

Consider the superposition $\overline{exp} \circ \bar{A}_e : \mathbb{R}^n \to \mathcal{D}^s(\mathcal{T}^n)$. By construction for an arbitrary $X \in \mathbb{R}^n$ we get that $\overline{exp} \circ \bar{A}_e(X)(m) = m + X$, i.e., the same the vector X is added to each point m.

Let $w(t)$ be a Wiener process in $\mathbb{R}^n$ given on some probability space $(\Omega, \mathcal{F}, \mathsf{P})$. Construct the random process

$$W^{(\sigma)}(t) = \overline{exp} \circ \bar{A}_e(\sigma w(t)) \tag{13}$$

on $\mathcal{D}^s(\mathcal{T}^n)$. By construction, for $\omega \in \Omega$ the corresponding sample trajectory $W^{(\sigma)}_\omega(t)$ is a diffeomorphism of the form $W^{(\sigma)}_\omega(t)(m) = m + \sigma w_\omega(t)$. Note that for given $\omega \in \Omega$ and given $t \in \mathbb{R}$ we get that $w(t)_\omega$ is a constant vector in $\mathbb{R}^n$ that is added to every point $m \in \mathcal{T}^n$ modulo factorization with respect to the integral lattice.

Denote by $W^{(B)}(t)$ the process given by the formula

$$W^{(B)}(t) = \overline{exp} \circ \bar{A}_e \left(\int_0^t B(s)dw(s) \right) \tag{14}$$

(see Section 1). By construction, for $\omega \in \Omega$ the corresponding sample path of $W^{(B)}_\omega(t)$ is the diffeomorphism of the form $W^{(B)}_\omega(t)(m) = m + \int_0^t B(s)dw(s)_\omega$. Note that for given $\omega \in \Omega$ and given $t \in \mathbb{R}$, as above, we get that $\int_0^t B(s)dw(s)_\omega$ is a constant vector in $\mathbb{R}^n$ that (as well as above) is added to every point $m \in \mathcal{T}^n$ modulo factorization with respect to the integral lattice. Note that both process (13) and process (14) are martingales.

Mean derivatives are also introduced in complete analogy to the usual finite-dimensional definition. We are interested in the operator of the second backward mean derivative D_*D_*. Recall that we understand this expression as the application operator D_* to the regression of the backward mean derivative of the random process (i.e. to a vector field).

3. The Main Result

Here and below we assume that $s > n/2 + 2$. So the diffeomorphisms from $\mathcal{D}^s(\mathcal{T}^n)$ are C^2-smooth as well as vector fields from $T_e\mathcal{D}^s(\mathcal{T}^n)$ on the torus. Everywhere below we use the same stochastic processes $W^{(\sigma)}(t)$ and $W^{(D)}(t)$ constructed from the Wiener process $w(t)$ in $\mathbb{R}^n$ by formulae (13) and (14).

Let $\bar{F}$ be a right-invariant vector field on $\mathcal{D}^s(\mathcal{T}^n)$. Consider on $\mathcal{D}^s(\mathcal{T}^n)$ the following system of equations with mean derivatives:

$$D_*D_*\xi(t) = \bar{F}, \tag{15}$$

$$D_*\xi(t) = 2D_S\xi(t). \tag{16}$$

It follows from equation (16) and from the definition of symmetric mean derivative, that $D\xi(t) = 0$ and so $\xi(t)$ is a martingale. Thus $\bar{F}$ must be equal to D_*D_* of this martingale. We will deal with two martingales $W^{(\sigma)}(t)$ and $W^{(B)}(t)$ introduced by (13) and (14).

First consider the case of $W^{(\sigma)}(t)$. Using Lemma 1, one can easily show that $D_*D_*W^{(\sigma)}(t) = 0$. Hence in this case system (15), (16) takes the form

$$D_*D_*\xi(t) = 0, \tag{17}$$

$$D_*\xi(t) = 2D_S\xi(t). \tag{18}$$

Translate both equations by right shifts to $T_e\mathcal{D}^s(\mathcal{T}^n)$, i.e, to the space of H^s vector fields on $\mathcal{T}^n$. In [6] by using formula (6) it is shown that equation (17) is transformed into the Burgers equation with viscosity $\frac{\sigma^2}{2}$ and zero external force, but taking into account formula (10) one can easily see that equation (18) is transformed into

$$\frac{\partial\rho^\xi(t,x)}{\partial t} = -\frac{1}{2}\mathrm{div}(\rho^\xi(t,x)v^\xi(t,x)). \tag{19}$$

We interpret (19) as a version of continuity equation. Thus system (17), (18) describes the above mentioned solution to the Burgers equaiton that in addition satisfies the analog of continuity equation.

Now consider the case on martingale $W^{(B)}(t)$. Recall that by Lemma 2

$$D_*D_*\int_0^t B(t)dw(t) = B'(t)\frac{w(t)}{t}$$

and that the regression of $B'(t)\frac{w(t)}{t}$ equals $B'(t)\frac{x}{2}$ that is ill-posed at $t = 0$. Denote by $\bar{G}$ the right-invariant random vector field on $\mathcal{D}^s(\mathcal{T}^n)$ obtained by translating $B'(t)\frac{w(t)}{t}$ by right shifts to all points of $\mathcal{D}^s(\mathcal{T}^n)$. Thus system (15), (16) is transformed into

$$D_*D_*\xi(t) = \bar{G}, \tag{20}$$

$$D_*\xi(t) = 2D_S\xi(t). \tag{21}$$

As well as above, translate both equations (20) and (21) by right shifts to $T_e\mathcal{D}^s(\mathcal{T}^n)$. Recall that in Remark 1 we have denoted the regression of $\bar{G}$, translated to $T_e\mathcal{D}^s(\mathcal{T}^n)$, by $\mathcal{B}$. In [7] by using formula (8) it is shown that the left-hand side of (20) is transformed into $\frac{1}{2}\mathfrak{B}(t)v + (v\cdot\nabla)v + \frac{\partial}{\partial t}v$.

We can construct the time-depended Riemannian metric on $\mathcal{T}^n$ from $B(t)\cdot B^*(t)$ as it is described in Section 1and apply formula (9) to get the form, into which equation (21) is transformed. Thus we obtain the system

$$\frac{1}{2}\mathfrak{B}(t)v + (v\cdot\nabla)v + \frac{\partial}{\partial t}v = \mathcal{B}, \tag{22}$$

$$\frac{\partial\rho^\xi(t,x)}{\partial t} = -\frac{1}{2}\mathrm{Div}_t(\rho^\xi(t,x)v^\xi(t,x)). \tag{23}$$

Thus this system describes a non-Newtonian fluid with external force $\mathcal{B}$ that in addition satisfies an analog of continuity equation. Note that the right-hand side of equation (22) is ill-posed at $t = 0$. So, the solution satisfies system (22), (23) only inside the open time interval.

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МОДЕЛИ ВЯЗКИХ ЖИДКОСТЕЙ, ПОРОЖДЕННЫЕ МАРТИНГАЛАМИ НА ГРУППАХ ДИФФЕОМОРФИЗМОВ

Ю. Е. Гликлих, Д. С. Сергеева

Мы изучаем два мартингала на группе соболевских диффеоморфизмов плоского n-мерного тора, оба мартингала заданы системами двух специальных уравнений с производными в среднем. Первый мартингал описывает решение уравнения Бюргерса на торе, которое также удовлетворяет аналогу уравнения неразрывности. Второй мартингал описывает некоторую неньютоновскую жидкость, удовлетворяющую некоторым специальным аналогам уравнения Бюргерса и уравнения неразрывности.

Ключевые слова: производные в среднем; плоский тор; группы диффеоморфизмов; вязкая гидродинамика.

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