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STUDY OF SOLVABILITY OF BOUNDARY VALUE PROBLEMS FOR ONE SINGULAR DIFFERENTIAL EQUATION

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In this paper, the abstract theory of functional-differential equations is applied to some singular second-order differential equation, which is a generalization of equations encountered in the theory of chemical reactions. The result is based on the properties of the Green's operator of the corresponding linear problem.

Keywords: functional differential equations; quasilinear boundary value problems; linear equation; singular equation; unique solvability; well-defined solvability; finite-dimensional parametrizability; Green's operator.

Introduction

Quasilinear boundary value problems for ordinary differential equations (ODEs) are the subject of intensive study since the time of S.N. Bernshtein and M. Nagumo. Interest in such objects is also explained by the fact that quasilinear boundary value problems arise in mathematical models of many real processes (in biology, chemistry, ecology, economics, etc.). Classical methods for studying quasilinear boundary value problems for ODEs are basically reduced to the following schemes: 1) obtaining a priori estimates of solutions with subsequent application of fixed point theorems to an auxiliary integral equation (Leray – Schauder scheme); 2) construction of successive approximations of the solution and proof of convergence. In addition, methods were developed for studying the solvability of quasilinear boundary value problems based on the use of specific properties of the problem under consideration (monotonicity in the sense of semi-ordering or according to Minty – Browder, etc.). Questions of solvability of quasilinear boundary value problems for ODEs were studied by many authors. We note the works of N.V. Azbelev, N.I. Vasil'ev, I.T. Kiguradze, B.L. Shekhter and others [1, 2, 3, 4].

Further development of the theory of quasilinear boundary value problems led to the need to consider problems for functional differential equations (FDEs). The fundamental works of N.V. Azbelev, V.P. Maksimov, and other authors [5, 6] are devoted to the development of the foundations of the theory of quasilinear boundary value problems for FDEs.

Following [5], by a functional differential equation $\dot{x} = Fx$ we mean an equation with an operator F defined on some set of absolutely continuous functions D. The FDE theory is based on the isomorphism of the space D to the direct product $L \times R^n$ and the decomposition of the linear operator $\mathcal{L} : D \to L$ into infinite-dimensional and finite-dimensional summands that follows from this isomorphism. Replacing the Lebesgue space L with the Banach space B allows to extend the FDE theory to other classes of equations, for example, to singular differential equations. Thus, we come to the theory of «abstract

FDE≫. We note the works on this issue written by N.V. Azbelev, L.F. Rakhmatullina, and others [1, 7, 8].

In the proposed work, the object of study is a quasilinear boundary value problem written as a system of two equations

$$\begin{aligned} \mathcal{L}x &= Fx, \\ lx &= \varphi x, \end{aligned} \tag{1}$$

where $\mathcal{L} : D \to B$ is a linear bounded operator, $F : D \to B$ is a continuous operator, $l : D \to R^n$ is a linear bounded vector functional, $\varphi : D \to R^n$ is a continuous vector functional. The Banach space D is isomorphic to the direct product of the Banach space B and R^n. Note that if the boundary value problem is written as system (1), then the second equation is called the boundary conditions of the problem. Note that [5] in form (1) we can write many relevant classes of quasilinear boundary value problems for ordinary differential equations, integro-differential equations, equations with a deviating argument, an equation with aftereffect, and other equations.

Classical schemes for studying the solvability of problem (1) use the Banach fixed point principle or the Schauder scheme. In the first case, both the operator F and the vector-functional φ must be Lipschitz. In the second case, complete continuity of the operator F is required. The paper proposes a scheme for studying the solvability of boundary value problem (1), which makes it possible to obtain less stringent solvability conditions than with the direct application of classical schemes.

The idea of one of the proposed schemes is based on the assumption of ≪finite-dimensional parametrizability≫ of the set of solutions to the equation

$$\mathcal{L}x = Fx. \tag{2}$$

The equation is finite-dimensionally parametrizable if there exists a one-to-one and mutually continuous correspondence between the set of solutions to the equation and some closed subset of a finite-dimensional space. Here n-dimensional parametrizability is a special case of ≪reducibility≫ of equation [7]. A linear equation is reducible if and only if the set of its solutions is finite-dimensionally parametrizable. The concept of reducibility was first introduced by N.V. Azbelev [1, 5].

Assuming n-dimensional parametrizability, the solution of equation (2) has the representation

$$x = M\alpha + X\alpha, \tag{3}$$

where the operator $M : R^n \to D$ is continuous, X is the fundamental vector of the equation $\mathcal{L}x = 0$, $\alpha \in R^n$. Substituting representation (3) into the boundary conditions of problem (1), we obtain the equation

$$l(M\alpha + X\alpha) = \varphi(M\alpha + X\alpha)$$

relative to α. And, if there exists α that satisfies the resulting equation, then problem (1) has a solution. In this paper, we use the scheme for studying problem (1) for correct solvability given in the papers [8, 9, 10]. This scheme is implemented for the singular differential equation

$$\psi(t)\ddot{x}(t) + \beta\dot{x}(t) = f(t, x). \tag{4}$$

Boundary value problems for equation (4) arise in mathematical models of some real processes. For example, we note processes taking place in chemical reactors in the presence of catalysts [2] or when describing the shape of the free surface of an axisymmetric liquid layer, taking into account body forces and surface tension. In this work, we obtain sufficient conditions for the existence and correct solvability of some boundary value problems for equation (4).

Known approaches to the study of singular equations can be conditionally classified into two directions: the classical one [11], which is based on the use of the method of a priori estimates and inequalities, and the approach based on the ideas of the theory of abstract functional differential equations [5], where a special space of solutions is constructed, and the fact of isomorphism of the given space and the direct product of some Banach space and R^n is used. Among works that use the construction of special spaces for singular equations, we note such papers as [6, 8, 9, 12, 13].

1. General Scheme for Studying Solvability of Boundary Value Problems for Equations Admitting Finite-Dimensional Parametrization

One of the features of uniquely solvable linear systems of ordinary differential equations $\mathcal{L}x = f$ is as follows: for each fixed right hand side, the set of all solutions admits an explicit finite-dimensional representation. For nonlinear equation (2), a similar representation is possible under special restrictions on the operator F. The main idea of the method proposed in this section is based on the use of such a representation.

Definition 1. *We say that equation (2) admits a finite-dimensional parametrization if:*

1) there exists a linear bounded vector-functional $l_1 : D \to R^n$ such that the linear boundary value problem

$$\begin{aligned} \mathcal{L}x &= f, \\ l_1 x &= \alpha \end{aligned} \tag{5}$$

is uniquely solvable for any $\alpha \in R^n$, $f \in B$, and

2) the quasilinear boundary value problem

$$\begin{aligned} \mathcal{L}x &= Fx, \\ l_1 x &= \alpha \end{aligned} \tag{6}$$

is correctly solvable, that is, uniquely solvable for any $\alpha \in R^n$, and its solution in the norm of the space D depends continuously on α.

Definition 2. *We say that the vector-functional φ (the operator F) is quasi-bounded if*

$$b_\varphi = \overline{\lim}_{\|u\|_D \to \infty} \frac{|\varphi u|}{\|u\|_D} < +\infty \quad \left(b_F = \overline{\lim}_{\|u\|_D \to \infty} \frac{\|Fu\|_B}{\|u\|_D} < +\infty \right).$$

The quantity b_φ (b_F) is said to be the quasi-norm of the vector-functional φ (the operator F).

Theorem 1. *Suppose that the following conditions hold:*

1) the operator M is quasi-bounded with the quasi-norm b_M;

2) $\det(lX) \neq 0$*;*
3) the vector functional φ *is quasi-bounded with the quasi-norm* b_φ*;*
4) $\|(lX)^{-1}\| \cdot (\|l\|_{D\to R^n} \cdot b_M + b_\varphi \cdot (b_M + \|X\|_{R^n\to D})) < 1.$
Then problem (1) has at least one solution.

Note that the main assumption in the above scheme is the correct solvability of problem (6). Correct solvability can be established by the Banach fixed point principle, in which the Lipschitz property of the operator F is used. In what follows, we assume that the constant b_F is the Lipschitz constant of the operator F.

Theorem 2. *Let the operator* F *be quasi-bounded with the quasi-norm* b_F*. If* $b_F < \|G\|_B^{-1} \to D$*, then*

$$\overline{\lim_{|\alpha|\to\infty}} \frac{\|M\alpha\|_D}{|\alpha|} \leq \frac{b_F \cdot \|G\|_{B\to D} \cdot \|X\|_{R^n\to D}}{1 - b_F \cdot \|G\|_{B\to D}},$$

where $G : B \to D$ *is the Green's operator of problem (5).*

Using the estimate of M obtained in Theorem 2, we formulate the following theorem.

Theorem 3. *Suppose that the following conditions hold:*
1) the conditions of Theorem 2 are fulfilled;
2) $\det(lX) \neq 0$*;*
3) the vector functional φ *is quasi-bounded with the quasi-norm* b_φ*;*
4) $\|(lX)^{-1}\| \cdot \|X\|_{R^n\to D} \cdot (b_F\|l\|_{D\to R^n} \cdot \|G\|_{B\to D} + b_\varphi) < 1 - b_F \cdot \|G\|_{B\to D}).$
Then problem (1) has at least one solution.

The obtained statements form the basis for the study of the solvability of boundary value problems for a singular differential equation of the second order.

2. Criteria for Well-Posed Solvability of Boundary Value Problems for Singular Differential Equation of Second Order

Consider the problem

$$\begin{aligned} \psi(t)\ddot{x} + \beta\dot{x} &= f(t,x), \\ x(a) = \alpha_1, \quad \dot{x}(b) &= \alpha_2, \end{aligned} \tag{7}$$

where $\alpha_{1,2} \in R$.

Consider the operator $T : L_p \to L_p$ $(1 \leq p \leq \infty)$ defined by the equality

$$(Tz)(t) = \int_t^b \frac{z(s)}{\psi(s)} ds, \quad t \in [a,b]. \tag{8}$$

We need the following auxiliary statements.

Lemma 1. *For every fixed* p $(1 \leq p \leq \infty)$*, the estimate*

$$\|T\|_{L_p\to L_p} \leq \|w\|_{L_p}$$

is true, where

$$w(t) = \begin{cases} \left(\int\limits_t^b \frac{ds}{|\psi(s)|^q} \right)^{1/q}, \quad 1 < q \leq \infty, \quad \frac{1}{p} + \frac{1}{q} = 1, \quad t \in [a, b]. \\ \operatorname{vraisup}\limits_{s \in [t,b]} \frac{1}{|\psi(s)|}, \quad p = 1. \end{cases}$$

For some particular case of the function ψ, the estimate of the norm of the operator T has the form described in Lemma 2.

Lemma 2. *If there exists* $k > 0$ *such that* $\psi(t) \geq k(t-a)$ *for all* $t \in [a, b]$, *then* $\|T\|_{L_p \to L_p} \leq \frac{q}{k(q-1)}$, *(*$1 < q \leq \infty$*).*

Consider the problem

$$\psi(t)\ddot{x} + \beta\dot{x} = f(t), \tag{9}$$

$$x(a) = 0, \quad \dot{x}(b) = 0. \tag{10}$$

Definition 3. *We say that* $x \in W_{\psi,p}$ *if* $x : [a, b] \to R$ *is an absolutely continuous function with a derivative absolutely continuous on* $(a, b]$, $\ddot{x}$ *exists almost everywhere on* $[a, b]$ *and* $\psi\ddot{x} \in L_p$.

Note that, under certain restrictions on the function ψ, the space $W_{\psi,p}$ is isomorphic to the direct product $L_p \times R^2$, and hence is a Banach space. The norm on the space $W_{\psi,p}$ is defined by the equality

$$\|x\|_{W_{\psi,p}} = |x(a)| + |\dot{x}(b)| + \|\psi\ddot{x}\|_{L_p}.$$

Lemma 3. *If the inequality*

$$\|T\|_{L_p \to L_p} \leq \frac{1}{|\beta|}$$

holds, then problem (9), (10) has a unique solution $x \in W_{\psi,p}$ *for each* $f \in L_p$.

Let $G : L_p \to W_{\psi,p}$ be the Green's operator of problem (9), (10). Let us find out the conditions for the correct solvability of problem (7).

Theorem 4. *Suppose that problem (9), (10) is uniquely solvable and* $f(t, u)$ *satisfies the Lipschitz condition in the second argument with the constant* k_f. *If*

$$k_f (b - a)^{1/p} \|G\|_{L_p \to C} < 1,$$

then boundary value problem (7) is correctly solvable.

Let us study the solvability of the boundary value problem

$$\psi(t)\ddot{x} + \beta\dot{x} = f(t, x), \tag{11}$$

$$lx = \varphi x, \tag{12}$$

where $l : W_{\psi,p} \to R^2$ is a linear bounded vector functional, $\varphi : W_{\psi,p} \to R^2$ is a continuous vector functional.

The scheme proposed above makes it possible to obtain criteria for the solvability of problem (11), (12) for a wide class of functionals l without specifying its specific form.

Let us consider problem (11), (12) under the following assumptions.

1) The function ψ satisfies the condition

$$(-Gz, z)_{L_2} \geq \gamma \|G\|^2_{L_2}.$$

2) The Nemytskii operator $N : W_{\psi,p} \to L^2$ defined by the equality $Nu \equiv f(t, u)$ is continuous, $f(t, 0) \equiv 0$.

Let us study the solvability of problem (11), (12) in the Banach space $W_{\psi,p}$. Let $l, \varphi : W_{\psi,p} \to R^2$. In this case, the fundamental vector of the equation $\psi\ddot{x} + \beta\dot{x} = 0$ has the form:

$$x \equiv \left(1, \frac{1}{r'(b)} r(t)\right), r'(t) = e^{-\beta \int\limits_{\varepsilon_0}^{t} \frac{d\tau}{\psi(t)}}, \quad \forall \varepsilon_0 > a.$$

Theorem 5. *Suppose that the following conditions hold:*

1) the conditions of Theorem 4 are satisfied;

2) $\det(lX) \neq 0$*;*

3) the vector functional φ *is quasi-bounded with the quasi-norm* b_φ*;*

4) $\|(lX)^{-1}\| \cdot \left(1 + \frac{|\beta\ vert}{r'(b)} \|r'(\cdot)\|_{L_p}\right) \cdot (k_f\|l\| + b_\varphi) < 1 - k_f$.

Then problem (11), (12) has at least one solution.

Consider the equation

$$\ddot{x} + \frac{\dot{x}}{t} + \beta_0 \exp\left(-\frac{1}{x+\tau}\right) = 0, \quad 0 < t \leq 1, \quad \beta_0 \geq 0, \quad \tau \geq 0. \tag{13}$$

Such an equation arises in the theory of chemical reactions. Here x is the dimensionless temperature, $\beta_0 \exp\left(-\frac{1}{x+\tau}\right)$ is some «reaction rate» [2, 3]. This equation is a special case of equation (11) with

$$f(t, x) = \beta_0 \exp\left(-\frac{1}{x+\tau}\right).$$

We can apply Theorem 5 to this equation and obtain sufficient conditions for the solvability of boundary value problems for the given singular equation (13). In conclusion, we note that the sufficient conditions obtained in the paper for the solvability of boundary value problems for singular differential equation (11) turn out to be less stringent than the conditions obtained by directly applying the Schauder scheme.

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ИССЛЕДОВАНИЕ РАЗРЕШИМОСТИ КРАЕВЫХ ЗАДАЧ ДЛЯ ОДНОГО СИНГУЛЯРНОГО ДИФФЕРЕНЦИАЛЬНОГО УРАВНЕНИЯ

А. В. Кунгурцева, И. А. Колесников

В работе абстрактная теория функционально-дифференциальных уравнений применяется для некоторого сингулярного дифференциального уравнения второго порядка, которое является обобщением уравнений, встречающихся в теории химических реакций. Результат базируется на свойствах оператора Грина соответствующей линейной задачи.

Ключевые слова: функционально-дифференциальные уравнения; квазилинейные краевые задачи; линейное уравнение; сингулярное уравнение; однозначная разрешимость; корректная разрешимость; конечномерная параметризуемость; оператор Грина.

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