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OPTIMAL CONTROL OF SOLUTIONS TO THE CAUCHY PROBLEM FOR AN INCOMPLETE SEMILINEAR SOBOLEV TYPE EQUATION OF THE SECOND ORDER

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The paper investigates the problem of optimal control of solutions to the Cauchy and Showalter–Sidorov problem for an incomplete semilinear second order Sobolev type equation in Banach spaces. Sobolev type equations are understood as operator-differential equations with an irreversible operator at the highest time derivative. Based on the theorem on the existence and uniqueness of a solution to an inhomogeneous equation, a theorem on the existence of a solution to the optimal control problem is proved. The solution is formally presented as a Galerkin sum and then, based on a priori estimates, the convergence of the Galerkin approximations in the *-weak sense is proved. To illustrate the abstract theory, a study of the optimal control problem in a mathematical model of wave propagation in shallow water under the condition of conservation of mass in the layer and taking into account capillary effects is presented. This mathematical model is based on the IMBq equation and the Dirichlet boundary conditions.

Keywords: mathematical model; modified Boussinesq equation; optimal control problem; numerical study; semilinear equation of Sobolev type of the second order.

Introduction

Let $\Omega \subset \mathbb{R}^n$ be a domain with boundary $\partial\Omega$ of class C^∞, $T \in \mathbb{R}_+$. In the cylinder $\Omega \times (0,T)$ consider inhomogeneous modified Boussinesq equation (IMBq)

$$(\lambda - \Delta)x_{tt} - \alpha^2 \Delta x - \Delta(x^3) = u(s,t), \quad (s,t) \in \Omega \times (0,T) \tag{1}$$

with homogeneous Dirichlet boundary condition

$$x(s,t) = 0, \quad (s,t) \in \partial\Omega \times (0,T), \tag{2}$$

where $\lambda, \alpha \in \mathbb{R}$.

Equation (1) has many applications in various fields of natural science. For example, it models the propagation of waves in shallow water, taking into account capillary effects. In this case, the function $x = x(s,t)$ determines the height of the wave. In [1] a (modified) mathematical model of wave propagation in shallow water in a one-dimensional domain was studied and a soliton solution to equation (1) was obtained. In [2] the existence of a unique global solution to the Cauchy problem for equation (1) was proved, with $\lambda = 1, \alpha = 1$. In [3] the interaction of shock waves is studied using equation (1).

In all the works listed above, an essential condition is the continuous invertibility of the operator at the highest derivative with respect to the variable t. However, the operator $\lambda - \Delta$ can be degenerate. Equations that are not solvable with respect to the highest derivative with respect to time, according to [4], are usually called Sobolev type equations. Using the theory of relatively $\mathfrak{p}$-bounded operators developed by G.A. Sviridyuk and his students [5, 6], it is shown [7] that in suitably chosen spaces the problem (1)–(2) can be reduced to an abstract semilinear Sobolev type equation of the second order

$$L\ddot{x} + Mx + N(x) = u, \tag{3}$$

where $\ddot{x}$ is the second derivative with respect to t.

Then, using the phase space method, a theorem on the existence of a unique local solution to the Cauchy problem and the Showalter–Sidorov problem is proved, and it is also shown that if the operator N is monotonic, the phase space is a simple manifold.

In this paper we study the problem of optimal control of solutions to the Cauchy problem

$$x(0) = x_0, \quad \dot{x}(0) = x_1 \tag{4}$$

and solutions to the Showalter-Sidorov problem

$$P(x(0) - x_0) = 0, \quad P(\dot{x}(0) - x_1) = 0 \tag{5}$$

for equation (3). Here P is some spectral projector along the kernel of the operator L. Obviously, problem (5) is more general than (4). In trivial case (existence of the inverse operator L) both problems coincide, that means that their solutions also coincide. A detailed review [8] shows that the Showalter–Sidorov problem for the Sobolev type equations is more natural than the Cauchy problem.

To formulate the optimal control problem, introduce the control space $\mathfrak{U}$ and select a non-empty, closed and convex set $\mathfrak{U}_{ad}$ in it, witch we called the set of admissible controls. Let us pose the optimal control problem as a condition for minimizing the functional

$$J(x, u) \to \inf, u \in \mathfrak{U}_{ad}. \tag{6}$$

The specific type of functional will be determined later.

The optimal control problem allows you to balance between proximity to the desired state and the amount of labor and energy costs. In Sobolev type models, the optimal control problem was first considered in [9]. For semilinear Sobolev type models of the first order, the optimal control problem was studied in [10, 11]. Problems of optimal control of oscillatory phenomena arise in such technical problems as problems of calming the motion of a ship, a crane boom, organizing vibration protection, and others. The importance of solving problems of optimal control of oscillatory processes has already been repeatedly noted in [12–15].

1. Phase space method

Previously, problem (4), (5) was studied by methods of the theory of relatively $\mathfrak{p}$-bounded operators. Let us present some of its statements. Let $\mathfrak{X}, \mathfrak{Y}$ be Banach spaces, the operator $L \in \mathcal{L}(\mathfrak{X}; \mathfrak{Y})$ (i.e. linear and continuous), and the operator $M \in \mathcal{C}l(\mathfrak{X}; \mathfrak{Y})$ (linear, closed and densely defined). The set

$$\rho^L(M) = \{\mu \in \mathbb{C} : (\mu L - M)^{-1} \in \mathcal{L}(\mathfrak{Y}; \mathfrak{X})\}$$

is called the *resolvent set* of the operator M with respect to the operator L (or, *L-resolvent set* operator of the M). The set $\mathbb{C} \backslash \rho^L(M) = \sigma^L(M)$ is called the *spectrum* of the operator M with respect to the operator L (or, *L-spectrum* operator of the M).

Operator functions $(\mu L - M)^{-1}, R^L_\mu = (\mu L - M)^{-1}L, L^L_\mu = L(\mu L - M)^{-1}$ with the domain $\rho^L(M)$ is called, respectively, *resolvent*, *right resolvent*, *left resolvent* operator of the M with respect to the operator L (in short, *L-resolvent, right L-resolvent, left L-resolution of the operator M*).

The operator M is called (L, σ)-*bounded* if

$$\exists a > 0\ \forall \mu \in \mathbb{C} : (|\mu| > a) \Rightarrow (\mu \in \rho^L(M)).$$

Let the operator M be (L, σ)-bounded. Then the operators

$$P = \frac{1}{2\pi i}\int\limits_\Gamma R^L_\lambda(M)d\lambda \text{ and } Q = \frac{1}{2\pi i}\int\limits_\Gamma L^L_\lambda(M)d\lambda$$

are projectors in the spaces $\mathfrak{X}$ and $\mathfrak{Y}$, respectively. Here $\Gamma = \{\lambda \in \mathbb{C} : |\lambda| = r > a\}$. Further, by virtue of the Splitting Theorem [6], the projectors P and Q split the spaces $\mathfrak{X}$, $\mathfrak{Y}$ into a direct sums: $\mathfrak{X} = \mathfrak{X}^0 \oplus \mathfrak{X}^1$ and $\mathfrak{Y} = \mathfrak{Y}^0 \oplus \mathfrak{Y}^1$, where $\ker P = \mathfrak{X}^0$, $\operatorname{im} P = \mathfrak{X}^1$, $\ker Q = \mathfrak{Y}^0$, $\operatorname{im} Q = \mathfrak{Y}^1$. And the operators $H = M_0^{-1}L_0 \in \mathcal{L}(\mathfrak{X}^0)$, $S = L_1^{-1}M_1 \in \mathcal{L}(\mathfrak{X}^1)$. If M is (L, σ)-*bounded* and the operator H is nilpotent of degree $\mathfrak{p}$, then the operator M is called $(L, \mathfrak{p})$-*bounded* .

Let $\mathfrak{P}$ be some Riemannian manifold without boundary modeled by the space $\mathfrak{X}$. We say that a pair (x, y) belongs to the tangent bundle of the set $\mathfrak{P}$ if x belongs to $\mathfrak{P}$ and the pair (x, y) belongs to the tangent space $T_x\mathfrak{P}$ at point x i.e.

$$(x, y) \in T\mathfrak{P} \Leftrightarrow x \in \mathfrak{P} \wedge (x, y) \in T_x\mathfrak{P}.$$

Call the set $\mathfrak{P}$ the phase space of equation (3) if
1) for any $(x_0, x_1) \in T\mathfrak{P}$ there is a unique solution to problem (3), (4);
2) any solution $x = x(t)$ to equation (3) lies in $\mathfrak{P}$ as a trajectory.

Let $\ker L \neq \{0\}$ and the operator M be $(L, 0)$-bounded, then, by virtue of the splitting theorem [6], equation (3) can be reduced to the equivalent system of equations

$$\begin{cases} (\mathbb{I} - Q)(M + N)(x) = (\mathbb{I} - Q)u, \\ \ddot{x}^1 = L_1^{-1}Q(M + N)(x) + L_1^{-1}Qu, \end{cases}$$

where $x^1 = Px$. Let $(\mathbb{I} - Q)u$ be independent of t, then the set

$$\mathfrak{P} = \{x \in \mathfrak{X} : (\mathbb{I} - Q)(M + N)(x) = (\mathbb{I} - Q)u\}$$

is the local phase manifold of equation (3). Note that is, in [7] the existence of a unique local solution was proven.

In addition, we need an auxiliary lemma.

Lemma 1.1. ([16]) If $f \in L^p(0, T; \mathfrak{X})$ and $\dot{f} \in L^p(0, T; \mathfrak{X})$ $(1 \leq p \leq \infty)$, then f, after perhaps changing on a zero measure set from the interval $(0, T)$, will be a continuous mapping $[0, T]$ in $\mathfrak{X}$.

2. Solving the Initial Problem for an Inhomogeneous Equation

First, we need to solve problem (3), (4). To do this, construct several functional spaces. Let $H = (H, \langle \cdot, \cdot \rangle)$ be a real separable Hilbert space. Let us define dual pairs of reflexive Banach spaces $(\mathfrak{X}, \mathfrak{X}^*)$ and (L^p, L^q) with respect to duality $\langle \cdot, \cdot \rangle$ such that there is a chain of dense and continuous embeddings

$$L^p \hookrightarrow \mathfrak{X} \hookrightarrow H \hookrightarrow \mathfrak{X}^* \hookrightarrow L^q. \tag{7}$$

In the given spaces define operators L, M, N satisfying the following conditions:
(C1) $L \in \mathcal{L}(\mathfrak{X}, \mathfrak{X}^*)$ is self-adjoint non-negatively defined and Fredholm operator;
(C2) $M \in \mathcal{L}(\mathfrak{X}, \mathfrak{X}^*)$ is self-adjoint non-negatively defined;
(C3) $N \in C^r(L^p, L^q), r \geq 1$ is s-monotone, p-coercive and homogeneous of order $p-1$ operator, with symmetric Frechet derivative.
Due to condition **(C3)**, the operator N satisfies the equality

$$\frac{d}{dt}\langle N(x), x \rangle = (p-1)\langle N(x), \dot{x} \rangle.$$

In addition, define the spaces of distributions (of functions with values in a Banach space) $L^\infty(0, T; \mathfrak{X} \cap L^p)$ and $L^\infty(0, T; \mathrm{coim} L)$, where $\mathfrak{X} = \ker L \oplus \mathrm{coim} L$. The spaces conjugate to them are constructed using the Dunford–Pettis theorem: $(L^\infty(0, T; \mathfrak{X} \cap L^p))^* \simeq L^1(0, T; \mathfrak{X}^* \cup L^q)$ and $(L^\infty(0, T; \mathrm{coim} L))^* \simeq L^1(0, T; \mathfrak{X}^*)$.

Let λ_k be the eigenvalues of the homogeneous Dirichlet problem for the operator L, numbered in nonincreasing order taking into account multiplicity, and let φ_k be the corresponding eigenfunctions, orthonormalized with respect to the inner product in H. Moreover, the linear span of $\{\varphi_1, \varphi_2, \ldots, \varphi_m\}$ for $m \to \infty$ is dense in $\mathfrak{X}$.

Theorem 2.1. Let conditions **(C1), (C2), (C3)** be satisfied and $u \in L^q(0, T; L^q)$. Then for any $(x_0, x_1) \in T_{x_0}\mathfrak{P}$ such that $x_0 \in \mathfrak{X} \cap L^p$, $x_1 \in \mathrm{coim} L$ there is a unique solution to problem (3), (4) $x = x(s, t)$ such that $x \in L^\infty(0, T; \mathfrak{X} \cap L^p)$ and $\dot{x} \in L^\infty(0, T; \mathrm{coim} L \cap \mathfrak{X})$.

Proof. Further symbol C will denote different constants. We will search a solution to problem (3), (4) in the form of the Galerkin approximation [16]

$$x^m(t) = \sum_{k=1}^{m} a_k^m(t)\varphi_k. \tag{8}$$

Find the coefficients $a_k^m(t)$ from the system of algebraic-differential equations

$$\langle L\ddot{x}^m, \varphi_k \rangle + \langle Mx^m, \varphi_k \rangle + \langle N(x^m), \varphi_k \rangle = \langle u, \varphi_k \rangle, \quad 1 \leq k \leq m. \tag{9}$$

$$\langle x^m(0), \varphi_k \rangle = \langle x_0, \varphi_k \rangle = \beta_k^m, \quad \langle \dot{x}^m(0), \varphi_k \rangle = \langle x_1, \varphi_k \rangle = \gamma_k^m, \quad 1 \leq k \leq m, \tag{10}$$

where $x_0^m = \sum\limits_{k=1}^{m} \beta_k^m \varphi_k \to x_0$ in $\mathfrak{X}$ for $m \to \infty$, and $x_1^m = \sum\limits_{k=1}^{m} \gamma_k^m \varphi_k \to x_1$ in $\mathfrak{X}$ for $m \to \infty$.

By classical results, there is a unique local solution $x^m = x^m(s, t)$, $t \in [0, t^m]$.

Let us obtain a priori estimates. Multiply equation (9) by $\dot{a}_k^m(t)$ $(1 \leq k \leq m)$ and sum over k from 1 to m. We get

$$\langle L\ddot{x}^m, \dot{x}^m \rangle + \langle Mx^m, \dot{x}^m \rangle + \langle N(x^m), \dot{x}^m \rangle = \langle u, \dot{x}^m \rangle. \tag{11}$$

In the space coim$L \cap \mathfrak{X}$ introduce the norm $|\dot{x}|^2 = \langle L\dot{x}, \dot{x}\rangle$. By virtue of the Courant principle this norm is equivalent to the norm in space $\mathfrak{X}$.

Using the self-adjointness of L, M, we obtain $2\langle L\ddot{x}^m, \dot{x}^m\rangle = \frac{d}{dt}\langle L\dot{x}^m, \dot{x}^m\rangle$, $2\langle Mx^m, \dot{x}^m\rangle = \frac{d}{dt}\langle Mx^m, x^m\rangle$. Due to the condition **(C3)** $(p-1)\langle N(x^m), \dot{x}^m\rangle = \frac{d}{dt}\langle N(x^m), x^m\rangle$. Then equation (11) takes the form

$$\frac{d}{dt}\left[|\dot{x}^m|^2 + \langle Mx^m, x^m\rangle + \frac{2}{p-1}\langle N(x^m), x^m\rangle\right] = 2\langle u, x^m\rangle. \tag{12}$$

Let us integrate it on the interval $[0,t], t \le t_m$

$$|\dot{x}^m|^2 + \langle Mx^m, x^m\rangle + \frac{2}{p-1}\langle N(x^m), x^m\rangle = 2\int_0^t \langle u, \dot{x}^m\rangle ds + |x_1^m|^2 + \langle Mx_0^m, x_0^m\rangle + \frac{2}{p-1}\langle N(x_0^m), x_0^m\rangle \tag{13}$$

Then

$$\begin{aligned} &|\dot{x}^m|^2 + \langle Mx^m, x^m\rangle + C_N\frac{2}{p-1}\|x^m\|^p_{L^p} \le \\ &\le 2\int_0^t \|u\|_{L^q}\|\dot{x}^m\|_{L^p}ds + |x_1^m|^2 + \langle Mx_0^m, x_0^m\rangle + C^N\frac{2}{p-1}\|N(x_0^m)\|^{p-1}_{L^q} \le \\ &\le \int_0^t \|u\|^2_{L^q}ds + \int_0^t \|\dot{x}^m\|^2_{L^p}ds + |x_1^m|^2 + \langle Mx_0^m, x_0^m\rangle + C^N\frac{2}{p-1}\|N(x_0^m)\|^{p-1}_{L^q}. \end{aligned} \tag{14}$$

Therefore

$$|\dot{x}^m|^2 \le C + \int_0^t \|\dot{x}^m\|^2_{L^p}ds \le C + \int_0^t \|\dot{x}^m\|^2_{\mathfrak{X}}ds \le C + \int_0^t |\dot{x}^m|^2 ds,$$

and, by virtue of Gronwall's inequality,

$$|\dot{x}^m|^2 \le Ce^t \le C, \quad t \in [0,T].$$

Since the right-hand side of the inequality (14) is bounded, then the inequality

$$|\dot{x}^m|^2 + \langle Mx^m, x^m\rangle + C_N\frac{2}{p-1}\|x^m\|^p_{L^p} \le C \tag{15}$$

holds. The constant C does not depend on t_m and, therefore, $t_m = T$.

Remark 2.1. The sequences x^m and $\dot{x}^m$ are bounded in the spaces $L^\infty(0,T;\mathfrak{X}\cap L^p)$ and $L^\infty(0,T;\mathrm{coim}L)$, respectively, being the dual spaces for separable Banach spaces $L^1(0,T;\mathfrak{X}^* \cup L^q)$ and $L^1(0,T;\mathfrak{X}^*)$. Therefore, we can choose *-weakly convergent subsequences x^{m_l} and $\dot{x}^{m_l}$ in them such that
$x^{m_l} \to x$ *-weakly in $L^\infty(0,T;\mathfrak{X}\cap L^p)$
$\dot{x}^{m_l} \to \dot{x}$ *-weakly in $L^\infty(0,T;\mathrm{coim}L)$. Here, $\dot{x}^{m_l}$ is understood as a generalized derivative in the space of distributions.

Since the operator N is p-coercive then

$$\langle N(x^m), x^m\rangle \le \|N(x^m)\|_{L^q}\|x^m\|_{L^p} \le C^N\|x^m\|^{p-1}_{L^p}\|x^m\|_{L^p} = C^N\|x^m\|^p_{L^p}, \qquad (16)$$

and, therefore, $N(x^m)$ are bounded in the space $L^q(0,T;\mathfrak{X}^*\cup L^q)$, then $N(x^{m_l}) \to g$ weakly in $L^q(0,T;\mathfrak{X}^*\cup L^q)$.

Let us show that $N(x) = g$. From the monotonicity of the operator N it follows that

$$X^{m_l} = \langle N(x^{m_l}) - N(z), x^{m_l} - z\rangle \ge 0, \quad \forall z \in L^p(0,T;\mathfrak{X}\cap L^p).$$

By (13)

$$X^{m_l} = 2\int\limits_0^t \langle u, \dot{x}^{m_l}\rangle ds + |x_1^{m_l}|^2 - |\dot{x}^{m_l}|^2 + \langle Mx_0^{m_l}, x_0^{m_l}\rangle - \langle Mx^{m_l}, x^{m_l}\rangle + \frac{2}{p-1}\langle N(x_0^{m_l}), x_0^{m_l}\rangle -$$

$$-\frac{2}{p-1}\langle N(x^{m_l}), z\rangle - \langle N(z), x^{m_l} - z\rangle.$$

Due to the properties of weakly convergent sequences, we have $\liminf\limits_{m_l\to\infty} |x^{m_l}(t)|^2 \ge |x(t)|^2$, therefore

$$\limsup X^{m_l} \le \langle g, x\rangle - \langle g, z\rangle - \langle N(z), x - z\rangle.$$

Then we get

$$\langle g, x\rangle - \langle g, z\rangle - \langle N(z), x - z\rangle \ge 0.$$

Put $z = x - hw, h > 0, w \in L^p(0,T;B)$, then

$$\langle g - N(x - hw), hw\rangle \ge 0,$$

$$\langle g - N(x - hw), w\rangle \ge 0.$$

Letting $h \to 0$, due to the continuity of N and Lebesgue theorem on the majorizing sequence, we obtain

$$\langle g - N(x), w\rangle \ge 0.$$

Due to the arbitrariness of the choice of w, we have $g = N(x)$.

Now we can go to the limit in equality (9), directing m_l to infinity. Let k be fixed and $m_l > k$, we get

$$\langle L\ddot{x}^{m_l}, \varphi_k\rangle + \langle Mx^{m_l}, \varphi_k\rangle + \langle N(x^{m_l}), \varphi_k\rangle = \langle u, \varphi_k\rangle. \qquad (17)$$

Thus, from (17) we get

$$\frac{d^2}{dt^2}\langle Lx, \varphi_k\rangle + \langle Mx, \varphi_k\rangle + \langle N(u), \varphi_k\rangle = \langle u, \varphi_k\rangle. \qquad (18)$$

Due to the density of the system of functions $\{\varphi_k\}_{k=1}^m$ in the space $\mathfrak{X}$ for $m \to \infty$, and the arbitrariness of the choice of φ_k, equality (18) holds for arbitrary $v \in \mathfrak{X}$

$$\frac{d^2}{dt^2}\langle Lx, v\rangle + \langle Mx, v\rangle + \langle N(x), v\rangle = \langle u, v\rangle. \qquad (19)$$

Due to the expansion of the initial functions in the series $x^{m_l}(0) = x_l^0 \to x_0$ weakly in $\mathfrak{X}$ and by Remarks 2.1 $x^{m_l}(0) \to x(0)$ in $\mathfrak{X}$, therefore $x(0) = x_0$.

Due to Remark 2.1

$$\langle \ddot{x}^{m_l}, \varphi_k \rangle \to \langle \ddot{x}, \varphi_k \rangle \quad \text{*-weakly in } L^\infty(0,T)$$

and, therefore, taking into account Lemma 1.1, we get

$$\langle \dot{x}^{m_l}(0), \varphi_k \rangle \to \langle \dot{x}(t), \varphi_k \rangle|_{t=0} = \langle \dot{x}(0), \varphi_k \rangle.$$

On the other hand, due to the expansion of the initial functions in the series

$$\langle \dot{x}^{m_l}(0), \varphi_k \rangle \to \langle x_1, \varphi_k \rangle.$$

Thus,

$$\langle \dot{x}(0), \varphi_k \rangle = \langle x_1, \varphi_k \rangle, \quad \forall\, k.$$

Thus, the function $x = x(s,t)$ satisfies the equation and initial conditions, i.e. it is a solution to (3), (4).

Uniqueness is proved by contradiction. Let $x_1(s,t)$ and $x_2(s,t)$ be two different solutions to problem (3), (4). Denote $w(s,t) = x_1(s,t) - x_2(s,t)$. Repeating the reasoning stated above, we arrive at the inequality

$$|\dot{w}^m|^2 \le \int_0^t \|\dot{w}^m\|^2_{L^p} ds \le \int_0^t \|\dot{w}^m\|^2_{\mathfrak{X}} ds \le \int_0^t |\dot{w}^m|^2 ds,$$

and by virtue of Gronwall's inequality,

$$|\dot{w}^m|^2 \le 0, \quad t \in [0,T].$$

Taking into account the zero initial condition, we obtain that $w(s,t) = 0$.

It is not difficult to prove the following theorem about the existence of a solution to the Showalter – Sidorov problem.

Theorem 2.2. Let conditions **(C1), (C2), (C3)** be satisfied and $u \in L^q(0,T;L^q)$. Then for any $x_0 \in \mathfrak{X} \cap L^p$, $x_1 \in \operatorname{coim} L$, there is a solution $x = x(s,t)$ to problem (3), (5) such that $x \in L^\infty(0,T;\mathfrak{X} \cap L^p)$ and $\dot{x} \in L^\infty(0,T;\mathfrak{X})$.

3. Optimal Control Problem

Consider the optimal control problem (3), (4), (6). Construct the space $\mathfrak{U} = L^2(0,T;L^q)$ and define a non-empty closed and convex subset $\mathfrak{U}_{ad}$ in it. Construct the space $\mathfrak{X}_1 = \{x | x \in L^\infty(0,T;\mathfrak{X} \cap L^p), \dot{x} \in L^\infty(0,T;\operatorname{coim} L \cap \mathfrak{X})$.

Definition 3.1. The pair $(\tilde{x}, \tilde{u}) \in \mathfrak{X}_1 \times \mathfrak{U}_{ad}$ is called *a solution to the optimal control problem* if

$$J(\tilde{x}, \tilde{u}) = \inf_{(x,u)} J(x,u),$$

where the pairs $(x,u) \in \mathfrak{X}_1 \times \mathfrak{U}_{ad}$ satisfy problem (3), (4). The function $\tilde{u}$ is called *an optimal control.*

Remark 3.1. The pair $(x, u) \in \mathfrak{X}_1 \times \mathfrak{U}_{ad}$ satisfying problem (3), (4) for which $J(x, u) < +\infty$ is called admissible element of the problem (3), (4), (6). Since the set is $\mathfrak{U}_{ad} \neq \varnothing$, then for any $u \in \mathfrak{U}_{ad} \subset \mathfrak{U}$ by Theorem 2.1. there is a unique solution $x = x(u)$ to problem (3), (4).

Let us formulate and prove a theorem on the existence of optimal control.

Theorem 3.1. Let conditions **(C1), (C2), (C3)** be satisfied. Then for any $(x_0, x_1) \in T\mathfrak{P}$, $T \in \mathbb{R}_+$, there is a solution to problem (3), (4), (6).

Proof. From the theorem on the existence of a unique solution to problem (3), (4) it follows that the operator

$$\left(L\frac{d}{dt^2} + M + N(\cdot)\right) : \mathfrak{X}_1 \to \mathfrak{U}$$

is a homeomorphism. Thus we can define the penalty functional used in (6) in the form

$$J(x, u) = J(u) = \beta(\|x(t) - z(t)\|^p_{L^p(0,T;\mathfrak{X})} + \|\dot{x}(t) - \dot{z}(t)\|^2_{L^2(0,T;\mathfrak{X})}) + \\ +(1 - \beta)\|u(t)\|^2_{L^2(0,T;L^q)}. \tag{20}$$

Let $\{u_m\} \subset \mathfrak{U}_{ad}$ be a sequence such that

$$\lim_{m\to+\infty} J(u_m) = \inf_{u\in\mathfrak{U}_{ad}} J(u).$$

Then from (20) it follows that

$$\|u(t)\|^2_{L^2(0,T;L^q)} \leq C \tag{21}$$

for all $m \in \mathbb{N}$. From (21) it follows that from the sequence $\{u_m\}$ we can choose a weakly convergent subsequence, which is denoted by $\{u_m\}$ again, such that $u_m \to \tilde{u}$. By Mazur's theorem, $\tilde{u} \in \mathfrak{U}_{ad}$. Denote by $x_m = x(u_m)$ the weakly generalized solution of equation

$$L\ddot{x}_m + Mx_m + N(x_m) = u_m. \tag{22}$$

A priori estimates can be obtained in the same way as in Theorem 2.1. Now we can go to the limit in equality (22), directing m to infinity

$$L\ddot{\tilde{x}} + M\tilde{x} + N(\tilde{x}) = \tilde{u}. \tag{23}$$

Thus, $\tilde{x} = \tilde{x}(\tilde{u})$ and $\liminf\limits_{m\to\infty} J(x_m) \leq J(\tilde{x})$. Therefore, $\tilde{x}$ is a solution to the optimal control problem (3), (4), (6).

It is not difficult to prove the theorem on the existence of a solution to the problem of optimal control of solutions to the Showalter–Sidorov problem (5). So, we can formulate the theorem

Theorem 3.2. Let conditions **(C1), (C2), (C3)** be satisfied. Then for any$x_0, x_1 \in \mathfrak{X}$, $T \in \mathbb{R}_+$, there is a solution to optimal control problem (3), (5), (6).

4. Application to the Study of Mathematical Model

Reduce problem (1), (2), (4) to problem (3), (4). For the space H take the Sobolev space $\overset{\circ}{W}{}_2^{-1}(\Omega)$ with inner product

$$\langle f_1, f_2\rangle = \int\limits_\Omega f_1\tilde{f}_2 ds, \quad \forall f_1, f_2 \in \overset{\circ}{W}{}_2^{-1}(\Omega), \tag{24}$$

where $\tilde{f_1}$ is a generalized solution to the Dirichlet problem (2) for the equation $\Delta\tilde{f_1} = f_1$ in the domain Ω.

Set $\mathfrak{X} = L^2$ and $\mathfrak{X}^* = (L^2)^*$ with respect to the duality (24). Then the chain of dense and continuous embeddings (7) takes place. Define the operator $L : \mathfrak{X} \to \mathfrak{X}^*$ by formula

$$\langle Lx, v\rangle = \int\limits_{\Omega} (xv + \lambda x\tilde{v})ds.$$

Condition **(C1)** is satisfied for $\lambda \geq \lambda_1$.

Condition **(C2)** is satisfied if the operator $M : \mathfrak{X} \to \mathfrak{X}^*$ is defined by formula

$$\langle Mx, v\rangle = \alpha^2 \int\limits_{\Omega} xvds.$$

Define the operator $N(x) : L^4 \to L^{\frac{4}{3}}$ by formula

$$\langle N(x), v\rangle = \int\limits_{\Omega} x^3vds.$$

Its Frechet derivative

$$|\langle N'_x(v), w\rangle| = 3\left|\int\limits_{\Omega} x^2vwdx\right| \geq \text{const}\|x\|_{L^2}\|v\ |_{L^4}\|w\|_{L^4}$$

is symmetric and bounded due to the Hölder inequality.

The operator N is s-monotone

$$\langle N'_s(v), v\rangle = 3\int\limits_{\Omega} x^2v^2ds \geq 0$$

and 4-coercive

$$\langle N(x), x\rangle = \int\limits_{\Omega} x^3xds = \|x\|^4_{L^4}, \quad \langle N(x), v\rangle = \int\limits_{\Omega} u^3vds = \|x\|^3_{L^4}\|v\|_{L^4} = \|x^3\|_{L^{\frac{4}{3}}}\|v\|_{L^4}.$$

Thus, condition **(C3)** is satisfied.

Let v be an eigenfunction corresponding to the eigenvalue λ_1 of operator L and $(\mathbb{I}-Q)u$ be independent of t. Then the phase space has the form

$$\mathfrak{P} = \begin{cases} \{x \in \mathfrak{X} : \langle -\alpha^2\Delta x - \Delta(x^3), v\rangle = \langle u, v\rangle\}, & \lambda = \lambda_1, \\ \mathfrak{X}, \ \ \lambda > \lambda_1. \end{cases}$$

Theorem 4.1. For any $(x_0, x_1) \in T\mathfrak{P}$, $T \in \mathbb{R}_+$, there is a solution to problem (1), (2), (4), (6).

5. Computational Experiment for Optimal Control Problem

Present the results of information processing using the developed algorithm, which was implemented in the Maple environment. Information processing was carried out on the basis of computational experiments.

Example 5.1. Let the domain $\Omega = [0, \pi]$ and the parameters of the equation $\lambda = -1, \alpha = 2$. Then the inhomogeneous modified Boussinesq equation takes the form

$$(1 - \Delta)x_{tt}(s,t) - 4\Delta x(s,t) - \Delta(x^3(s,t)) = u(s,t), \quad (s,t) \in \quad [0,\pi] \times (0,T) \tag{25}$$

with homogeneous Dirichlet boundary condition

$$x(0,t) = x(\pi,t) = 0, \quad t \in (0,T) \tag{26}$$

and the Cauchy initial conditions

$$x(s,0) = \sin(s) - 0.5\sin(2s), \quad x_t(s,0) = \sin(2s), \quad s \in (0,\pi). \tag{27}$$

To solve control problem (25)–(27), (6) numerically, we use the decomposition method and linearize equation (25)

$$\begin{aligned}&(1 - \Delta)x_{tt}(s,t) - 4\Delta x(s,t) - \Delta(y^3(s,t)) = u(s,t),\\ &y(s,t) = x(s,t).\end{aligned}$$

Define the penalty functional from (6) as follows (the symbol $'$ denotes the derivative with respect to s).

$$\begin{aligned} J(x,u) = \beta\theta \int_0^T (\|x(s,t) - z(s,t)\|^4_{L^4} + \|x'(s,t) - z'(s,t)\|^2_{L^2})dt + \\ +\beta(1-\theta)\int_0^T (\|y(s,t) - z(s,t)\|^4_{L^4} + \|y'(s,t) - z'(s,t)\|^2_{L^2})dt + \\ +(1-\beta)\int_0^T \|u(t)\|^2_{\mathfrak{Y}} dt + r\int_0^T (\|y(s,t) - x(s,t)\|^4_{L^4} + \|y'(s,t) - x'(s,t)\|^2_{L^2})dt. \end{aligned} \tag{28}$$

Search the solution to problem (25)–(27), (6) in the form of an expansion into the Galerkin sum up to the second term by eigenfunctions of problem (26) for the Laplace operator

$$x(s,t) = \tfrac{2}{\pi}\sum_{k=1}^{2} a_k(t)\sin(ks),$$
$$y(s,t) = \tfrac{2}{\pi}\sum_{k=1}^{2} b_k(t)\sin(ks), \quad u(s,t) = \tfrac{2}{\pi}\sum_{k=1}^{2} c_k(t)\sin(ks).$$

The coefficients of the Galerkin sums for the auxiliary function $y(s,t)$ and the control function $u(s,t)$ according to the Ritz method are represented as polynomials

$$b_k(t) = \sum_{j=0}^{2} b_{kj}t^j, \quad c_k(t) = \sum_{j=0}^{2} c_{kj}t^j, \quad k = 1,2.$$

It must be taken into account that

$$b_{k0} = a_k(0), \quad b_{k1} = \frac{da_k}{dt}(0), \quad k = 1, 2.$$

Note that equation (25) is degenerate. In order to fulfilled the condition

$$(\mathbb{I} - Q)u \textit{ is not depend on } t, \quad \forall t \in (0, T).$$

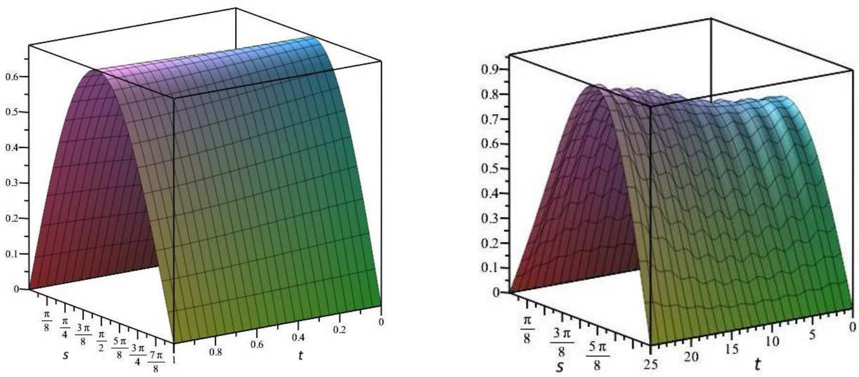
Set set $c_{11} = 0, c_{12} = 0$. Then the control function will take the form $u(s,t) = \frac{2}{\pi}(c_{10}\sin(s) + c_2(t)\sin(2s))$. The phase manifold of equation (25) has the form

$$\mathfrak{P} = \left\{ x(s,t) : 4a_1(t) + \left\langle \left(\frac{2}{\pi} \sum_{k=1}^{2} a_k(t)\sin(ks) \right)^3, \sin s \right\rangle = c_{10} \right\}. \tag{29}$$

Set the parameters θ, β from the interval $(0,1)$, and take the parameter r as large as possible, so that the solution $x(s,t)$ and the auxiliary function $y(s,t)$ are close enough, for example put $r = 100$.

Using the branch and bound method find the minimum value of the functional and its minimum point.

Using the developed method, the information was processed and the minimum value of the functional was found. For the time interval $[0,1]$ $J_{min} = 16.728$, and for the time interval $[0,25]$ $J_{min} = 2511.045$. In both cases, an approximate solution to the optimal control problem, i.e. a pair of functions: optimal control and system state was found.



a) b)

Fig. 1. Graph of function $x(s,t)$ a) for $t \in [0,1]$; b) for $t \in [0,25]$

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ОПТИМАЛЬНОЕ УПРАВЛЕНИЕ РЕШЕНИЯМИ ЗАДАЧИ КОШИ ДЛЯ НЕПОЛНОГО ПОЛУЛИНЕЙНОГО УРАВНЕНИЯ СОБОЛЕВСКОГО ТИПА ВТОРОГО ПОРЯДКА

А. А. Замышляева, Е. В. Бычков

В работе исследована задача оптимального управления решениями задачи Коши и Шоуолтера – Сидорова для неполного полулинейного уравнения соболевского типа второго порядка в банаховых пространствах. Под уравнениями соболевского типа понимаются операторно-дифференциальные уравнения с необратимым оператором при старшей производной по времени. На основе теоремы о существовании и единственности решения неоднородного уравнения доказана теорема о существовании решения задачи оптимального управления. Решение формально представляется в виде галеркинской суммы и затем, на основе априорных оценок, доказывается сходимость галеркинских приближений в *-слабом смысле. Для иллюстрации абстрактной теории проведено исследование задачи оптимального управления в математической модели распространения волн на мелкой воде при условии сохранения массы в слое и с учетом капиллярных эффектов. Данная математическая модель основана на уравнении IMBq, краевых условиях Дирихле.

Ключевые слова: математическая модель; модифицированное уравнение Буссинеска; задача оптимального управления; численное исследование; полулинейное уравнение соболевского типа второго порядка.

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