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NUMERICAL INVESTIGATION OF THE NON-UNIQUENESS OF SOLUTIONS OF THE SHOWALTER–SIDOROV PROBLEM FOR THE HOFF MATHEMATICAL MODEL ON A RECTANGLE

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The article is devoted to the question of the uniqueness or non-uniqueness of solutions of the Showalter–Sidorov–Dirichlet problem for the Hoff equation on a rectangle. To study this issue, the phase space method was used, which was developed by G.A. Sviridyuk. An algorithm is constructed to identify the conditions of multiplicity and uniqueness of solutions, which allows numerically solving the Showalter–Sidorov–Dirichlet problem based on the modified Galerkin method. The article considers cases where the dimension of the operator kernel with a time derivative is equal to 1 or 2. Computational experiments demonstrating the non-uniqueness of solutions to the Showalter–Sidorov problem depending on the values of the problem parameters are presented.

Keywords: Sobolev type equations; Showalter–Sidorov problem; the Hoff equation; non-uniqueness of solutions; phase space method; the Galerkin method.

Introduction

Let $\Omega \subset \mathbb{R}$ be a bounded domain with a smooth boundary of the class C^∞. In the cylinder $Q = \Omega \times \mathbb{R}_+$ consider the Showalter – Sidorov problem:

$$(\lambda + \Delta)\,(u(x,0) - u_0(x)) = 0,\ x \in \Omega, \tag{1}$$

or the Hoff equation

$$(\lambda + \Delta)u_t = \alpha u + \beta u^3,\ x \in \Omega, t \in (0,T), \tag{2}$$

with the Dirichlet condition

$$u(x,t) = 0,\ (x,t) \in \partial\Omega \times \mathbb{R}_+. \tag{3}$$

The initial boundary value problems for the equation (3) were studied earlier in the framework of the problems of Sobolev type equations in the following papers [1–6]. In the works [7, 8], the case is considered when $\dim\ker(\lambda + \Delta) = 1$ and there is a unique solution to the problem. The articles [9–15] describe cases of non-uniqueness of solutions of the Showalter–Sidorov problem for various Sobolev type equations in the case of $\dim\ker(\lambda + \Delta) = 1$. In the work of G.A. Sviridyuk and T.G. Sukacheva a system of Oskolkov equations was considered for the case when $\dim\ker(\lambda+\Delta) = 2$ and the conditions for the existence of several solutions were found [15].

The purpose of this paper is to study the problem (1) – (3) on a rectangle and to identify conditions imposed on the parameters α, β, under which there are several solutions to the problem in question in cases when $\dim\ker(\lambda+\Delta)=1$ and $\dim\ker(\lambda+\Delta)=2$. The work, in addition to the introduction and the list of references, contains three paragraphs. The first section presents an algorithm for a numerical method for finding the conditions of uniqueness and non-uniqueness of solutions to the Showalter–Sidorov problem on a rectangle. The second paragraph is a description of the software package for the implementation of the algorithm presented in paragraph 1. The results of computational experiments in the case when $\dim\ker(\lambda+\Delta)=1$ and $\dim\ker(\lambda+\Delta)=2$ are set out in the third paragraph.

1. Algorithm a Numerical Method for Finding the Conditions of Uniqueness and Non-uniqueness of the Solution to the Showalter–Sidorov Problem

Let $\Omega=(0,\pi)\times(0,\pi)$, consider the Showalter – Sidorov problem

$$\lambda\left(u(x,y,0)-u_0(x,y)\right)+\left(u_{xx}(x,y,0)+u_{yy}(x,y,0)-u_0(x,y)\right)=0, x\in(0,\pi), y\in(0,\pi) \quad (4)$$

or the Hoff equation

$$\lambda u_t+u_{xxt}+u_{yyt}=\alpha u+\beta u^3,\ t\in(0,T), \quad (5)$$

with the Dirichlet condition

$$\begin{aligned} u(0,y,t)=u(\pi,y,t)=0,\ y\in(0,\pi),\ t\in(0,T),\\ u(x,0,t)=u(x,\pi,t)=0,\ x\in(0,\pi),\ t\in(0,T). \end{aligned} \quad (6)$$

Let us describe an algorithm for the numerical method for finding the conditions of uniqueness and non-uniqueness of the solution to the Showalter–Sidorov problem. The algorithm allows you to find approximate solutions to the problem using the Galerkin method. Consider the homogeneous Dirichlet problem for the equation

$$-u_{xx}-u_{yy}=\lambda u, \quad (7)$$

$$\begin{aligned} u(0,y)=u(l_1,y)=0,\ y\in(0,l_1),\\ u(x,0)=u(x,l_2)=0,\ x\in(0,l_2), \end{aligned} \quad (8)$$

in a rectangle $(0,l_1)\times(0,l_2)$. Solution of the Sturm–Liouville problem (7), (8) for a given domain $\Omega=(0,\pi)\times(0,\pi)$ has the form:

$$\lambda_{k_1,k_2}=k_1^2+k_2^2, \quad (9)$$

$$\varphi_{k_1,k_2}(x,y)=\sin(k_1x)\sin(k_2y). \quad (10)$$

The eigenvalues of the problem (7), (8) are single or double. In the numerical study of the problem (4) – (6) we get two cases.

(a) If $\lambda = \lambda_{k_1,\ k_2}$ and $k_1 = k_2$, then $\dim\ker(\lambda + \Delta) = 1$, then the considered eigenvalue $\lambda_{k_1,\ k_2}$ corresponds to one eigenfunction $\varphi_{k_1,\ k_2}$. In this case, the function $u(x, y, 0)$ will be represented as $u(x, y, 0) = s_1\varphi_{k_1,\ k_2} + u^{\perp}$, $u^{\perp} \in \mathfrak{U}^{\perp} = \{u \in L_4(\Omega) : \langle u, \varphi_{k_1,\ k_2}\rangle = 0\}$.

(b) If $\lambda = \lambda_{k_1,\ k_2}$ and $k_1 \neq k_2$, then $\dim\ker(\lambda + \Delta) = 2$, then the considered eigenvalue $\lambda_{k_1,\ k_2}$ corresponds to two eigenfunction $\varphi_{k_1,\ k_2}$ and $\varphi_{k_2,\ k_1}$. In this case, the function $u(x, y, 0)$ will be represented as $u(x, y, 0) = s_1\varphi_{k_1,\ k_2} + s_2\varphi_{k_2,\ k_1} + u^{\perp}$, $u^{\perp} \in \mathfrak{U}^{\perp} = \{u \in L_4(\Omega) : \langle u, \varphi_{k_1,\ k_2}\rangle = 0, \langle u, \varphi_{k_2,\ k_1}\rangle = 0\}$.

Check the uniqueness or multiplicity of the solution of the Showalter–Sidorov problem for given initial function $u(x, y, 0)$: For case (a) we will consider the set $\mathfrak{B}$:

$$\begin{aligned}\mathfrak{B} = \{(s_1, u^{\perp}) \in \mathbb{R} \times L_4(\Omega) : s_1^3\|\varphi_{k_1,\ k_2}\|^4_{L_4(\Omega)} + 3s_1^2\int_0^{\pi}\int_0^{\pi}\varphi^3_{k_1,\ k_2}u^{\perp}\ dxdy+ \\ + s_1\ (3\int_0^{\pi}\int_0^{\pi}\varphi^2_{k_1,\ k_2}(u^{\perp})^2\ dxdy + \alpha\beta^{-1} + \int_0^{\pi}\int_0^{\pi}\varphi_{k_1,\ k_2}(u^{\perp})^3\ dxdy = 0\}.\end{aligned} \tag{11}$$

The equation defining the set $\mathfrak{B}$ are cubic equation of general form

$$as_1^3 + bs_1^2 + cs_1 + d = 0. \tag{12}$$

According to Cardano's formulas, any cubic equation of general form with the help of replacement $s_1 = y - \frac{b}{3a}$ can be reduced to canonical form $y^3 + py + q = 0$ with coefficients

$$\begin{aligned}&a = \|\varphi_{k_1,\ k_2}\|^4_{L_4(\Omega)},\ b = 3\int_0^{\pi}\int_0^{\pi}\varphi^3_{k_1,\ k_2}u^{\perp}\ dxdy, \\ &c = 3\int_0^{\pi}\int_0^{\pi}\varphi^2_{k_1,\ k_2}(u^{\perp})^2\ dxdy + \alpha\beta^{-1}, d = \int_0^{\pi}\int_0^{\pi}\varphi_{k_1,\ k_2}(u^{\perp})^3 dxdy, \\ &Q(s_1, u) = \left(\frac{3ac - b^2}{9a^2}\right)^3 + \frac{1}{4}\left(\frac{2b^3}{27a^3} - \frac{bc}{3a^2} + \frac{d}{a}\right)^2.\end{aligned} \tag{13}$$

(1) The uniqueness or multiplicity of the solution of the Showalter–Sidorov problem will depend on the following cases: For $Q < 0$ the equation defining the set (11), the problem has three solutions, therefore, the system of algebra-differential equations will have three solutions. In this case, all subsequent steps must be performed three times for each of the functions $u_{k_1,k_2}(t)$ with its own set of initial values.

(2) For $Q > 0$, the equation defining the set (11) has one solution, therefore, the system of algebra-differential equations will have one solution.

In case (b), the selection of conditions for uniqueness or multiplicity of solutions to the Showalter-Sidorov problem is individual for each specific eigenvalue $\lambda_{k_1,k_2}, k_1 \neq k_2$.

Following the Galerkin method, we search for approximate solution for an approximate solution of the problem under consideration as sums

$$u_m(x, y, t) = \sum_{k_1=1}^{m}\sum_{k_2=1}^{m} u_{k_1,\ k_2}(t)\varphi_{k_1,\ k_2}(x, y). \tag{14}$$

Substitute Galerkin sums in (4). Then multiply the resulting equation scalar in $L_2(\Omega)$ on eigenfunctions $\varphi_{k_1,\ k_2}(x,y), k_1,\ k_2 = \overline{1,m}$, and get a system of equations with respect to the unknowns $u_{k_1,\ k_2}(t)$.

At the same time, depending on the parameter λ, the equations in this system can be differential or algebraic. Consider these cases in more detailes:

(i) If $\lambda \neq \lambda_{k_1,\ k_2}$, then in this case all the equations of the system will be ordinary differential equations of the first order. To solve this system relatively $u_{k_1,\ k_2}(t), k_1,\ k_2 = \overline{1,m}$, from the conditions (6), multiplying them scalar in $L_2(\Omega)$ on eigenfunctions$\varphi_{k_1,\ k_2}(x,y), k_1,\ k_2 = \overline{1,m}$, we find m initial conditions. Next, the resulting system of nonlinear first-order differential equations with initial conditions is solved numerically, and unknown functional coefficients are found $u_{k_1,\ k_2}(t), k_1,\ k_2 = \overline{1,m}$, in an approximate solution $u_m(x,y,t)$.

(ii) If $\lambda = \lambda_{k_1,\ k_2}$, then then one or two equations is algebraic, and the rest ones are differential. Consider separately a system composed of first-order differential equations and an algebraic equations. To solve a system of first-order ordinary differential equations with respect to $u_{k_1,\ k_2}(t), k_1,\ k_2 = \overline{1,m}$, from the conditions (4), multiplying them scalar in $L_2(\Omega)$ on eigenfunctions $\varphi_{k_1,\ k_2}(x,y), k_1,\ k_2 = \overline{1,m}$, we find $(m-1)$ or $(m-2)$ initial conditions depending on the number of algebraic equations of the system. Let us proceed to the numerical solution of a system of algebra-differential equations with initial conditions $(m-1)$ or $(m-2)$.

Description of the Operation of Computer Program

The algorithm of the numerical method built in clause 1 was implemented in the Maple 2017 computer mathematics system for Windows 7, 8.1, 10 in the form of a software package. This system of computer mathematics differs from its analogues in the presence of a built-in apparatus for analytical calculations of integrals **student**, a package of commands for solving differential equations, including systems, **DEtools**. The software package is designed to find an approximate solution to the Showalter–Sidorov problem for the I-beam deformation model in the case of uniqueness or multiplicity of solutions on a rectangle. The modified Galerkin method and the phase space method are implemented in the program.

The coefficients of the equation α, β, function of the initial value $u_0(x,y)$ for the initial Showalter–Sidorov condition are fed to the input of the software package. At the output, the program outputs approximate solutions of $u_m(x,y,t)$ and plots them. The scheme of the algorithm of the software package is shown in Fig. 1.

The following steps are performed while the program is running.

Step 1. The coefficients of the equation α, β, the function of the initial value $u_0(x,y)$ for the initial Showalter–Sidorov condition are introduced, as well as the number of Galerkin approximations m.

Step 2. The procedure **unapply** allows you to present the desired approximate solutions in the form of sums

$$u := unapply(u_{k_1,\ k_2}(t)\varphi_{k_1,\ k_2}(x,y) + u_{k_2,\ k_1}(t)\varphi_{k_2,\ k_1}(x,y) + \ldots + u_m(t)\varphi_{m,m}(x,y)). \quad (15)$$

Step 3. A check is made for the degeneracy of the equation, that is, whether λ is the eigenvalue of the operator $(-\Delta)$. If the verification condition is met, we solve the resulting algebraic equation with respect to the unknowns $u_{k_1,k_2}, k_1, k_2 = \overline{1,m}$, using the built-in procedures **subs, solve**. The solution of the system of differential equations is found using the built-in procedure **dsolve**.

Step 4. Using the built-in procedure **if**... **else** ... **fi**, a one-time or two-time check of the own solutions of the Sturm–Liouville problem is performed. In the case of one-time own solutions, $u(x, y, 0)$ will be set using the procedure **unapply** as follows:

$$u0 := unapply(s_1\varphi_{k_1,\ k_2}(x, y) + u^{\perp}(x, y). \tag{16}$$

If the proper solutions of the Sturm–Liouville problem are twofold, then $u(x, y, 0)$ will be set as

$$u0 := unapply(s_1\varphi_{k_1,\ k_2}(x, y) + s_2\varphi_{k_2,\ k_1}(x, y) + u^{\perp}(x, y)). \tag{17}$$

Step 5. Substitute the obtained values into formulas (13). Using the built-in procedure **if**... **else** ... **fi**, the presence of one or more solutions to the Showalter–Sidorov problem is checked under given initial conditions.

Step 6. The expressions compiled in step 3 are substituted into the algebraic equation of the system and in the nested loop **for** k_1 **to 1 do** m **end do** and **for** k_2 **to 1 do** m **end do** the resulting equation is multiplied on the eigenfunctions φ_{k_1,k_2} and is integrated in the domain under consideration Ω using the procedure **int**. Using the built-in procedures **subs** and **solve**, with the setting **RealDomain**, we solve the resulting system of algebraic equations with respect to the unknowns $u_{k_1,k_2}(0)$, $k_1, k_2 = \overline{1,m}$.

In the case when the system of equations has three solutions, we get three functions $u_{0_1}(x, y, 0)$, $u_{0_2}(x, y, 0)$, $u_{0_3}(x, y, 0)$ for each of the solutions, accordingly. All subsequent steps must be done three times for each of the functions. To realize the possibility of finding three different solutions using the built-in procedure **save**, the initial conditions are saved in the file **usl.mw**, the first function $u_{0_1}(x, y, 0)$ is stored in the file **resh1.mw**, the second function $u_{0_2}(x, y, 0)$ is stored in the file **resh2.mw**, the third function $u_{0_3}(x, y, 0)$ is stored in the file **resh3.mw**.

In the case when the system of equations has nine solutions, we get nine functions $u_{0_1}(x, y, 0), .., u_{0_9}(x, y, 0)$ for each of the solutions, respectively. All subsequent steps must be done nine times for each of the functions. To realize the possibility of finding nine different solutions using the built-in procedure **save**, the initial conditions are saved in the file **usl.mw**, the first function $u_{0_1}(x, y, 0)$ is stored in the file **resh1.mw** and etc.

Step 7. The built-in procedure **read** reads the initial conditions and $u_{0_1}(x, y, 0)$, .., $u_{0_9}(x, y, 0)$ stored in files **resh1.mw**, .., **resh9.mw**. In the double loop **for** k_1 **to 1 do for** k_2 **to 1 do** m **end do** m **end do**, the left and right sides of the differential equation obtained in the third step are multiplied by the proper function φ_{k_1,k_2} and are integrated **int**). As a result of steps 5 and 6, we obtain a system of algebra-differenqialn equations for determining the approximation coefficients u_{k_1,k_2}.

Step 8. The system obtained in step 6 is solved with the initial conditions saved in the file **resh1.mw**, ..,**resh9.mw** using the built-in procedure **dsolve**.

Step 9. The solution is compiled and displayed on the screen as a graph by the built-in procedures **plot3d**.

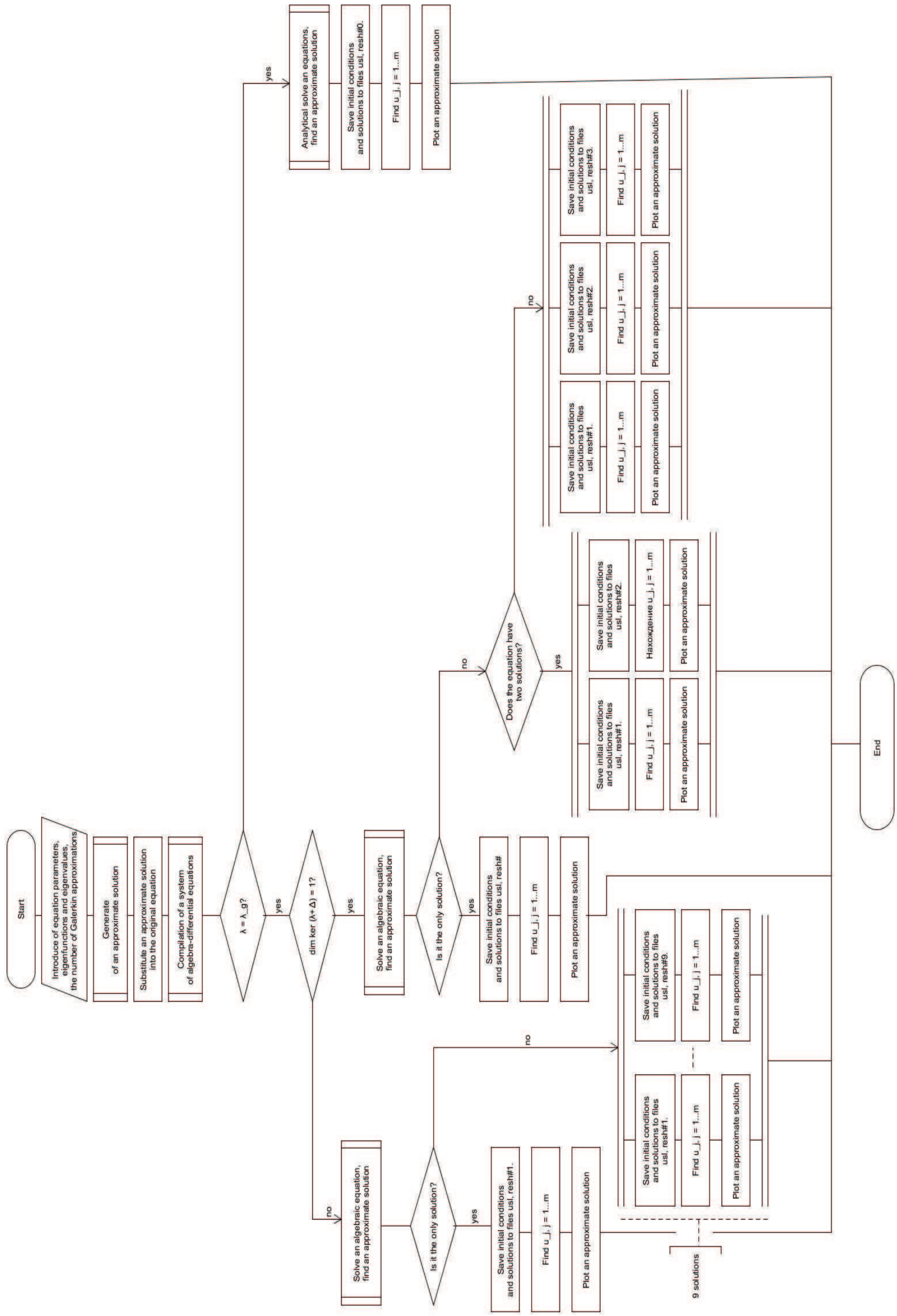


Fig. 1. Diagram of the algorithm

2. Numerical Experiment

Let us consider examples of numerical investigation of the non-uniqueness of solutions to the Showalter–Sidorov problem for the I-beam deformation model based on the implementation of the algorithm and program described above.

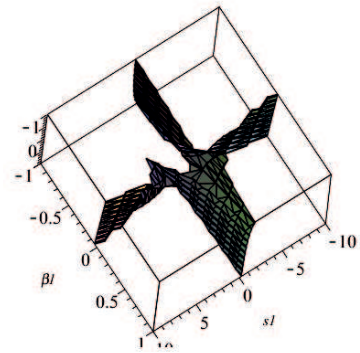


Fig. 2. The phase space of the equation (18)

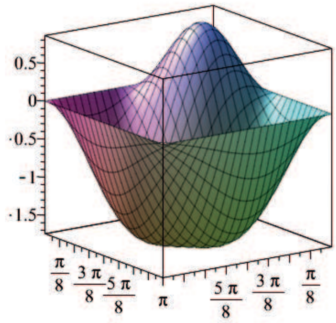


Fig. 3. Numerical solution of the $u(x, y, t)$ problem (18) – (20) in the case of uniqueness of the solution at time $t = 0$

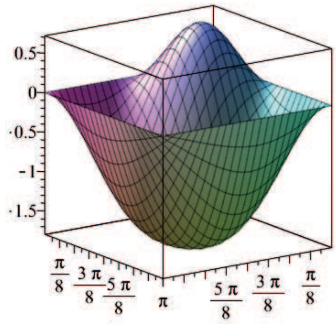


Fig. 4. Numerical solution of the $u(x, y, t)$ problem (18) – (20) in the case of uniqueness of the solution at time $t = 1$

Example 1. It is required to find a numerical solution to the Showalter–Sidorov problem

$$\begin{aligned}&2\left(u(x,y,0)-\tfrac{2}{\pi}\sin x\sin y\right)+\\&+\left(u_{xx}(x,y,0)+u_{yy}(x,y,0)-\tfrac{2}{\pi}\sin x\sin y\right)=0,\ x\in(0,\pi),y\in(0,\pi)\end{aligned} \tag{18}$$

for the equation

$$2u_t+u_{xxt}+u_{yyt}=\alpha u+\beta u^3,\ t\in(0,1) \tag{19}$$

with Dirichlet boundary condition

$$\begin{aligned}u(0,y,t)=u(\pi,y,t)=0,\ y\in(0,\pi),\ t\in(0,1),\\u(x,0,t)=u(x,\pi,t)=0,\ x\in(0,\pi),\ t\in(0,1).\end{aligned} \tag{20}$$

Since under the conditions of this experiment λ coincides with the first eigenvalue $\lambda_1=2$ of the homogeneous Dirichlet problem for $(-\Delta)$, the considered eigenfunction φ_g correspond to single eigenvalues, satisfying the condition $\dim\ker(\lambda+\Delta)=1$. Following the algorithm described in paragraph 1, we represent the function $u(x,y,0)$ at the initial moment of time as $u(x,y,0)=s_1\varphi_g+u^{\perp}$, where $\varphi_g=\frac{2}{\pi}\sin x\sin y$ and $u^{\perp}=\frac{2}{\pi}\sin x\sin 2y+\frac{2}{\pi}\sin 2x\sin y+\frac{2}{\pi}\sin 2x\sin 2y$. By virtue of $\left\langle\varphi_g,u^{\perp}\right\rangle=0$, we get that the set $\mathfrak{B}$ will take the following form:

$$\mathfrak{B}=\{s_1\in\mathbb{R}:-\alpha s_1-\tfrac{9\beta s_1^3}{4\pi^2}-\tfrac{12\beta s_1}{\pi^2}-\tfrac{6\beta}{\pi^2}=0\}. \tag{21}$$

Using the formulas (13) for the equation describing the set (21), we find Q. Following the algorithm described above, we obtain conditions imposed on the parameters of the problem (18) – (20), under which there may be several solutions to this problem

$$\frac{3845.556777\alpha^3+14026.90912\alpha^2\beta+17054.67641\alpha\beta^2+9099\beta^3}{\beta}>0. \tag{22}$$

Consider a special case when $\alpha=1,\beta=-0.5$, then the condition (22) is met, the task (18) – (20) will have one solution. The phase space is shown in Fig. 2. For clarity, Fig. 3 one numerical solution $u(x,y,t)$ of this problem at time $t=0$ is presented, Fig. 4 is represented at time $t=1$. When $\alpha=1,\beta=-0.1$, then the conditions (22) are not met, the task (18) – (20) will have three solutions. In Fig. 5 presents three numerical solutions $u(x,y,t)$ of this problem at time $t=0$, in Fig. 6 presents three numerical solutions of this problem at time $t=1$.

Example 2. It is required to find a numerical solution to the Showalter–Sidorov problem

$$\begin{aligned}&5\left(u(x,y,0)-\tfrac{2}{\pi}\sin x\sin y-\tfrac{2}{\pi}\sin 2x\sin 2y\right)+(u_{xx}(x,y,0)+\\&+\left(u_{yy}(x,y,0)-\tfrac{2}{\pi}\sin x\sin y-\tfrac{2}{\pi}\sin 2x\sin 2y\right)=0,\ x\in(0,\pi),y\in(0,\pi)\end{aligned} \tag{23}$$

for the Hoff's equation

$$5u_t+u_{xxt}+u_{yyt}=\alpha u+\beta u^3,\ t\in(0,1) \tag{24}$$

with Dirichlet boundary condition

$$\begin{aligned}u(0,y,t)=u(\pi,y,t)=0,\ y\in(0,\pi),\ t\in(0,1),\\u(x,0,t)=u(x,\pi,t)=0,\ x\in(0,\pi),\ t\in(0,1).\end{aligned} \tag{25}$$

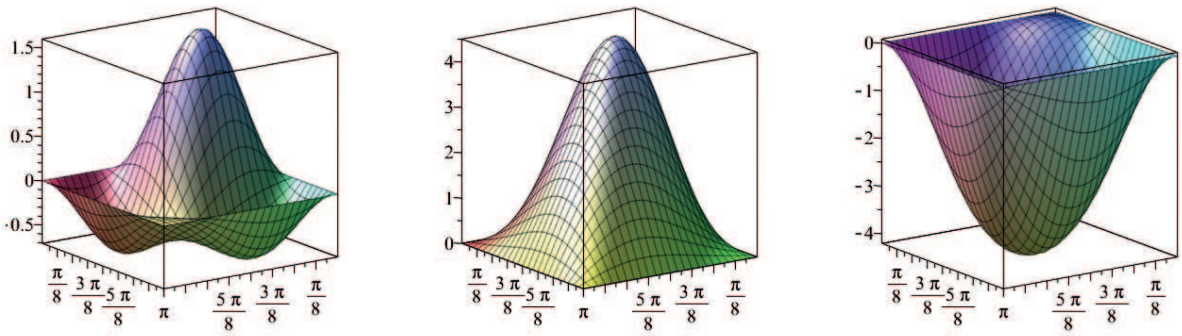


Fig. 5. Numerical solution of the $u(x, y, t)$ problem (18) – (20) in the case of non-uniqueness of the solution at time $t = 0$

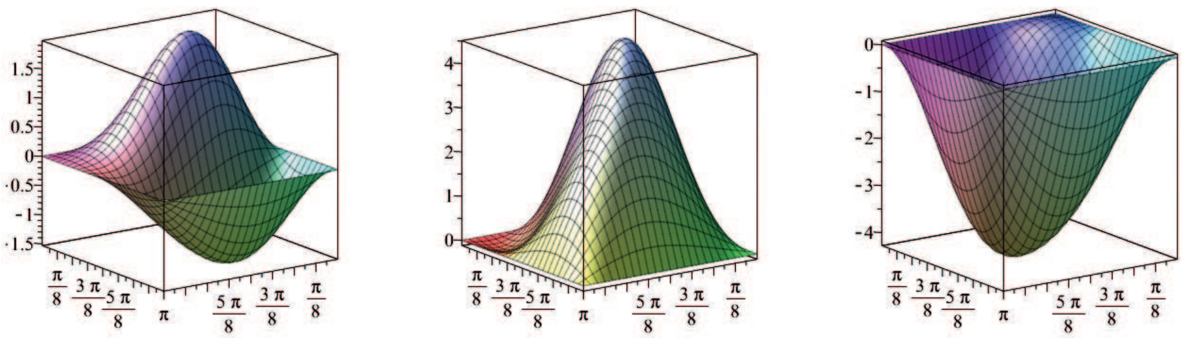


Fig. 6. Numerical solution of the $u(x, y, t)$ problem (18) – (20) in the case of non-uniqueness of the solution at time $t = 1$

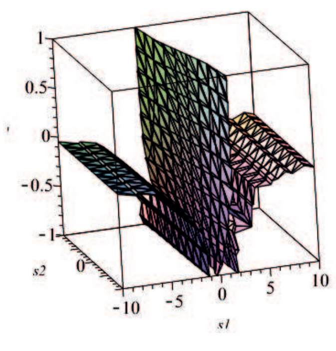
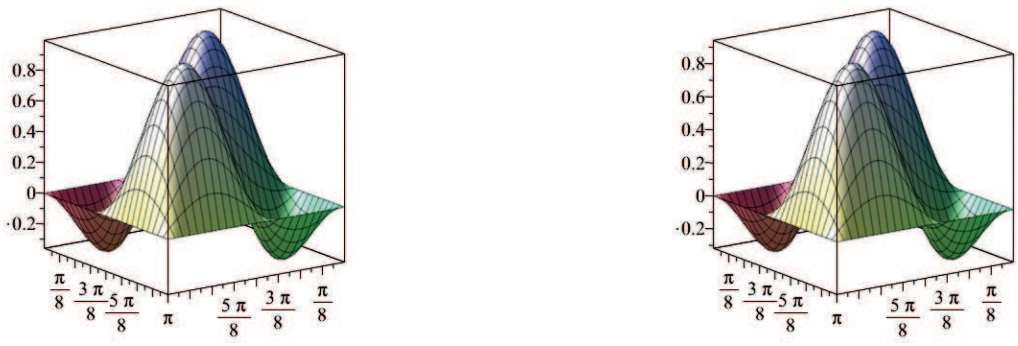


Fig. 7. The phase space of the equation (23)

Since under the conditions of this experiment λ coincides with the second eigenvalue $\lambda_2 = 5$ of the homogeneous Dirichlet problem for $(-\Delta)$, the considered eigenfunctions φ_{g_1} and φ_{g_2} correspond to double eigenvalues, satisfying the condition $\dim \ker(\lambda + \Delta) = 2$.



a) b)

Fig. 8. Numerical solution of the $u(x,y,t)$ problem (23) – (25) in the case of uniqueness of the solution at time $t=0$ and time $t=1$

Represent the function $u(x,y,0)$ at the initial moment of time for (23) – (25) as $u(x,y,0) = s_1\varphi_{g_1} + s_2\varphi_{g_2} + u^{\perp}$, where $\varphi_{g_1} = \frac{2}{\pi}\sin x \sin 2y, \varphi_{g_2} = \frac{2}{\pi}\sin 2x \sin y$ and $u^{\perp} = \frac{2}{\pi}\sin x \sin y + \frac{2}{\pi}\sin 2x \sin 2y$. By virtue of $\langle\varphi_{g_1},\varphi_{g_1}\rangle = 1$, $\langle\varphi_{g_1},\varphi_{g_2}\rangle = 0$ and $\langle\varphi_{g_1},u^{\perp}\rangle = 0$, we get that the sets $\mathfrak{B}_1$ и $\mathfrak{B}_2$ will take the following form:

$$\mathfrak{B}_1 = \{s_1,s_2 \in \mathbb{R} : \alpha s_1 + \frac{3\beta s_1 s_2^2}{\pi^2} + \frac{9\beta s_1}{\pi^2} + \frac{6\beta s_2}{\pi^2} + \frac{9\beta s_1^3}{4\pi^2} = 0\}, \tag{26}$$

$$\mathfrak{B}_2 = \{s_1,s_2 \in \mathbb{R} : \alpha s_2 + \frac{3\beta s_1^2 s_2}{\pi^2} + \frac{6\beta s_1}{\pi^2} + \frac{9\beta s_2}{\pi^2} + \frac{9\beta s_2^3}{4\pi^2} = 0\}. \tag{27}$$

Following the algorithm described in paragraph 1, the problem can have several solutions. To identify the conditions of non-uniqueness of solutions, we perform the following operations on the equations (26) and (27) describing the sets $\mathfrak{B}_1$ и $\mathfrak{B}_2$. Let's express from each equation $(\alpha + \frac{6\beta}{\pi^2})$:

$$\begin{aligned} s_1\left(\alpha + \frac{6\beta}{\pi^2}\right) &= -\frac{3\beta s_1 s_2^2}{\pi^2} - \frac{3\beta s_1}{\pi^2} - \frac{6\beta s_2}{\pi^2} - \frac{9\beta s_1^3}{4\pi^2}, \\ s_2\left(\alpha + \frac{6\beta}{\pi^2}\right) &= -\frac{3\beta s_1^2 s_2}{\pi^2} - \frac{3\beta s_2}{\pi^2} - \frac{6\beta s_1}{\pi^2} - \frac{9\beta s_2^3}{4\pi^2}, \end{aligned} \tag{28}$$

$$\begin{aligned} \left(\alpha + \frac{6\beta}{\pi^2}\right) &= -\frac{3\beta s_2^2}{\pi^2} - \frac{3\beta}{\pi^2} - \frac{6\beta s_2}{\pi^2 s_1} - \frac{9\beta s_1^2}{4\pi^2}, \\ \left(\alpha + \frac{6\beta}{\pi^2}\right) &= -\frac{3\beta s_1^2}{\pi^2} - \frac{3\beta}{\pi^2} - \frac{6\beta s_1}{\pi^2 s_2} - \frac{9\beta s_2^2}{4\pi^2}. \end{aligned} \tag{29}$$

We equate the resulting expressions to each other:

$$-\frac{3\beta s_2^2}{\pi^2} - \frac{3\beta}{\pi^2} - \frac{6\beta s_2}{\pi^2 s_1} - \frac{9\beta s_1^2}{4\pi^2} = -\frac{3\beta s_1^2}{\pi^2} - \frac{3\beta}{\pi^2} - \frac{6\beta s_1}{\pi^2 s_2} - \frac{9\beta s_2^2}{4\pi^2}, \tag{30}$$

$$-\frac{3\beta s_1^2}{4\pi^2} + \frac{3\beta s_2^2}{4\pi^2} + \frac{6\beta s_2}{\pi^2 s_1} - \frac{6\beta s_1}{\pi^2 s_2} = 0. \tag{31}$$

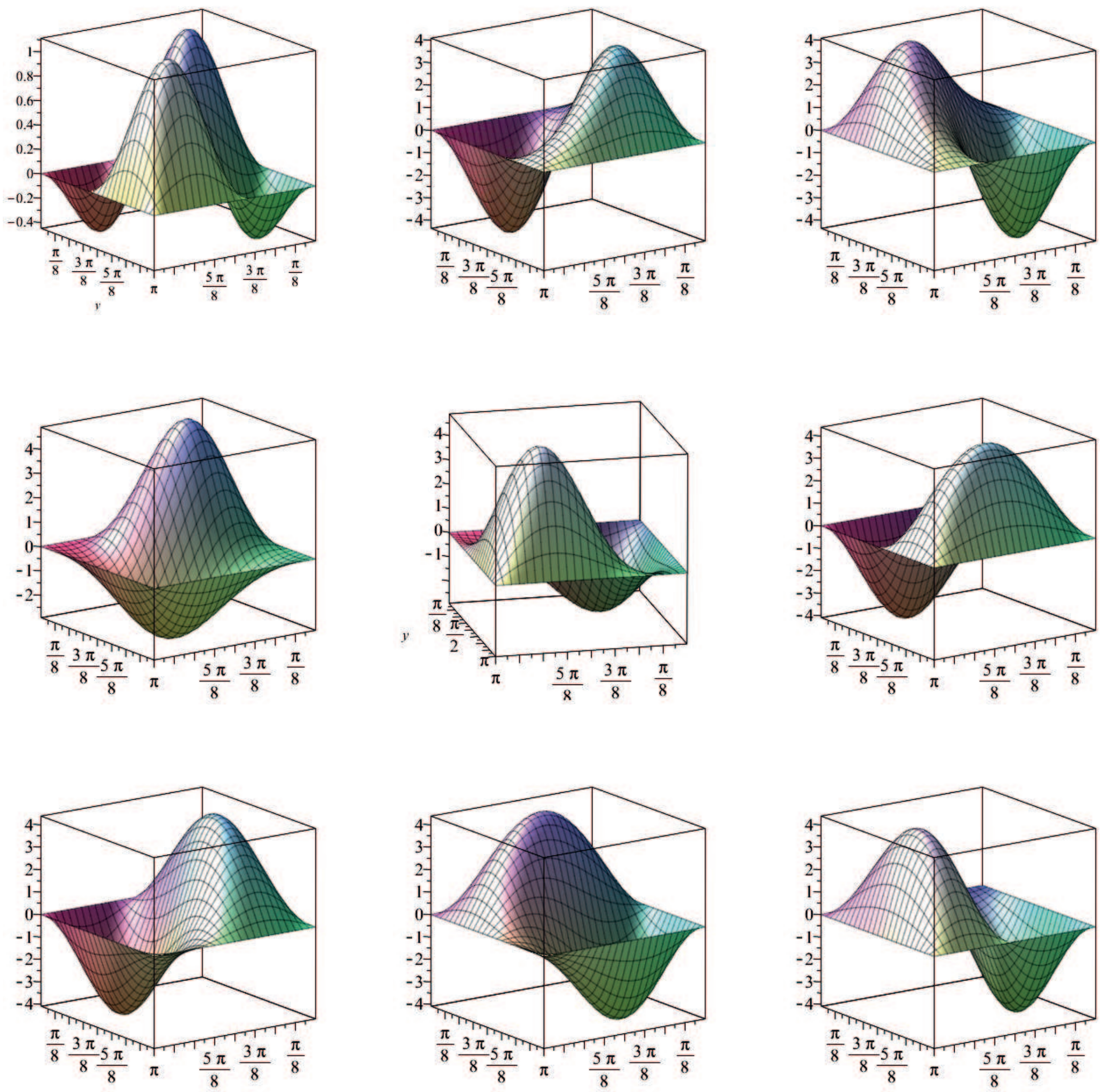


Fig. 9. Numerical solution of the $u(x, y, t)$ problem (23) – (25) in the case of non-uniqueness of the solution at time $t = 0$

Take out $\frac{3\beta}{\pi^2}$ and get:

$$-\frac{s_1^2}{4} + \frac{s_2^2}{4} + \frac{2s_2}{s_1} - \frac{2s_1}{s_2} = 0. \tag{32}$$

Let's group the terms and bring them to a common denominator:

$$\frac{s_2^2 - s_1^2}{4} + 2\frac{s_2^2 - s_1^2}{s_1 s_2} = 0. \tag{33}$$

Let's take $s_2^2 - s_1^2$ out of brackets:

$$\left(s_2^2 - s_1^2\right)\left(\frac{4s_1 s_2 + 2}{s_1 s_2}\right) = 0. \tag{34}$$

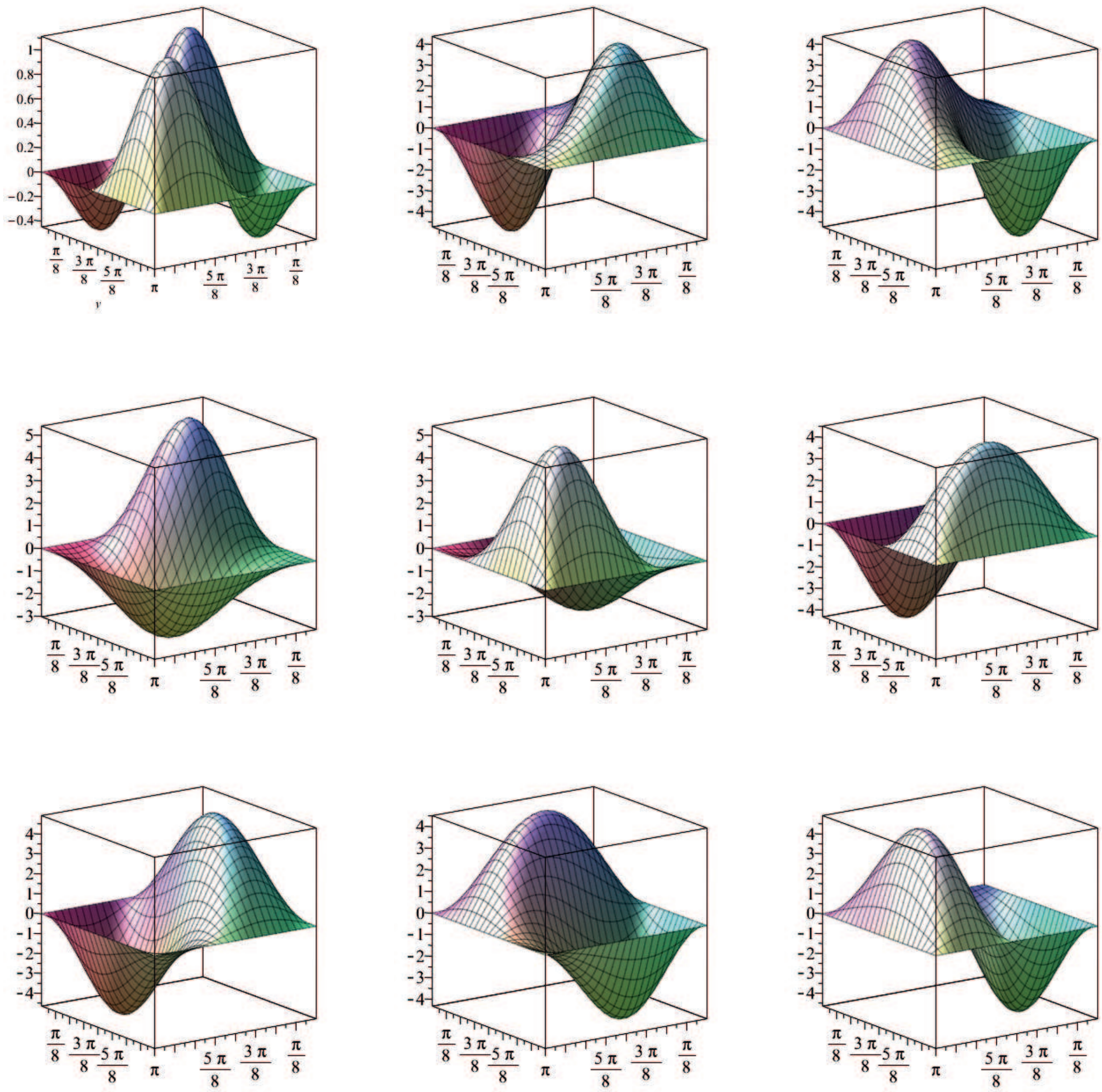


Fig. 10. Numerical solution of the $u(x, y, t)$ problem (23) – (25) in the case of non-uniqueness of the solution at time $t = 1$

Based on this, we get the following solution:

$$\begin{aligned} &s_1 = \pm s_2, \\ &s_1 s_2 = -\tfrac{1}{2}. \end{aligned} \tag{35}$$

Substituting (35) into (26), (27), we obtain the following conditions imposed on the parameters of the problem (23) – (25), under which there may be several solutions to this problem:

$$-\frac{4\pi^2\alpha + 12\beta}{21\beta} > 0. \tag{36}$$

Consider a special case when $\alpha = 1, \beta = -3.5$, then the condition (36) is not fulfilled, the task (23) – (25) will have one solution. The phase space is shown in Fig. 7. For clarity, Fig. 8 one numerical solution $u(x, y, t)$ of this problem is presented at time $t = 0$ and at time $t = 1$ respectively.

If $\alpha = 1, \beta = -0.1$, then the condition (36) is met, the task (23) – (25) will have nine solutions. For clarity, Fig. 9 and Fig. 10 presents nine numerical solutions $u(x, y, t)$ of this problem at time $t = 0$ and time $t = 1$ accordingly.

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ЧИСЛЕННОЕ ИССЛЕДОВАНИЕ ВОПРОСА НЕЕДИНСТВЕННОСТИ РЕШЕНИЯ ЗАДАЧИ ШОУОЛТЕРА – СИДОРОВА ДЛЯ МАТЕМАТИЧЕСКОЙ МОДЕЛИ ХОФФА НА ПРЯМОУГОЛЬНИКЕ

Н.Г. Николаева, О.В. Гаврилова, Н.А. Манакова

Статья посвящена исследованию единственности или неединственности решений задачи Шоуолтера – Сидорова – Дирихле для уравнения Хоффа на прямоугольнике. Для исследования данного вопроса использован метод фазового пространства, который был разработан Г.А. Свиридюком. Построен алгоритм выявления условий множественности и единственности решений, который позволяет численно решить задачу Шоуолтера – Сидорова – Дирихле на основе модифицированного метода Галеркина. В статье рассмотрены случаи размерность ядра оператора при производной по времени равна 1 или 2. Представлены вычислительные эксперименты, демонстрирующие неединственность решений задачи Шоуолтера – Сидорова в зависимости от значений параметров задачи.

Ключевые слова: уравнения соболевского типа; задача Шоуолтера – Сидорова; уравнение Хоффа; неединственность решений; метод фазового пространства; метод Галеркина.

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