MSC 35G61 DOI: 10.14529/jcem230302

THE LINEAR OSKOLKOV SYSTEM OF NON-ZERO ORDER IN THE AVALOS–TRIGGIANI PROBLEM

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The Avalos–Triggiani problem for a linear Oskolkov system of non-zero order and a system of wave equations is investigated. The mathematical model contains a linear Oskolkov system of non-zero order and a wave vector equation corresponding to some structure immersed in the Kelvin–Voight incompressible viscoelastic fluid. The theorem of the existence of the unique solution to the Avalos–Triggiani problem for the indicated systems is proved using the method proposed by the authors of this problem. The results of this article generalize the results received earlier.

Keywords: Avalos–Triggiani problem; incompressible viscoelastic fluid; linear Oskolkov system.

Formulation of the problem

Let Ω be a bounded domain in $\mathbb{R}^n, n = 2, 3$, with sufficiently smooth boundary $\partial\Omega$. Let $u = \mathrm{col}(u_1, u_2, ..., u_n)$ be a n-dimensional velocity vector $n = 2, 3$, the scalar function p be a pressure, and the vector $w = \mathrm{col}(w_1.w_2, ..., w_n)$ be a vector of displacement of a body, which occupies the domain Ω_s, and is immersed in a fluid occupying the domain Ω_f. Therefore, $\Omega = \Omega_s \cup \Omega_f, \overline{\Omega}_s \cap \overline{\Omega}_f = \partial\Omega_s \equiv \Gamma_s$ is the common boundary of Ω_s, and Ω_f. Let us denote the outer boundary of Ω_f by Γ_f (see Fig. 1). Our goal is to investigate the Avalos–Triggiani problem [1, 2] for the case when the fluid in Ω_f is an incompressible viscoelastic Kelvin–Voigt fluid of the non-zero order [3]. The mathematical model in question is defined by the system

$$(1 - \lambda\nabla^2)u_t - \eta\nabla^2 u - \sum_{l=1}^{K} \beta_l \nabla^2 \mathbf{w}_l + \nabla p = 0 \qquad \forall (t, x) \in \mathbb{R} \times \Omega_f \equiv \Omega_{\mathbb{R}f}, \tag{1}$$

$$\frac{\partial \mathbf{w}_l}{\partial t} = u + \alpha_l \mathbf{w}_l, \qquad \alpha_l \in \mathbb{R}_-, \qquad \beta_l \in \mathbb{R}_+, \qquad l = \overline{1,\ K}, \qquad \forall (t, x) \in \Omega_{\mathbb{R}f}, \tag{2}$$

$$\nabla \cdot u = 0, \qquad \forall (t, x) \in \Omega_{\mathbb{R}f}, \tag{3}$$

$$w_{tt} - \nabla^2 w + w = 0 \qquad \forall (t, x) \in \mathbb{R} \times \Omega_s \equiv \Omega_{\mathbb{R}s} \tag{4}$$

with the boundary value conditions

$$u|_{\Gamma_f} \equiv 0, \qquad \forall (t, x) \in \mathbb{R} \times \Gamma_f \equiv \Gamma_{\mathbb{R}f}, \tag{5}$$

$$\mathbf{w}_l|_{\Gamma_f} \equiv 0, \qquad \forall (t, x) \in \Gamma_{\mathbb{R}f}, \tag{6}$$

$$u \equiv w_t, \qquad \forall (t, x) \in \mathbb{R} \times \Gamma_s \equiv \Gamma_{\mathbb{R}s}, \tag{7}$$

$$\frac{\partial u}{\partial \nu} - \frac{\partial w}{\partial \nu} = p\nu \qquad \forall (t,x) \in \Gamma_{\mathbb{R}s} \tag{8}$$

and the initial value condition

$$(w(0,\cdot), w_t(0,\cdot), \mathbf{w}_1(0,\cdot), \ldots, \mathbf{w}_K(0,\cdot), u(0,\cdot)) = (w_0, w_1, \mathbf{w}_{10}, \ldots, \mathbf{w}_{K0}, u_0) \in \mathbf{H}, \tag{9}$$

where $\mathbf{H} = (H^1(\Omega_s))^n \times (L^2(\Omega_s))^n \times \mathcal{H}_1 \times \ldots \times \mathcal{H}_K \times \mathcal{H}_f$ and $\mathcal{H}_l = (L^2(\Omega_s))^n, l = \overline{1,\ K}$, $\mathcal{H}_f = \{f \in (L^2(\Omega_f))^n : \nabla \cdot f = 0 \text{ in } \Omega_f \text{ and } [f \cdot \nu]|_{\Gamma_f} = 0\}$.

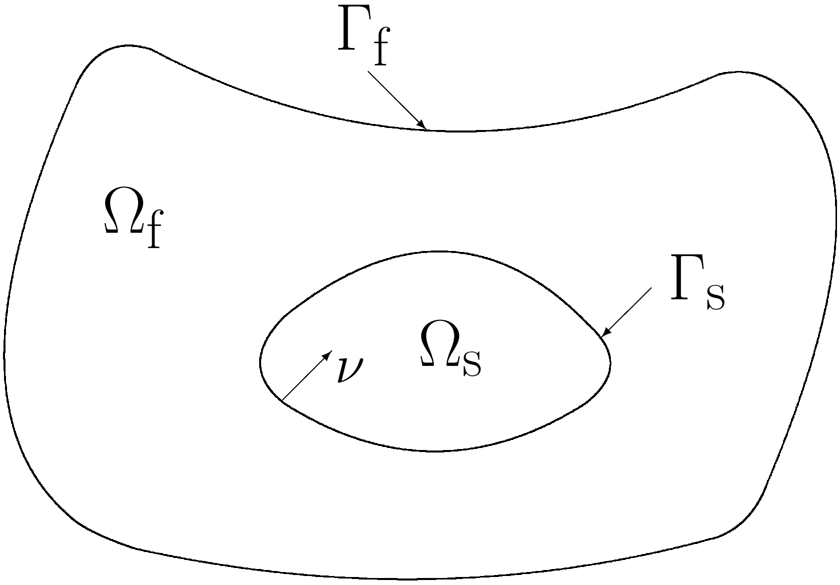

Fig. 1. Physical model

In system (1), the parameters λ and η characterize the elastic and viscous properties of the fluid, respectively, the parameters $\beta_l,\ l = \overline{1,\ K}$ determine the time of pressure retardation (delay), ν is a single normal vector. In the case of $K = 0, \lambda = 0$, problem (1)–(8) was investigated in [1,2], and for $K = 0, \lambda \neq 0$ in [4,5]. The case of $K \neq 0, \lambda \neq 0$ is investigated for the first time and generalizes the results for the case $K = 0$ in [5].

1. Reduction to the Cauchy problem and its solvability

Following [1,2], we assume that $p(t)$ satisfies the following elliptic problem:

$$\begin{aligned} &\Delta p = 0 \quad \text{in} \quad \Omega_{\mathbb{R}f}, \\ &p = \frac{\partial u}{\partial \nu}\cdot\nu - \frac{\partial w}{\partial \nu}\cdot\nu \quad \text{on} \quad \Gamma_{\mathbb{R}s}, \\ &\frac{\partial p}{\partial \nu} = \Delta u \cdot \nu \quad \text{on} \quad \Gamma_{\mathbb{R}f}. \end{aligned} \tag{10}$$

Then the pressure p can be represented as follows:

$$p(t) = D_s\{(\frac{\partial u(t)}{\partial \nu}\cdot\nu - \frac{\partial w(t)}{\partial \nu}\cdot\nu)_{\Gamma_{\mathbb{R}s}}\} + N_f((\Delta u(t)\cdot\nu)_{\Gamma_{\mathbb{R}f}}) \quad \text{in} \quad \Omega_{\mathbb{R}f};$$

where the Dirichlet map D_s is defined by the relations

$$h = D_s(g) \Leftrightarrow \begin{cases} \Delta h = 0 \quad \text{in} \quad \Omega_f, \\ h = g \quad \text{on} \quad \Gamma_s, \\ \dfrac{\partial h}{\partial \nu} = 0 \quad \text{on} \quad \Gamma_f, \end{cases}$$

and the Neumann map N_f is defined by the relations

$$h = N_f(g) \Leftrightarrow \begin{cases} \Delta h = 0 \quad \text{in} \quad \Omega_f, \\ h = 0 \quad \text{on} \quad \Gamma_s, \\ \dfrac{\partial h}{\partial \nu} = g \quad \text{on} \quad \Gamma_f. \end{cases}$$

Then original system (1)–(4), which describes the interaction of the fluid and the body immersed in the fluid, may be presented in the form

$$(1-\lambda\nabla^2)u_t - \eta\nabla^2 u - \sum_{l=1}^{K}\beta_l\nabla^2\mathbf{w}_l - G_1 w - G_2 u = 0 \qquad \forall (t,x) \in \Omega_{\mathbb{R}f}, \tag{11}$$

$$\frac{\partial \mathbf{w}_l}{\partial t} = u + \alpha_l \mathbf{w}_l, \qquad \alpha_l \in \mathbb{R}_-, \qquad \beta_l \in \mathbb{R}_+, \qquad l = \overline{1,\ K}, \tag{12}$$

$$\nabla \cdot u = 0, \tag{13}$$

$$w_{tt} - \nabla^2 w + w = 0 \qquad \forall (t,x) \in \Omega_{\mathbb{R}s} \tag{14}$$

with the boundary value conditions

$$u|_{\Gamma_f} \equiv 0, \qquad \forall (t,x) \in \Gamma_{\mathbb{R}f}, \tag{15}$$

$$\mathbf{w}_l|_{\Gamma_f} \equiv 0, \qquad \forall (t,x) \in \Gamma_{\mathbb{R}f}, \tag{16}$$

$$u \equiv w_t, \qquad \forall (t,x) \in \Gamma_{\mathbb{R}s}, \tag{17}$$

where

$$G_1 w \equiv \nabla\{D_s\{(\frac{\partial w(t)}{\partial \nu} \cdot \nu)_{\Gamma_{\mathbb{R}s}}\}\} \quad \text{in} \quad \Omega_{\mathbb{R}f},$$

$$G_2 u \equiv -\nabla\{D_s\{(\frac{\partial u(t)}{\partial \nu} \cdot \nu)_{\Gamma_{\mathbb{R}s}}\} + N_f((\Delta u(t) \cdot \nu)_{\Gamma_{\mathbb{R}f}})\} \quad \text{in} \quad \Omega_{\mathbb{R}f}.$$

Let us rewrite problem (11)–(17), in which pressure is excluded, in the form of an abstract Cauchy problem:

$$L\dot{v} = Mv, \quad v(0) = v_0, \tag{18}$$

where the operators L and M are defined by the matrices respectively

$$\begin{pmatrix} I & O & O & O & \dots & O & O \\ O & I & O & O & \dots & O & O \\ O & O & I & O & \dots & O & O \\ O & O & O & I & \dots & O & O \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ O & O & O & O & \dots & I & O \\ O & O & O & O & \dots & O & A_\lambda \end{pmatrix}, \begin{pmatrix} O & I & O & O & \dots & O & O \\ \Delta - I & O & O & O & \dots & O & O \\ O & O & \alpha_1 & O & \dots & O & I \\ O & O & O & \alpha_2 & \dots & O & I \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ O & O & O & O & \dots & \alpha_K & I \\ G_1 & O & \beta_1\Delta & \beta_2\Delta & \dots & \beta_K\Delta & \eta\Delta + G_2 \end{pmatrix}$$

and $v = \mathrm{col}(w, w_t, \mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_K, u)$, $A_\lambda = 1-\lambda\nabla^2$, I is a unit operator whose domain is clear out of context. We study problem (18) based on the results obtained in [6–9].

Lemma 1. *Let $\lambda \in \mathbb{R}$, $\eta \in \mathbb{R}_+$, the operators L and M be linear continuous operators from $\boldsymbol{G}$ to $\boldsymbol{H}$ ($L, M \in \mathcal{L}(\boldsymbol{G}, \boldsymbol{H})$), then there exists $L^{-1} \in \mathcal{L}(\boldsymbol{H})$. Here is the space $\boldsymbol{G} = (H^2(\Omega_s))^n \times (H^2(\Omega_s))^n \times \mathcal{G}_1 \times \ldots \times \mathcal{G}_K \times \mathcal{G}_f$, where $\mathcal{G}_l = (H^2(\Omega_s))^n, l = \overline{1,\ K}$, $\mathcal{G}_f$ is closure according to the norm of the space $(H^2(\Omega_s))^n$ spaces of infinitely differentiable solenoid functions such that (15)–(17) are fulfilled.*

Theorem 1. *For any $\lambda \in \mathbb{R}, \eta \in \mathbb{R}_+$ and $v_0 \in \boldsymbol{G}$, there is a unique solution to the problem (18) $v \in C^\infty(\mathbb{R}, \boldsymbol{G})$*

Remark 1. Received results can be generalized to the Avalos-Triggiani problem with the linear Oskolkov system of the highest order [3].

Remark 2. We intend to develop our research in the direction indicated in [10–12].

This work was supported by the laboratory of differential equations and mathematical physics of the Center for Applied Mathematics of the Yaroslav the Wise Novgorod State University. The authors express their gratitude to Professor Georgy A. Sviridyuk for his attention to the work and discussion of the results.

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Received May 7, 2023

УДК 517.958 **DOI: 10.14529/jcem230302**

ЛИНЕЙНАЯ СИСТЕМА ОСКОЛКОВА НЕНУЛЕВОГО ПОРЯДКА В ЗАДАЧЕ АВАЛОС – ТРИДЖИАНИ

А.О. Кондюков, Т.Г. Сукачева

Исследуется задача Авалос – Триджиани для линейной системы Осколкова ненулевого порядка и системы волновых уравнений. Математическая модель содержит линейную систему Осколкова ненулевого порядка и волновое уравнение, соответствующее некоторой структуре, погруженной в несжимаемую вязкоупругую жидкость Кельвина–Фойгта. С помощью метода, предложенного авторами этой задачи, доказывается теорема существования единственного решения задачи Авалос–Триджиани для указанных систем. Результаты данной статьи обобщают результаты, полученные ранее.

Ключевые слова: задача Авалос – Триджиани; несжимаемая вязкоупругая жидкость; линейная система Осколкова.

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Поступила в редакцию 7 мая 2023 г.