MSC 35R60 DOI: 10.14529/jcem230204

# NUMERICAL SOLUTIONS FOR THE CAUCHY PROBLEM FOR THE OSKOLKOV EQUATION IN THE SPACES OF DIFFERENTIAL FORMS WITH STOCHASTIC COEFFICIENTS

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The article contains a research of the solvability of the Cauchy problem for the linear Oskolkov equation in specially given spaces, namely the spaces of differential forms with stochastic coefficients defined on some Riemannian manifold without boundary. This work presents graphs of coefficients of differential forms that are solutions to the Cauchy problem for Oskolkov equations. Since the equations are studied in space of differential forms, the operators themselves are understood in a special form, in particular, instead of the Laplace operator, we take its generalization that is the Laplace–Beltrami operator. Graphs of coefficients of differential forms obtained within other computational experiments are presented for various values of parameters of the Oskolkov equation.

*Keywords: Sobolev type equation; differential forms; Riemannian manifold; Laplace – Beltrami operator; numerical solution.*

## Introduction

Consider the Oskolkov linear homogeneous equation [1]:

$$(1-\kappa\Delta)\Delta\varphi_t = \alpha\Delta^2\varphi, \tag{1}$$

If we use substitution

$$u = \Delta\varphi, \tag{2}$$

then we reduce it for the Barenblatt–Zheltov–Kochina equation [2]

$$(\lambda-\Delta)u_t = \alpha\Delta u. \tag{3}$$

In the functional spaces $\mathfrak{U}$, $\mathfrak{F}$ chosen by us, (3) are reduced [3] to the linear equation of Sobolev type

$$L\dot{u} = Mu \tag{4}$$

with the irreversible operator $L$.

Consider the Cauchy problem [4]

$$u(0) = u_0 \tag{5}$$

for equations (1) after substitution (2) when it seems (3).

The paper [5] propose a transition of (4) to the stochastic Sobolev type equations

$$L\overset{\circ}{\eta} = M\eta \tag{6}$$

with the condition Cauchy

$$\eta(0) = \eta_0 \tag{7}$$

in spaces of Wiener stochastic processes in the case of an abstract $(L, p)$-bounded operator $M$. Since Wiener processes are continuous, but nondifferentiable in the usual sense at each point, we use the Nelson–Gliklikh derivative [6]. In this article, we study numerical solutions to the Oskolkov linear equation [7] in spaces of differential forms defined on a torus as (6).

## 1. Structure of Differentiable «Noises» Spaces

Consider the complete probability space $\Omega = (\Omega, \Sigma, P)$ with the probability measure $P$ associated with the $\sigma$-algebra $\Sigma$ of subsets of the space $\Omega$. If $\mathbb{R}$ is the set of real numbers endowed with a $\sigma$-algebra, then the mapping $\xi : \Sigma \mapsto \mathbb{R}$ is called a random variable. The set of random variables $\xi$, the mathematical expectation of which is equal to zero, i.e. $M\xi = 0$, while variance is finite, i.e. $D\xi < \infty$, form the Hilbert space $\mathbf{L_2}$ with the scalar product $(\xi_1, \xi_2) = M\xi_1\xi_2$ and with the norm denoted by $\|\xi\|_{\mathbf{L_2}}$. If we take the subalgebra $\Sigma_0$ of the $\sigma$-algebra $\Sigma$, then we obtain the subspace of random variables $\mathbf{L_2^0} \subset \mathbf{L_2}$ measurable with respect to $\Sigma_0$.

A measurable mapping $\eta = \eta(t, \omega) : J \times \Sigma \mapsto \mathbb{R}$, where $J = (a, b) \subset \mathbb{R}$, is called a stochastic process, a random variable $\eta(\cdot, \omega), \omega \in \Omega$ is said to be a section of the stochastic process, and a function $\eta(t, \cdot), t \in J$ is said to be a trajectory of the stochastic process. The stochastic process $\eta = \eta(t, \omega)$ is called continuous, if the trajectories $\eta = \eta(t, \omega_0)$ are continuous functions almost sure (i.e. for a.a. (almost all) $\omega_0 \in \Sigma$). The set $\eta = \eta(t, \omega)$ of continuous stochastic processes forms a Banach space $\mathbf{CL_2}$.

By the Nelson – Gliklikh derivative of the stochastic process $\eta \in \mathbf{CL_2}$ at the point $t \in J$ we mean the random variable

$$\overset{\circ}{\eta} = \frac{1}{2}\left(\lim_{\Delta t \mapsto 0+} M_t^\eta \left(\frac{\eta(t+\Delta t, \cdot) - \eta(t, \cdot)}{\Delta t}\right) + \lim_{\Delta t \mapsto 0+} M_t^\eta \left(\frac{\eta(t, \cdot) - \eta(t-\Delta t, \cdot)}{\Delta t}\right)\right), \tag{8}$$

if the limit exists in the sense of a uniform metric on $t \in J$. Here $M_t^\eta$ is the expectation on a subalgebra of the $\sigma$-algebra $\Sigma$ that is generated by the random variable $\eta = \eta(t, \omega)$. If there exist the Nelson – Gliklikh derivatives $\overset{\circ}{\eta}(\cdot, \omega)$ of the stochastic process $\eta$ at almost all points of the interval $J$, then we say that there exists the Nelson – Gliklikh derivative $\overset{\circ}{\eta}(\cdot, \omega)$ almost sure on $J$.

The set of continuous stochastic processes with continuous Nelson – Gliklikh derivatives $\overset{\circ}{\eta}$ form the Banach space $\mathbf{C^1L_2}$. Further, by induction, we obtain the Banach spaces $\mathbf{C^\ell L_2}$, $\ell \in \mathbb{N}$ of the stochastic processes having continuous Nelson – Gliklikh derivatives on $J$ up to the order $\ell \in \mathbb{N}$ inclusively with the norms of the form

$$\|\eta\|_{\mathbf{C^\ell L_2}} = \sup_{t \in J}\left(\sum_{k=0}^{\ell} D\overset{\circ}{\eta}^{(k)}(t, \omega)\right)^{\frac{1}{2}},$$

where $\overset{\circ}{\eta}^{(0)}(t, \omega) = \eta(t, \omega)$.

Note that such processes we also can investigate in complex numbers [8].

## 2. Resolving Groups of Operators

Let $\mathfrak{U}$ and $\mathfrak{F}$ be real separable Hilbert spaces. Denote by $\mathcal{L}(\mathfrak{U};\mathfrak{F})$ the space of linear bounded operators, and by $Cl(\mathfrak{U};\mathfrak{F})$ the space of linear closed and densely defined operators. Let us construct the Hilbert spaces $\mathbf{U_K L_2}$ and $\mathbf{F_K L_2}$, where $\mathbf{K} = \{\lambda_k\} \subset \mathbb{R}$ is a monotone sequence of numbers such that $\sum\limits_{k=1}^{\infty} \lambda_k^2 < +\infty$.

The operator $M$ is called *spectrally bounded with respect to the operator* $L$ (or, shortly, $(L,\sigma)$*-bounded*), if $\exists r > 0 \quad \forall \mu \in \mathbb{C} \quad (|\mu| > r) \Rightarrow (\mu \in \rho^L(M))$.

In complex plane $\mathbb{C}$, for the $(L,\sigma)$-bounded operator $M$, we choose a closed circuit of the form $\gamma = \{\mu \in \mathbb{C} : |\mu| = R > r\}$. Then the integrals

$$P = \frac{1}{2\pi i}\int\limits_{\gamma} R_\mu^L(M)d\mu, \quad Q = \frac{1}{2\pi i}\int\limits_{\gamma} L_\mu^L(M)d\mu$$

make sense as the integrals of analytic functions on a closed circuit. Moreover, the operators $P : \mathfrak{U} \to \mathfrak{U}$ and $Q : \mathfrak{F} \to \mathfrak{F}$ are projectors [3]. Denote by $L_k$, $M_k$ the restrictions of the operators $L$, $M$ on the subspace $\mathfrak{U}^k$, $k = 0, 1$.

**Theorem 1.** [3] *Let the operator $M$ be $(L,\sigma)$-bounded. Then*
(i) $L_k, M_k \in \mathcal{L}(\mathfrak{U}^k;\mathfrak{F}^k)$, $k = 0, 1$;
(ii) *there exist operators* $L_1^{-1} \in \mathcal{L}(\mathfrak{F}^1;\mathfrak{U}^1)$, $M_0^{-1} \in \mathcal{L}(\mathfrak{F}^0;\mathfrak{U}^0)$.

If the operator $M$ is $(L,\sigma)$-bounded, then by virtue of Theorem 1, there exist the operators $H = M_0^{-1}L_0 \in \mathcal{L}(\mathfrak{U}^0)$ and $S = L_1^{-1}M_1 \in \mathcal{L}(\mathfrak{U}^1)$.

**Definition 1.** The $(L,\sigma)$-bounded operator $M$ is called
(i) $(L,0)$*-bounded*, if $H = \mathbb{O}$;
(ii) $(L,p)$*-bounded*, if $H^p \neq \mathbb{O}$ and $H^{p+1} = \mathbb{O}$ for some $p \in \mathbb{N}$.

**Theorem 2.** [7] *Let the operator $M$ be $(L,p)$-bounded, $p \in \{0\} \cup \mathbb{N}$. Then there exists an analytical group of the operators on the space* $\mathbf{U_K L_2}, \mathbf{F_K L_2}$.

The stochastic Sobolev type equation

$$L \overset{\circ}{\eta} = M\eta \tag{9}$$

can be reduced to two equations of the form

$$A \overset{\circ}{\nu} = B\nu.$$

Let us formulate

**Lemma 1.** *The following statements are true:*
(i) *the operator* $A \in \mathcal{L}(\mathfrak{U};\mathfrak{F})$ *exactly if* $A \in \mathcal{L}(\mathbf{U_K L_2};\mathbf{F_K L_2})$;
(ii) *the operator* $B \in Cl(\mathfrak{U};\mathfrak{F})$ *exactly if* $B \in Cl(\mathbf{U_K L_2};\mathbf{F_K L_2})$.

Denote $\mathfrak{U}^1 = \{\mathbf{U_K L_2}\}$ ($\mathfrak{F}^1 = \{\mathbf{F_K^1 L_2}\}$), which form a closure $\mathrm{im}R^L(M)$ ($\mathrm{im}L^L(M)$) in the norm of the space $\mathfrak{U} = \mathbf{U_K L_2}$ ($\mathfrak{F} = \mathbf{F_K L_2}$). The spaces $\mathbf{U_K L}_2$ and $\mathbf{F_K L}_2$ are splitted into the direct sum

$$\mathbf{U_K L}_2 = \mathbf{U_K^0 L}_2 \oplus \mathbf{U_K^1 L}_2, \mathbf{F_K L}_2 = \mathbf{F_K^0 L}_2 \oplus \mathbf{F_K^1 L}_2. \tag{10}$$

The following theorem takes place.

**Theorem 3.** *If the operator $M$ is $(L,p)$-bounded and there exist splittings* (10)*, then* $\mathrm{im}U^* = \mathbf{U_K^1 L_2}$ *and* $\mathrm{im}F^* = \mathbf{F_K^1 L_2}$.

Previously, the Cauchy problem

$$\eta(0) = \eta_0 \tag{11}$$

was investigated [7] in the spaces $\mathfrak{U} = \mathbf{U_K L_2}$ ($\mathfrak{F} = \mathbf{F_K L_2}$), where there exist representations of the form

$$\eta(t,\cdot) = \sum_{k=0}^{+\infty} \lambda_k \xi_k(t,\cdot)\varphi_k. \tag{12}$$

**Theorem 4.** *Suppose that the operator $M$ is $(L,p)$-bounded and there exist splittings* (10)*, then $\forall \eta_0 \in \mathfrak{U}^1 \subset \mathfrak{U}$ there exists the unique solution to problem* (9)*,* (11)*.*

## 3. Differential Forms and Computational Experiments

Consider a two-dimensional torus obtained by the direct product of two segments $\mathbb{T} = [0,\pi] \times [0, 2\pi]$. The torus is a 2-dimensional smooth compact oriented Riemannian manifold without boundary. Using theory presented in Sections 1 and 2, we construct spaces of smooth differential $q$-forms with stochastic processes as the coefficient:

$$\omega(t,\omega,x_1,x_2) = \sum_{|i_1,\dots,i_q|=q} \chi_{i_1,\dots,i_q}(t,\omega,x_1,x_2)dx_{i_1} \wedge \dots \wedge dx_{i_2}, \tag{13}$$

where $|i_1,\dots,i_q|$ is a multi-index, and, according to (12), the coefficients have the form

$$\chi_{i_1,i_2,\dots,i_q}(t,\omega,x_1,x_2) = \sum_{k=1}^{\infty} \lambda_k \xi_{k,i_1,\dots,i_q}(t)\varphi_k.$$

Take the Oskolkov linear homogeneous equation [1]:

$$(1-\kappa\Delta)\Delta\varphi_t = \alpha\Delta^2\varphi. \tag{14}$$

If we define this equation in the space of differential form [9] we must use Laplace–Beltrami operator

$$\Delta u = d * d * u + *d * du, \tag{15}$$

where $*$ –Hodge operator and $d$ –differential of differential forms. It change sign before $\Delta$.

In these spaces it is possible to introduce the inner product

$$(\chi,\xi)_0 = \int_{\mathbb{T}} \chi \wedge *\xi d\tau, \chi,\xi \in \mathbf{C^4 L_2}. \tag{16}$$

Because of Hodge–Kodair theorem, spaces of differential forms of norms current and complete in accordance with the norm obtained with the help of inner product (16) splitting in direct sum of potential, solenoidal and harmonic differential form. If we use substitutions

$$u = \Delta\varphi, \lambda = 1/\kappa, \tag{17}$$

then we reduce (14) for the homogeneous Barenblatt–Zheltov–Kochina equation [2], but only in the case when the space contains differential forms orthogonal to harmonic ones.

$$(\lambda + \Delta)u_t = \alpha \Delta u. \quad (18)$$

Then we take the stochastic version of the Barenblatt–Zheltov–Kochina equation

$$(\lambda + \Delta) \overset{\circ}{\eta} = \alpha \Delta \eta. \quad (19)$$

Denote the operators $L = (\lambda + \Delta),\ M = \alpha\Delta$ and arrive at

$$L \overset{\circ}{\eta} = M\eta. \quad (20)$$

Those operator $L, M : \mathbf{C^4 L_2} \mapsto \mathbf{C^0 L_2}$.

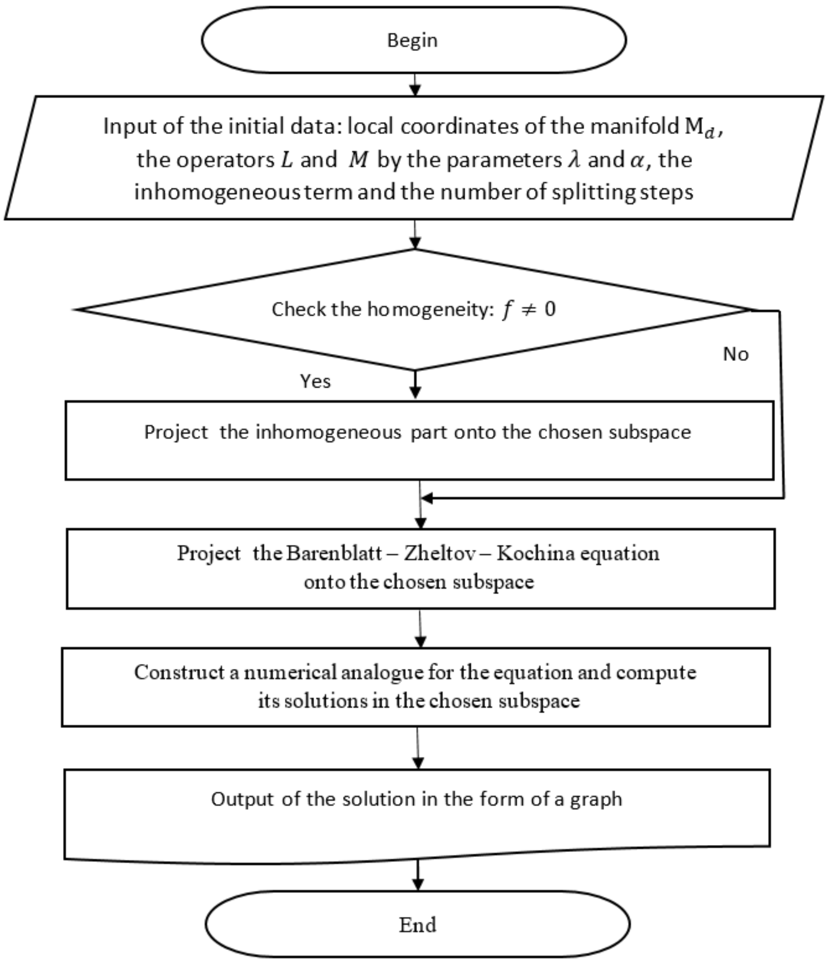


**Fig. 1**. Block diagram of the algorithm for the equation (18)

For (20) equation, the paper [7] proves $(L, 0)$-boundedness of the operator $M$ and constructs the relative spectrum

$$\mu_t = \frac{\alpha \lambda_k}{\lambda + \lambda_k}.$$

In [7] the algorithm of solution of the Cauchy problem for the Barenblatt–Zheltov–Kochina equation was presented by the block diagram given in Fig. 1.

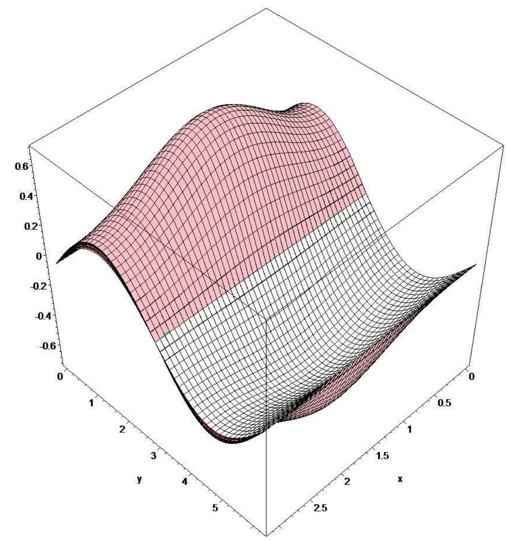


**Fig. 2**. The graph of the solution in space 0-form for (20) with $\lambda = 3, \alpha = 0.5$

An implementation of this algorithm has been registered under the name ≪Numerical Solutions in the Stochastic Barenblatt–ZHeltov–Kochina Model in Spaces of Differential Forms≫ and certificate number 2022661554 RU of state registration of a computer program was issued. The graphs show the solutions at the time moments $t_k, k = 1, ..., 8$, by the corresponding colors: pink, green, blue, black, yellow, brown, red, white.

Fig. 2 shows the graph of the solution to the Cauchy problem with $\lambda = 3, \alpha = 0.5$ in 0-form space.

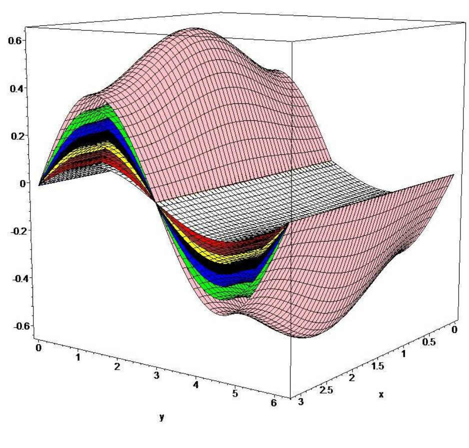


**Fig. 3**. The graph $dx_1$ coefficient of the Cauchy problem solution for (20) with $\lambda = 1, \alpha = 1$

Fig. 3 draw before $dx_1$ coefficient and Fig. 4 draw before $dx_2$ coefficient of the solution to the Cauchy problem with $\lambda = 1, \alpha = 1$ in 1-form space.

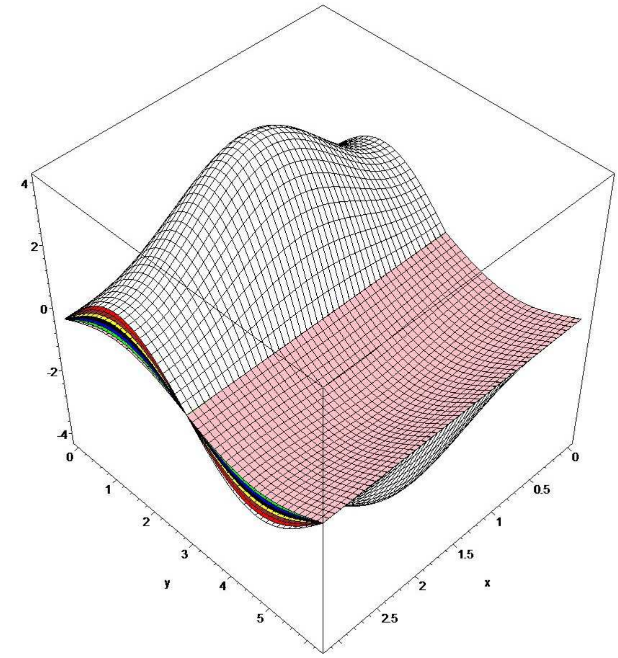


**Fig. 4**. The graph $dx_2$ coefficient of the Cauchy problem solution for equation (20) with $\lambda = 1, \alpha = 1$.

Further on Fig. 5, we present graph coefficient before $(dx_1 \wedge dx_2)$ of the solution for equation (20) with $\lambda = 1, \alpha = 2$ in 2-form space.

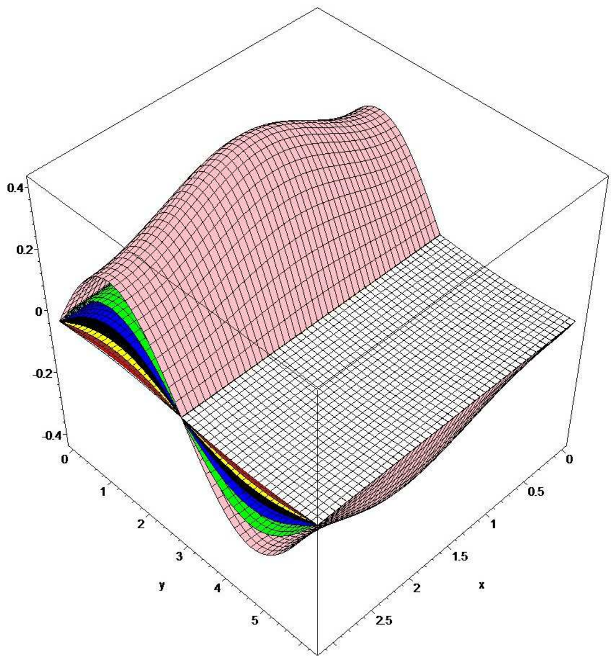


**Fig. 5**. The graph coefficient before $(dx_1 \wedge dx_2)$ of the Cauchy problem solution for equation (20) with $\lambda = 1, \alpha = 2$

## Conclusion

Numerical solutions of the Cauchy problem for the linear Oskolkov equation were studied. The linear equation of Oskolkov linearized Oskolkov's system of equations. In the future, we should study the Cauchy and Showalter – Sidorov problems for the general Okolkov's system of equation or it linearization in spaces of differential forms with stochastic coefficients.

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*Received* May 7, 2023

УДК 517.95 DOI: 10.14529/jcem230204

# ЧИСЛЕННЫЕ РЕШЕНИЯ ЗАДАЧИ КОШИ ДЛЯ ЛИНЕЙНОГО УРАВНЕНИЯ ОСКОЛКОВА В ПРОСТРАНСТВАХ ДИФФЕРЕНЦИАЛЬНЫХ ФОРМ СО СТОХАСТИЧЕСКИМИ КОЭФФИЦИЕНТАМИ

*Д. Е. Шафранов*

Статья содержит исследование разрешимости задачи Коши для линейного уравнения Осколкова в специально заданных пространствах, а именно пространствах дифференциальных форм со стохастическими коэффициентами, определенных на некотором римановом многообразии без края. В данной работе представлены рисунки коэффициентов дифференциальных форм являющихся решениями задачи Коши для уравнения Осколкова. Поскольку уравнения изучаются в пространстве дифференциальных форм, сами операторы понимаются в специальной форме, в частности, вместо оператора Лапласа берется его обобщение – оператор Лапласа–Бельтрами. Графики коэффициентов дифференциальных форм полученные при проведении вычислительных экспериментов приведены для различных значений параметров уравнения Осколкова.

*Ключевые слова: уравнение соболевского типа; дифференциальные формы; риманово многообразие; оператор Лапласа – Бельтрами; численное решение.*

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*Поступила в редакцию* 7 мая 2023 г.