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ANALYSIS OF THE SYSTEM OF WENTZELL EQUATIONS IN THE CIRCLE AND ON ITS BOUNDARY

***N. S. Goncharov*[1],** Goncharov.NS.krm@yandex.ru
***G. A. Sviridyuk*[1],** sviridiukga@susu.ru
[1] South Ural State University, Chelyabinsk, Russian Federation

In this paper the system of Wentzell equations, which is represented by two differential equations, namely, the Barenblatt – Zheltov – Kochina equation describing the heat conduction process at two temperatures inside a circle with the dynamic boundary condition ofWentzel, represented as a heat conduction equation with the Laplace – Beltrami operator, set on the boundary of the circle. Meanwhile, in the classical theory of boundary value problems, the boundary condition is understood as an equation on the boundary in which the order of derivatives on spatial variables is at least one less than the order of derivatives in the equation given in the domain. Therefore the study of Wentzell's system of equations opens the door to a new direction in research, where the equations can have derivatives of any order on both spatial and temporal variables.

Keywords: Barenblatt – Zheltov – Kochina equation; Wentzell boundary condition; Wentzell system.

Introduction

Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$ be a connected bounded region with boundary Γ, provided with the structure of a Riemannian manifold of class C^∞. On the compact $\Omega \cup \Gamma$ we consider a system of equations:

$$(\lambda - \Delta)v_t(t,x) = \alpha\Delta v(t,x) + \beta v(t,x),\ (t,x) \in \mathbb{R}_+ \times \Omega, \tag{1}$$

$$w_t(t,x) = \gamma\Delta w(t,x) + \partial_\nu v(t,x) + \delta w(t,x),\ (t,x) \in \mathbb{R}_+ \times \Gamma, \tag{2}$$

$$tr\ v(t,x) = w(t,x). \tag{3}$$

$$v(0,x) = v_0(x), x \in \Omega, \tag{4}$$

$$w(0,x) = w_0(x), x \in \Gamma. \tag{5}$$

The system (1) – (3) describes the process of thermal conductivity in the region $\mathbb{R} \times \Omega$ at two temperatures [1], and at its boundary $\mathbb{R}_+ \times \Gamma$ is the classical process of thermal conductivity with one temperature. The symbol Δ in (1) denotes the Laplace operator, and in the equation (2) Δ is the Laplace – Beltrami operator, and equation (2) is considered exclusively in the space of 0-forms. The symbol ∂_ν denotes the derivative with respect to the external (relative to the region Ω) normal to the boundary Γ. The parameters $\alpha, \beta, \delta \in \mathbb{R}$, $\gamma \in \mathbb{R}_+$ characterize the properties of the medium both in the region Ω, and at its boundary Γ.

By now, there is a tradition (see, for example, [2-4]) to call the stationary equation (2) the «boundary Wentzell condition». Meanwhile, in classical boundary value theory (see for example [5]) the boundary condition is an equation on a boundary in which the order of derivatives on the spatial variables is at least one less than the order of derivatives in the equation given in the domain. Therefore, the study of the system of equations (1) – (3) opens the door to a new direction in the study of systems of Wentzell equations, where equations can have derivatives of any order both in spatial and temporal variables.

Our approach to the study of the Wentzell system (1) – (5) is traditional. Using the classical theory of elliptic operators, we reduce it to two Sobolev type equations in the domain and on its boundary, after which we apply a coupling condition that allows us to obtain unambiguous solvability of the system. The article, in addition to the introduction and the list of references, contains one section in which the theorem on the solvability of the Wentzell system in a circle and on its boundary is described.

1. Solvability of the Wentzel system in a circle

For instance, we move on to the solvability of the system of Wentzell equations in a circle $K_R = \{(x, y) : x^2 + y^2 \leq R^2\}$ with radius R. To do this, perform the following replacement

$$\partial_\nu v = \varphi(x, y),\ (x, y) \in \partial K_R, \tag{6}$$

and consider the solution of the Cauchy problem for Barenblatt – Zheltov –Kochina (7) – (8) with nonzero Neumann condition (6)

$$(\lambda - \Delta)v_t(t, x, y) = \alpha\Delta v(t, x, y) + \beta v(t, x, y),\ (t, x, y) \in \mathbb{R}_+ \times K_R, \tag{7}$$

$$v(0, x, y) = v_0(x, y), (x, y) \in K_R. \tag{8}$$

We look for a solution in the following form

$$v(t, x, y) = u(t, x, y) + q(t, x, y),$$

where $q(t, x, y)$ is a harmonic function that satisfies the following condition

$$\partial_\nu q = \varphi(x, y).$$

Since here, in a polar coordinate system,

$$q(t, \rho, \psi) = C + \sum_{n=1}^{\infty} \frac{\rho^n}{nR^{n-1}}\Big(a_n(t)\cos n\psi + b_n(t)\sin n\psi\Big),$$

$$a_n(t) = \frac{1}{\pi}\int_0^{2\pi} \varphi(t, \psi)\cos n\psi d\psi, \qquad b_n(t) = \frac{1}{\pi}\int_0^{2\pi} \varphi(t, \psi)\sin n\psi d\psi.$$

we substitute this replacement into the original problem and get

$$\begin{aligned}(\lambda - \Delta)u_t(t, \rho, \psi) = \alpha\Delta u(t, \rho, \psi) + \beta u(t, \rho, \psi) - \lambda\sum_{n=1}^{\infty} \frac{\rho^n}{nR^{n-1}}\Big(a_n'(t)\cos n\psi + \\ + b_n'(t)\sin n\psi\Big) + \beta\sum_{n=1}^{\infty} \frac{\rho^n}{nR^{n-1}}\Big(a_n(t)\cos n\psi + b_n(t)\sin n\psi\Big),\end{aligned} \tag{9}$$

$$\partial_\nu u = 0, \tag{10}$$

$$u(0,\rho,\psi) = v_0(\rho,\psi) - q_0(\rho,\psi). \tag{11}$$

Let us consider the differential operator A

$$Au(\rho,\psi) = \Delta u(\rho,\psi), (\rho,\psi) \in K_R. \tag{12}$$

with the Neumann boundary condition

$$\partial_\nu u = 0, (\rho,\psi) \in \partial K_R. \tag{13}$$

Using formulas (12) – (13) we define the linear operator $A : \mathfrak{U} \to \mathfrak{F}$ in the corresponding Banach spaces $\mathfrak{U}$ and $\mathfrak{F}$. Here, $\mathfrak{F}$ is the Sobolev space $W_2^l(K_R)$ for some $l \in \{0\} \cup \mathbb{N}$, and the space $\mathfrak{U} = \{u \in W_2^{l+2}(K_R) : \partial_\nu u = 0, (\rho,\psi) \in \partial K_R\}$. Let us assume $\sigma(A) = \{\lambda_k\}$, where the eigenvalues of $\{\lambda_k\}$ are numbered on a non-increase basis taking into account their multiplicity. By $\{\varphi_k\} \subset C^\infty(K_R)$ we denote the family of proper functions of the operator A, orthonormal in the sense of the space $\mathfrak{F}$. Let us put $L = \lambda - A$ and $M = \alpha A + \beta$. Then

$$(\mu L - M)^{-1} = \sum_{k=1}^{\infty} \frac{\langle \cdot, \varphi_k \rangle \varphi_k}{\mu(\lambda - \lambda_k) - \alpha\lambda_k - \beta}\,. \tag{14}$$

The sequence in (14) converges absolutely and uniformly on any compact in $\mathbb{C}$, that does not contain points of the L-spectrum $\sigma^L(M)$ of the operator M

$$\mu_k = \frac{\alpha\lambda_k - \beta}{\lambda_k - \lambda}\,,\ k \in \mathbb{N}\,. \tag{15}$$

Thus the following theorem applies.

Theorem 1. *Let $\lambda \in \sigma(A)$. Then*

(i) *for* $f = -\lambda \sum\limits_{n=1}^{\infty} \frac{\rho^n}{nR^{n-1}} \Big(a_n'(t) \cos n\psi + b_n'(t) \sin n\psi \Big) +$
$+ \beta \sum\limits_{n=1}^{\infty} \frac{\rho^n}{nR^{n-1}} \Big(a_n(t) \cos n\psi + b_n(t) \sin n\psi \Big) \in C^1((0,\tau);\mathfrak{F}^0) \cap C^0([0,\tau];\mathfrak{F}^1)$ *и* $u_0 \in \mathfrak{U}$ *such, that*

$$\sum_{\lambda_k=\lambda} \langle u_0, \varphi_k \rangle \varphi_k = \sum_{\lambda_k=\lambda} \frac{\langle f(0), \varphi_k \rangle \varphi_k}{\alpha\lambda + \beta},$$

there is a unique solution $u \in C^1((0,\tau);\mathfrak{U}) \cap C^0([0,\tau];\mathfrak{U})$ of Cauchy problem (9) – (11);

(ii) *the solution $u = u(t,\rho,\psi)$ of the problem has the form*

$$u(t) = \sum_{k=1}^{\infty}{}' e^{\frac{\alpha\lambda_k - \beta}{\lambda_k - \lambda} t} \langle u_0, \varphi_k \rangle \varphi_k + \sum_{\lambda_k=\lambda} \frac{\langle f(t), \varphi_k \rangle \varphi_k}{\alpha\lambda + \beta} + \sum_{k=1}^{\infty}{}' \left(e^{\frac{\alpha\lambda_k - \beta}{\lambda_k - \lambda} t} - 1 \right) \frac{\langle f(s), \varphi_k \rangle}{\alpha\lambda_k + \beta} \varphi_k,$$

where the stroke at the sign of the sum means the absence of terms with numbers k such that $\lambda = \lambda_k$.

Let us introduce a condition

$$\left.\begin{array}{c}\text{The coefficients } \alpha \in \mathbb{R} \text{ и } \beta \in \mathbb{R}_+ \text{ are such that no} \\ \text{eigenvalue } \lambda_k \in \sigma(A) \text{ is not the root of the equation } \alpha\xi - \beta = 0.\end{array}\right\} \tag{$*$}$$

Theorem 2. *Let $\lambda \notin \sigma(A)$ and the condition (*) is satisfied.*

Then for $f = -\lambda \sum\limits_{n=1}^{\infty} \frac{\rho^n}{nR^{n-1}} \left(a_n'(t) \cos n\psi + b_n'(t) \sin n\psi \right) + \beta \sum\limits_{n=1}^{\infty} \frac{\rho^n}{nR^{n-1}} \left(a_n(t) \cos n\psi + b_n(t) \sin n\psi \right) \in C([0,\tau];\mathfrak{F})$ *u* $u_0 \in \mathfrak{U}$ *there is a unique solution* $u \in C^1((0,\tau);\mathfrak{U}) \cap C([0,\tau];\mathfrak{U})$, $u = u(t)$ *of the Cauchy problem (9) – (11), which also has following form*

$$u(t,\rho,\psi) = \sum_{k=1}^{\infty} e^{\frac{\alpha\lambda_k - \beta}{\lambda_k - \lambda} t} \langle u_0, \varphi_k \rangle \varphi_k + {\sum_{k=1}^{\infty}}' \left(e^{\frac{\alpha\lambda_k - \beta}{\lambda_k - \lambda} t} - 1 \right) \frac{\langle f(s), \varphi_k \rangle}{\alpha\lambda_k + \beta} \varphi_k.$$

At the second step, we consider the solution of problem (2), having previously performed the decomposition of the introduced function $\varphi(t,x)$ on the boundary of the domain Γ, using the Fourier method, according to the eigenfunctions $\{\psi_l : l \in \mathbb{N}\}$ of the Laplace – Beltrami operator and the corresponding eigenvalues $\{\varkappa_l : l \in \mathbb{N}\}$, orthonormal and numbered in a non-increasing order, taking into account the multiplicity, respectively,

$$\varphi(t,x,y) = \sum_{l=0}^{\infty} \alpha_l(t) \psi_l(x,y),$$

$$\omega(t,x,y) = \sum_{l=0}^{\infty} \omega_l(t) \psi_l(x,y).$$

Substituting the expressions obtained above into equation (2), we obtain the following system

$$-\sum_{l=0}^{\infty} \omega_l'(t) \psi_l(x,y) - \gamma \sum_{l=0}^{\infty} \varkappa_l \omega_l(t) \psi_l(x,y) + \delta \sum_{l=0}^{\infty} \omega_l(t) \psi_l(x,y) = -\sum_{l=0}^{\infty} \alpha_l(t) \psi_l(x,y).$$

Equating the coefficients with the corresponding eigenfunctions, we obtain the Cauchy problem for first-order differential equations with respect to the previously introduced parameters $\omega_l(t)$ and $\alpha_l(t)$:

$$\begin{cases} -\omega_l'(t) - \gamma \varkappa_l \omega_l(t) + \delta \omega_l(t) = -\alpha_l, \\ \omega_l(0) = \omega_{0,l}(0). \end{cases}$$

Solving a linear inhomogeneous differential equation, we obtain the following function

$$w_l(t) = \left(w_{0,l}(0) + \int_0^t e^{-(\delta - \gamma\varkappa_l)\xi} \alpha_l(\xi) d\xi \right) e^{(\delta - \varkappa_l \gamma)t}.$$

Thus, the solution of the Cauchy problem for equation (2) has the form

$$w_l(t,x,y) = \sum_{l=0}^{\infty} \left\{ \left(w_{0,l}(0) + \int_0^t e^{-(\delta - \gamma\varkappa_l)\xi} \alpha_l(\xi) d\xi \right) e^{(\delta - \varkappa_l \gamma)t} \right\} \psi_l(x,y).$$

Let us proceed to the connection condition using the solutions found above, depending on whether λ lies in the spectrum of the Laplace operator $\sigma(\Delta)$

$$tr\ v(x,y) = w(x,y).$$

For simplicity of calculations, let us suppose that $\lambda \notin \sigma(\Delta)$, we have,

$$\sum_k v_{0,k} e^{\frac{t(\alpha\lambda_k-\beta)}{\lambda-\lambda_k}} \langle u_0, \varphi_k\rangle \varphi_k + \sum_k{}' \Big(e^{\frac{\alpha\lambda_k-\beta}{\lambda_k-\lambda}t} - 1\Big) \frac{\langle f(s), \varphi_k\rangle}{\alpha\lambda_k+\beta}\varphi_k = \sum_l \Bigg\{\Bigg(w_{0,l} +$$

$$+ \int_0^t e^{-(\delta-\gamma\varkappa_l)\xi}\alpha_l(\xi)d\xi\Bigg) e^{(\delta-\varkappa_l\gamma)t}\Bigg\}\psi_l(x,y),$$

or

$$\sum_k v_{0,k} e^{\frac{t(\alpha\lambda_k-\beta)}{\lambda-\lambda_k}} \langle u_0, \varphi_k\rangle \sum_l \varphi_{0,l}\psi_l(x,y) + \sum_k{}' \Big(e^{\frac{\alpha\lambda_k-\beta}{\lambda_k-\lambda}t} - 1\Big) \frac{\langle f(s), \varphi_k\rangle}{\alpha\lambda_k+\beta} \sum_l \varphi_{0,l}\psi_l(x,y) =$$

$$= \sum_l \Bigg\{\Bigg(w_{0,l} + \int_0^t e^{-(\delta-\gamma\varkappa_l)\xi}\alpha_l(\xi)d\xi\Bigg) e^{(\delta-\varkappa_l\gamma)t}\Bigg\}\psi_l(x,y),$$

where $\varphi_{0,l} = (\varphi_k(x), \psi_l(x))$, which is equivalent to the equation with respect to $\alpha_l(t)$

$$\sum_k v_{0,k} e^{\frac{t(\alpha\lambda_k-\beta)}{\lambda-\lambda_k}} \langle u_0, \varphi_k\rangle \varphi_{0,l} + \sum_k{}' \Big(e^{\frac{\alpha\lambda_k-\beta}{\lambda_k-\lambda}t} - 1\Big) \frac{\langle f(s), \varphi_k\rangle}{\alpha\lambda_k+\beta}\varphi_{0,l} =$$

$$= \Bigg(w_{0,l} + \int_0^t e^{-(\delta-\gamma\varkappa_l)\xi}\alpha_l(\xi)d\xi\Bigg) e^{(\delta-\varkappa_l\gamma)t}.$$

Applying the Laplace transform to both parts of the equation, we obtain

$$\sum_k v_{0,k} \mathcal{L}\Big\{e^{\frac{t(\alpha\lambda_k-\beta)}{\lambda-\lambda_k}}\Big\} \langle u_0, \varphi_k\rangle \varphi_{0,l} + \sum_k{}' \Big(\mathcal{L}\Big\{e^{\frac{\alpha\lambda_k-\beta}{\lambda_k-\lambda}t} - 1\Big\}\Big) \frac{\langle f(s), \varphi_k\rangle}{\alpha\lambda_k+\beta}\varphi_{0,l} =$$

$$= \mathcal{L}\Big\{w_{0,l}e^{(\delta-\varkappa_l\gamma)t}\Big\} + \mathcal{L}\Big\{\int_0^t e^{(\delta-\gamma\varkappa_l)(t-\xi)}\alpha_l(\xi)d\xi\Big\},$$

which is equivalent to

$$\sum_k v_{0,k} \frac{\lambda-\lambda_k}{(\lambda-\lambda_k)p-\alpha\lambda_k+\beta}\langle u_0, \varphi_k\rangle\varphi_{0,l} + \sum_k{}' \Big(\frac{\lambda-\lambda_k}{(\lambda-\lambda_k)p-\alpha\lambda_k+\beta} - \frac{1}{p}\Big)\frac{\langle f(s), \varphi_k\rangle}{\alpha\lambda_k+\beta}\varphi_{0,l} =$$

$$= \frac{w_{0,l}}{p-\delta+\varkappa_l\gamma} + \frac{1}{p-(\delta-\gamma\varkappa_l)}\mathcal{L}\Big\{\alpha_l(t)\Big\}$$

and

$$\mathcal{L}\Big\{\alpha_l(t)\Big\} = (p-\delta+\varkappa_l\gamma)\Big\{-\frac{w_{0,l}}{p-\delta+\varkappa_l\gamma} + \sum_k v_{0,k}\frac{\lambda-\lambda_k}{(\lambda-\lambda_k)p-\alpha\lambda_k+\beta}\langle u_0, \varphi_k\rangle\varphi_{0,l} +$$

$$+\sum_k{}'\left(\frac{\lambda-\lambda_k}{(\lambda-\lambda_k)p-\alpha\lambda_k+\beta}-\frac{1}{p}\right)\frac{\langle f(s),\varphi_k\rangle}{\alpha\lambda_k+\beta}\varphi_{0,l}\Bigg\}.$$

Applying the inverse Laplace transform, we obtain the desired coefficients $\alpha_l(t)$

$$\alpha_l(t)=-w_{0,l}\delta(t)+\sum_k v_{0,k}\mathcal{L}^{-1}\left\{\frac{\lambda-\lambda_k}{(\lambda-\lambda_k)p-\alpha\lambda_k+\beta}\right\}\langle u_0,\varphi_k\rangle\varphi_{0,l}+$$

$$+\sum_k{}'\mathcal{L}^{-1}\left\{\frac{\lambda-\lambda_k}{(\lambda-\lambda_k)p-\alpha\lambda_k+\beta}-\frac{1}{p}\right\}\frac{\langle f(s),\varphi_k\rangle}{\alpha\lambda_k+\beta}\varphi_{0,l}.$$

Thus the following theorem applies.

Theorem 3. *Let $\lambda\notin\sigma(A)$. Then for $f=-\lambda\sum\limits_{n=1}^{\infty}\frac{\rho^n}{nR^{n-1}}\Bigg(a_n'(t)\cos n\psi+b_n'(t)\sin n\psi\Bigg)+$ $+\beta\sum\limits_{n=1}^{\infty}\frac{\rho^n}{nR^{n-1}}\Bigg(a_n(t)\cos n\psi+b_n(t)\sin n\psi\Bigg)\in C^1((0,\tau);\mathfrak{F}^0)\cap C^0([0,\tau];\mathfrak{F}^1)$ and $v_0,\omega_0\in\mathfrak{U}$ such that*

$$\alpha_l(t)=-w_{0,l}\delta(t)+\sum_k v_{0,k}\mathcal{L}^{-1}\left\{\frac{\lambda-\lambda_k}{(\lambda-\lambda_k)p-\alpha\lambda_k+\beta}\right\}\langle u_0,\varphi_k\rangle\varphi_{0,l}+$$

$$+\sum_k{}'\mathcal{L}^{-1}\left\{\frac{\lambda-\lambda_k}{(\lambda-\lambda_k)p-\alpha\lambda_k+\beta}-\frac{1}{p}\right\}\frac{\langle f(s),\varphi_k\rangle}{\alpha\lambda_k+\beta}\varphi_{0,l},$$

there is a unique solution $u\in C^1((0,\tau);\mathfrak{U})\cap C([0,\tau];\mathfrak{U})$, $u=u(t,\rho,\psi)$ of the Cauchy problem in the Wentzell system, which also has the following form

$$u(t,\rho,\psi)=\begin{cases}\sum\limits_{k=1}^{\infty}e^{\frac{\alpha\lambda_k-\beta}{\lambda_k-\lambda}t}\langle u_0,\varphi_k\rangle\varphi_k+\sum\limits_{k=1}^{\infty}{}'\left(e^{\frac{\alpha\lambda_k-\beta}{\lambda_k-\lambda}t}-1\right)\frac{\langle f(s),\varphi_k\rangle}{\alpha\lambda_k+\beta}\varphi_k,(\rho,\psi)\in K_R;\\ \sum\limits_{l=0}^{\infty}\left\{\left(w_{0,l}(0)+\int\limits_0^t e^{-(\delta-\gamma\varkappa_l)\xi}\alpha_l(\xi)d\xi\right)e^{(\delta-\varkappa_l\gamma)t}\right\}\psi_l(x,y),(\rho,\psi)\in\partial K_R.\end{cases}\qquad(16)$$

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Nikita S. Goncharov, postgraduate student, Department of Equations of Mathematical Physics, South Ural State University (Chelyabinsk, Russian Federation), Goncharov.NS.krm@yandex.ru

Georgy A. Sviridyuk, DSc (Math), Full Professor, Department of Equations of Mathematical Physics, South Ural State University (Chelyabinsk, Russian Federation), sviridiukga@susu.ru

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АНАЛИЗ СИСТЕМЫ УРАВНЕНИЙ ВЕНТЦЕЛЯ В КРУГЕ И НА ЕГО ГРАНИЦЕ

Н. С. Гончаров, Г. А. Свиридюк

В работе рассмотрена система уравнений Венцеля, которая представлена двумя дифференциальными уравнениями, а именно, уравнением Баренблатта – Желтова – Кочиной, описывающим процесс теплопроводности при двух температурах внутри круга с динамическим краевым условием Венцеля, представленным в виде уравнения теплопроводности с оператором Лапласа – Бельтрами, заданным на границе круга. Между тем, в классической теории краевых задач под краевым условием понимается уравнение на границе, в котором порядок производных по пространственным переменным на единицу меньшего порядка производных в уравнении, заданной в области. Поэтому изучение системы уравнений Венцеля открывает дверь новому направлению в исследовании, где уравнения могут иметь производные любого порядка как по пространственным переменным, так и временному переменному.

Keywords: уравнение Баренблатта – Желтова – Кочиной; краевое условие Вентцеля; система Вентцеля.

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Гончаров Никита Сергеевич, аспирант, кафедра уравнений математической физики, Южно-Уральский государственный университет (г. Челябинск, Российская Федерация), Goncharov.NS.krm@yandex.ru

Свиридюк Георгий Анатольевич, доктор физико-математических наук, профессор кафедры уравнений математической физики, Южно-Уральский государственный университет (г. Челябинск, Российская Федерация), sviridiukga@susu.ru

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