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METHOD OF ITERATIVE EXTENSIONS FOR ANALYSIS OF A SCREENED HARMONIC SYSTEMS

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In this paper, mixed boundary value problem for screened Poisson equation is considered in a geometrically complex domain. The asymptotically optimal method of iterative extensions is described. An analysis of screened harmonic system is carried out with the method of iterative extensions. An algorithm is written that implements the method of iterative extensions in matrix form. An example of calculating the bending of a membrane on an elastic base is given.

Keywords: fictitious domain method; method of iterative extensions; screened harmonic systems; screened Poisson equation.

Introduction

This work devotes to the development of a scientific direction, which bases on the method of iterative extensions for the analysis of a screened harmonic system – a mixed boundary value problem for an inhomogeneous screened Poisson equation on vertical displacements of membrane points located horizontally on elastic base, under the vertical pressure with homogeneous Dirichlet boundary condition and homogeneous Neumann boundary condition.

$$\breve{u}: \ -\Delta\breve{u} + \kappa\breve{u} = \breve{f}|_{\Omega}, \ \Omega \subset \mathbb{R}^2, \tag{1}$$
$$\breve{u}|_{\Gamma_1} = 0,$$
$$\frac{\partial\breve{u}}{\partial n}|_{\Gamma_2} = 0,$$

where

$$\partial\Omega = \overline{s}, \ s = \Gamma_1 \cup \Gamma_2, \ \Gamma_1 \cap \Gamma_2 = \emptyset.$$

Difficulties in solving these problems arise due to the complex geometry of the domain, the presence of the Dirichlet boundary condition, and the high order of differential equations. Promising for solving such problems fictitious domain method [3] turns out to be not asymptotically optimal. The authors use an explicit construction of operators for the continuation of discrete functions from a curvilinear boundary with preservation of the norm and obtain an asymptotically optimal fictitious domain method for a second-order elliptic equation. Without diminishing the achievements of this approach, we can say that this method is quite difficult to implement and is not universal. Thus, we can conclude that the solution to these problems have important practical applications, have problems associated with this method. For constructing new asymptotically optimal method, we use the well-known fictitious domain method. In this method, on example of physical system from theory of elasticity, we increase the response of the underlying surface and the rigidity of the material in continuation. We minimize the error in a stronger norm than the energy norm of $\check{f}$. Thus, to select the iterative parameters, we use the method of minimum residuals [4] and specify the conditions sufficient for the convergence of the iterative process. In this paper, we reduce the second-order elliptic problem to a discrete analogue of the screened Poisson equation in a rectangular domain with a homogeneous Dirichlet boundary condition on two adjacent sides and a homogeneous Neumann boundary condition on the other two sides of the rectangle. Solution to this problem exists, and is unique. The question of existence and uniqueness of such problem considered, for example, in [1, 2].

In this paper, we use reductions of boundary value problems in variational form to mathematical systems of a discrete form, which accurately preserves the properties of the original boundary value problems at the difference level, using the method of summation identities, the method of approximation by parts, and the finite element method [5].

We consider it important to reduce solutions of the studied boundary value problems to solutions of systems of linear algebraic equations with matrices. The number of non-zero elements in each row does not exceed five elements in these matrices. We can obtain solutions to these systems, using well-known marching methods [6, 7]. In this paper, we reduce the solutions to the original problems to the numerical solutions to discrete analogues of the screened Poisson equation in a rectangular domain, and finally to the solution to systems of linear algebraic equations with five-diagonal matrices.

The paper presents an analysis of a screened harmonic system – a boundary value problem with the Dirichlet boundary condition for the screened Poisson equation. We consider the screened harmonic system and its continuation in the Sobolev space. We carry out the analysis of the extended screened harmonic system by the method of iterative extensions in the Euclidean space. We write out the algorithm for the method of iterative extensions, which solves the problems of a screened harmonic system in the Euclidean space. We give an example of calculating the bending of a membrane on an elastic base.

1. Screened Harmonic System in the Sobolev Space

We consider the screened harmonic system in the Sobolev space. From the theory of elasticity [2, 6], the energy of a deformed membrane is:

$$\check{E}_\omega(\check{u}_\omega) = \frac{1}{2}\check{T}_\omega \int\limits_{\Omega_\omega} (\check{u}^2_{\omega_x} + \check{u}^2_{\omega_y}) d\Omega_\omega + \frac{1}{2}\int\limits_{\Omega_\omega} \check{K}_\omega \check{u}^2_\omega d\Omega_\omega - \int\limits_{\Omega_\omega} \check{P}_\omega \check{u}_\omega d\Omega_\omega,\ \omega \in \{1, \mathrm{II}\},$$

where $\check{P}_\omega$ is pressure, $\check{K}_\omega$ is coefficient of stiffness of the elastic base, $\check{T}_\omega$ is coefficient of tension of a membrane, Ω_ω is flat bounded domain with piecewise smooth boundary of C^2 class without self-contacts and self-intersections, $\partial\Omega_\omega = \overline{s}_\omega,\ \overline{s} = \Gamma_{\omega,1} \cup \Gamma_{\omega,2},\ \Gamma_{\omega,i} \cap \Gamma_{\omega,j} = \emptyset$, if $i \neq j,\ i,j = 1,2,\ \Gamma_{\omega,i},\ i = 1,2$ is the union of a finite number of open, disjoint subsets on a boundary $\partial\Omega_\omega$ of arcs of smooth curves of C^2 class, $\check{u}_\omega$ is desired movement of the membrane. Membrane energy variation equates to zero

$$\delta\check{E}_\omega(\check{u}_\omega) = \check{T}_\omega \int\limits_{\Omega_\omega} (\check{u}_{\omega_x}\check{v}_{\omega_x} + \check{u}_{\omega_y}\check{v}_{\omega_y}) d\Omega_\omega + \int\limits_{\Omega_\omega} \check{K}_\omega \check{u}_\omega \check{v}_\omega d\Omega_\omega - \int\limits_{\Omega_\omega} \check{P}_\omega \check{v}_\omega d\Omega_\omega = 0,$$

if $\check{v}_\omega = \delta\check{u}_\omega,\ \kappa_\omega = \check{K}_\omega/\check{T}_\omega,\ \check{f}_\omega = \check{P}_\omega/\check{T}_\omega$, then

$$\int\limits_{\Omega_\omega} (\check{u}_{\omega_x}\check{v}_{\omega_x} + \check{u}_{\omega_y}\check{v}_{\omega_y} + k_\omega\check{u}_\omega\check{v}_\omega)d\Omega_\omega = \int\limits_{\Omega_\omega} \check{f}_\omega\check{v}_\omega d\Omega_\omega.$$

We integrate by parts and get

$$\int\limits_{\Omega_\omega} (-\Delta\check{u}_\omega + \kappa_\omega\check{u}_\omega)\check{v}_\omega d\Omega_\omega + \int\limits_{s_\omega} \frac{\partial\check{u}_\omega}{\partial n_\omega}\check{v}_\omega ds_\omega = \int\limits_{\Omega_\omega} \check{f}_\omega\check{v}_\omega d\Omega_\omega,$$

where n_ω is outer normal to $\partial\Omega_\omega$. If membrane fixed on $\Gamma_{\omega,1}$, and free on $\Gamma_{\omega,2}$, then we obtain boundary value problem with mixed homogeneous boundary conditions:

$$-\Delta\check{u}_\omega + \kappa_\omega\check{u}_\omega = \check{f}_\omega, \tag{2}$$

$$\check{u}_\omega\Big|_{\Gamma_{\omega,1}} = 0,\ \frac{\partial\check{u}_\omega}{\partial n_\omega}\Big|_{\Gamma_{\omega,2}} = 0.$$

Let us formulate a variational boundary value problem with mixed homogenous boundary conditions:

$$\check{u}_\omega \in \check{H}_\omega \colon A_\omega(\check{u}_\omega, \check{v}_\omega) = F_\omega(\check{v}_\omega)\ \forall\check{v}_\omega \in \check{H}_\omega,\ \check{F}_\omega \in \check{H}'_\omega, \tag{3}$$

where the Sobolove space

$$\check{H}_\omega = \check{H}_\omega(\Omega_\omega) = \left\{\check{v}_\omega \in W_2^1(\Omega_\omega) \colon \check{v}_\omega\Big|_{\Gamma_{\omega,1}} = 0\right\}$$

on flat bounded domain Ω_ω with piecewise smooth boundary of C^2 class without self-contacts and self-intersections.

Sufficient to assume the existence and uniqueness for problem (3)

$$\exists c_1, c_2 \in (0; +\infty) \colon c_1\|\check{v}_\omega\|^2_{W_2^1(\Omega_\omega)} \leq A_\omega(\check{v}_\omega, \check{v}_\omega) \leq c_2\|\check{v}_\omega\|^2_{W_2^1(\Omega_\omega)}\ \forall\check{v}_\omega \in \check{H}_\omega,$$

This condition always guarantees the uniqueness of the solution for $\kappa_\omega \in (0; +\infty)$ and any combinations of given boundary conditions.

We solve mixed boundary value problem for the screened Poisson equation with the Dirichlet boundary condition i. g. with $\omega = 1,\ \kappa_1 \geq 0,\ \Gamma_{1,1} \neq \emptyset$, which we consider as a harmonic system

$$-\Delta\check{u}_1 + \kappa_1\check{u}_1 = \check{f}_1, \tag{4}$$

$$\check{u}_1\Big|_{\Gamma_{1,1}} = 0,\ \frac{\partial\check{u}_1}{\partial\check{n}_1}\Big|_{\Gamma_{1,2}} = 0.$$

Additionally, we consider a mixed homogeneous fictitious problem for the screened Poisson equation, i. g. with $\omega = \mathrm{II},\ \check{f}_{\mathrm{II}} = 0,\ \check{u}_{\mathrm{II}} = 0$

$$-\Delta\check{u}_{\mathrm{II}} + \kappa_{\mathrm{II}}\check{u}_{\mathrm{II}} = 0, \tag{5}$$

$$\check{u}_{\mathrm{II}}\Big|_{\Gamma_{\mathrm{II},1}} = 0,\ \frac{\partial\check{u}_{\mathrm{II}}}{\partial\check{n}_{\mathrm{II}}}\Big|_{\Gamma_{\mathrm{II},2}} = 0.$$

We propose fictitious continuation of the original boundary value problem for the screened Poisson equation in variational form:

$$\check{u} \in \check{V} \colon A_1(\check{u}, I_1\check{v}) + A_{\text{II}}(\check{u}, \check{v}) = F_1(I_1\check{v}) \; \forall \check{v} \in \check{V}, \tag{6}$$

second domain supplements solution domain of the original problem in the first domain on the plane to a rectangular domain

$$\Omega_1 \cup \Omega_{\text{II}} = \Pi, \; \Omega_1 \cap \Omega_{\text{II}} = \emptyset, \; \Omega_1, \Omega_{\text{II}} \subset \mathbb{R}^2.$$

In the continued problem, we use the operator of non-orthogonal projection of the extended space onto the subspace of the solution to the continued problem

$$I_1 \colon \check{V} \mapsto \check{V}_1, \; \check{V}_1 = \text{im} I_1, \; I_1 = I_1^2.$$

We introduce subspaces of the extended solution space

$$\check{V}_3 = \check{V}_3(\Pi) = \left\{ \check{v}_3 \in \check{V} \colon \check{v}_3 \Big|_{\Pi \setminus \Omega_{\text{II}}} = 0 \right\}, \; \check{V}_0 = \check{V}_1 \oplus \check{V}_3,$$

$$\check{V}_2 = \check{V}_2(\Pi) = \{ \check{v}_2 \in \check{V} \colon A(\check{v}_2, \check{v}_0) = 0 \; \forall \check{v}_0 \in \check{V}_0 \},$$

$$\check{V} = \check{V}_1 \oplus \check{V}_2 \oplus \check{V}_3 = \check{V}_1 \oplus \check{V}_{\text{II}}, \; \check{V}_{\text{I}} = \check{V}_1 \oplus \check{V}_2, \; \check{V}_{\text{II}} = \check{V}_2 \oplus \check{V}_3.$$

We consider direct sums of subspaces in the scalar product generated by the bilinear form

$$A(\check{u}, \check{v}) = A_1(\check{u}, \check{v}) + A_{\text{II}}(\check{u}, \check{v}) \quad \forall \check{u}, \check{v} \in \check{V}.$$

We assume that the bilinear form gives a normalization of the extended solution space equivalent to the normalization of the Sobolev space

$$\exists c_1, c_2 > 0 \colon c_1 \|\check{v}\|^2_{W_2^1(\Pi)} \le A(\check{v}, \check{v}) \le c_2 \|\check{v}\|^2_{W_2^1(\Pi)} \quad \forall \check{v} \in \check{V}.$$

Thus, solution to continued problem exists, and is unique. This is the solution to the original problem on the first domain with zero continuation on the rest of the rectangular domain.

Proposition 1. *The following equalities take place:*

$$A_\omega(\check{u}_0, \check{v}_2) = A_\omega(\check{v}_2, \check{u}_0) = 0 \; \forall \check{u}_0 \in \check{V}_0, \; \forall \check{v}_2 \in \check{V}_2, \; \omega \in \{1, \text{II}\}.$$

Solution to problem (6) $\check{u} \in \check{V}_1$ exists, is unique on Ω_1, and matches with the solution to problem (3) with $\omega = 1$, and equals zero on Ω_{II}.

Let us consider the discretization of the problem of the continued screened harmonic system on a finite-dimensional subspace with the following type of boundary conditions, when

$$\Pi = (0; b_1) \times (0; b_2), \; \Gamma_1 = \{b_1\} \times (0; b_2) \cup (0; b_1) \times \{b_2\},$$

$$\Gamma_2 = \{0\} \times (0; b_2) \cup (0; b_1) \times \{0\}, \; b_1, b_2 \in (0; +\infty).$$

In a rectangular domain, we introduce a grid

$$(x_i; y_j) = \big((i - 1,5)h_1; (j - 1,5)h_2\big),$$

$h_1 = b_1/(m-1,5),\ h_2 = b_2/(n-1,5),\ i = 1,2,\ldots,m,\ j = 1,2,\ldots,n,\ m-2, n-2 \in \mathbb{N}.$
We introduce grid functions on the set of nodes of the grid

$$v_{i,j} = v(x_i; y_j) \in \mathbb{R},\ i = 1,2,\ldots,m,\ j = 1,2,\ldots,n,\ m-2, n-2 \in \mathbb{N}.$$

We apply the completion of grid functions, taking into account the selected boundary conditions, using linear basis functions

$$\Phi^{i,j}(x;y) = \Psi^{1,i}(x)\Psi^{2,j}(y),\ i = 2,\ldots,m-1\ ,j = 2,\ldots,n-1\ ,m-2, n-2 \in \mathbb{N},$$

$$\Psi^{1,i}(x) = [2/i]\ \Psi(x/h_1 - i + 3,5) + \Psi(x/h_1 - i + 2,5),$$
$$\Psi^{2,j}(y) = [2/j]\ \Psi(y/h_2 - j + 3,5) + \Psi(y/h_2 - j + 2,5),$$
$$\Psi(z) = \begin{cases} z, & z \in [0;1], \\ 2-z, & z \in [1;2], \\ 0, & z \notin (0;2). \end{cases}$$

Here, $[\bullet]$ is integer part of number. We assume that the values of the basis functions equal to zero outside the rectangular domain

$$\Phi^{i,j}(x;y) = 0,\ (x;y) \notin \Pi,\ i = 2,\ldots,m-1,\ j = 2,\ldots,n-1,\ m-2, n-2 \in \mathbb{N}.$$

Combinations of basis functions are a finite-dimensional subspace in the solution space of the extended problem

$$\hat{V} = \left\{ \hat{v} = \sum_{i=2}^{m-1} \sum_{j=2}^{n-1} v_{i,j}\Phi^{i,j}(x;y) \right\} \subset \check{V}.$$

We are aware of convergence estimation of the following form [1]:

$$\|\check{u} - \hat{u}\|_{W_2^{m_1}(\Pi)} \le ch^{m_2-m_1}\|\check{u}\|_{W_2^{m_2}(\Pi)},\ \lim_{h\to 0}\|\check{u} - \hat{u}\|_{W_2^1(\Pi)} = 0,\ h = max\{h_1, h_2\}.$$

Let us present the continued problem on the introduced finite-dimensional subspace in the variational form

$$\hat{u} \in \hat{V}\colon A_1(\hat{u}, I_1\hat{v}) + A_{\Pi}(\hat{u}, \hat{v}) = F_1(I_1\hat{v})\ \forall \hat{v} \in \hat{V}. \tag{7}$$

The solution to the continued problem exists, and is unique on a finite-dimensional subspace. On a finite-dimensional subspace, this is the solution to the original problem on the first domain with zero continuation on the rest of the rectangular domain.

2. Analysis of the Continued Screened Harmonic System

Approximating the problem of the continued screened harmonic system using a finite-dimensional subspace, we obtain the system of equations:

$$\overline{u} \in \mathbb{R}^N\colon B\overline{u} = \overline{f},\ \overline{f} \in \mathbb{R}^N. \tag{8}$$

We also assume that the projection operator onto the solution space of the continued problem nullifies the coefficients of basis functions whose supports does not contain completely in the first domain. We define the continued matrix and the continued right side of the system, and get continued problem as the system in matrix form

$$\langle B\overline{u}, \overline{v}\rangle = A_1(\hat{u}, I_1\hat{v}) + A_{\rm II}(\hat{u}, \hat{v})\ \forall \hat{u}, \hat{v} \in \hat{V},\ \langle \overline{f}, \overline{v}\rangle = F_1(I_1\hat{v})\ \forall \hat{v} \in \hat{V},$$

$$\langle \overline{f}, \overline{v}\rangle = (\overline{f}, \overline{v})h_1h_2 = \overline{f}\overline{v}h_1h_2,\ \overline{v} = (v_1, v_2, \ldots, v_N)' \in \mathbb{R}^N,\ N = (m-2)(n-2).$$

In this case, we enumerate first the basis functions whose supports lie completely in the first domain. Then we enumerate the basis functions whose supports cross the boundary of the first domain and the second domain together. Last we enumerate the basis functions whose supports lie completely in the second domain. With this numbering, the resulting vectors have the following structure

$$\overline{v} = (\overline{v}_1', \overline{v}_2', \overline{v}_3')',\ \overline{u} = (\overline{u}_1', \overline{0}', \overline{0}')',\ \overline{f} = (\overline{f}_1', \overline{0}', \overline{0}')'.$$

Here, $'$ indicates column vector. We define the matrices generated by the corresponding bilinear forms

$$\langle A_{\rm I}\overline{u}, \overline{v}\rangle = A_1(\hat{u}, \hat{v}),\ \langle A_{\rm II}\overline{u}, \overline{v}\rangle = A_{\rm II}(\hat{u}, \hat{v})\ \forall \hat{u}, \hat{v} \in \hat{V}.$$

These matrices have the following structure:

$$A_{\rm I} = \begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{21} & A_{20} & 0 \\ 0 & 0 & 0 \end{bmatrix},\ A_{\rm II} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & A_{02} & A_{23} \\ 0 & A_{32} & A_{33} \end{bmatrix}.$$

We define extended matrix

$$A = A_{\rm I} + A_{\rm II} = \begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{21} & A_{22} & A_{23} \\ 0 & A_{32} & A_{33} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{21} & A_{20} & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & A_{02} & A_{23} \\ 0 & A_{32} & A_{33} \end{bmatrix}.$$

In fictitious domain method, we solve continued problem in matrix form

$$B\overline{u} = \overline{f},\ \begin{bmatrix} A_{11} & A_{12} & 0 \\ 0 & A_{02} & A_{23} \\ 0 & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} \overline{u}_1 \\ \overline{0} \\ \overline{0} \end{bmatrix} = \begin{bmatrix} \overline{f}_1 \\ \overline{0} \\ \overline{0} \end{bmatrix}.$$

This is the solution to the original problem in matrix form and this is the zero solution to the fictitious problem in matrix form

$$A_{11}\overline{u}_1 = \overline{f}_1,\ \begin{bmatrix} A_{02} & A_{23} \\ A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} \overline{u}_2 \\ \overline{u}_3 \end{bmatrix} = \begin{bmatrix} \overline{0} \\ \overline{0} \end{bmatrix},\ \begin{bmatrix} \overline{u}_2 \\ \overline{u}_3 \end{bmatrix} = \begin{bmatrix} \overline{0} \\ \overline{0} \end{bmatrix}.$$

For solving problem (8) we use new method of iterative extensions [9, 10, 11]. We define the extended matrix in a new way, as the sum of the first matrix and the second matrix multiplied by a positive parameter

$$C = A_{\rm I} + \gamma A_{\rm II},\ \begin{bmatrix} C_{11} & C_{12} & 0 \\ C_{21} & C_{22} & C_{23} \\ 0 & C_{32} & C_{33} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{21} & A_{20} & 0 \\ 0 & 0 & 0 \end{bmatrix} + \gamma \begin{bmatrix} 0 & 0 & 0 \\ 0 & A_{02} & A_{23} \\ 0 & A_{32} & A_{33} \end{bmatrix},\ \gamma \in (0; +\infty).$$

We also assume that the conditions still satisfy on the continuation of functions in the following form

$$\exists\gamma_1 \in (0;+\infty),\ \gamma_2 \in [\gamma_1;+\infty)\colon \gamma_1^2\langle C\overline{v}_2, C\overline{v}_2\rangle \le \langle A_{\mathrm{II}}\overline{v}_2, A_{\mathrm{II}}\overline{v}_2\rangle \le \gamma_2^2\langle C\overline{v}_2, C\overline{v}_2\rangle \quad \forall\overline{v}_2 \in \overline{V}_2,$$

$$\exists\alpha \in (0;+\infty)\colon \langle A_{\mathrm{I}}\overline{v}_2, A_{\mathrm{I}}\overline{v}_2\rangle \le \alpha^2\langle A_{\mathrm{II}}\overline{v}_2, A_{\mathrm{II}}\overline{v}_2\rangle \quad \forall\overline{v}_2 \in \overline{V}_2.$$

The method of iterative extensions is a generalization of the fictitious domain methods, when we use an additional parameter in extended matrix, and select iterative parameters with the method of minimum residuals

$$\overline{u}^k \in \mathbb{R}^N\colon C(\overline{u}^k - \overline{u}^{k-1}) = -\tau_{k-1}(B\overline{u}^{k-1} - \overline{f}),\ k \in \mathbb{N}, \tag{9}$$

$$\forall\overline{u}^0 \in \overline{V}_1,\ \gamma > \alpha,\ \tau_0 = 1,\ \tau_{k-1} = \langle\overline{r}^{k-1}, \overline{\eta}^{k-1}\rangle / \langle\overline{\eta}^{k-1}, \overline{\eta}^{k-1}\rangle,\ k \in \mathbb{N}\backslash\{1\},$$

where, to calculate the iterative parameters, we sequentially calculate the residuals, corrections, and equivalent residuals

$$\overline{r}^{k-1} = B\overline{u}^{k-1} - \overline{f},\ \overline{w}^{k-1} = C^{-1}\overline{r}^{k-1},\ \overline{\eta}^{k-1} = B\overline{w}^{k-1},\ k \in \mathbb{N}.$$

We define the norm generated by extended matrix

$$\|\overline{v}\|_{C^2} = \sqrt{\langle C^2\overline{v}, \overline{v}\rangle} \quad \forall\overline{v} \in \mathbb{R}^N.$$

Lemma 1. *For iterative process from* (9)*, estimation is*

$$\|\overline{u}^1 - \overline{u}\|_{C^2} \le 2\|\overline{u}^0 - \overline{u}\|_{C^2}.$$

Theorem 1. *For method of iterative extensions from* (9)*, convergence estimation is*

$$\|\overline{u}^k - \overline{u}\|_{C^2} \le \varepsilon\|\overline{u}^0 - \overline{u}\|_{C^2},\ \varepsilon = 2(\gamma_2/\gamma_1)(\alpha/\gamma)^{k-1},\ k \in \mathbb{N}.$$

We estimate the sequence of relative errors in a stronger norm than the energy norm from above by a converging geometric progression.

Remark 1. In iterative process from (9) error belongs to subspace, i. g. $\overline{\psi}^k \in \overline{V}_2 \quad \forall k \in \mathbb{N}$, approximation belongs to subspace, i. g. $\overline{u}^k \in \overline{V}_1 \quad \forall k \in \mathbb{N}$.

3. Algorithm of the Method of Iterative Extensions

For solving problem (8) we use method of iterative extensions. We apply the matrices generated by the corresponding bilinear forms

$$\langle A_{\mathrm{I}}\overline{u}, \overline{v}\rangle = A_{\mathrm{I}}(\hat{u}, \hat{v}),\ \langle A_{\mathrm{II}}\overline{u}, \overline{v}\rangle = A_{\mathrm{II}}(\hat{u}, \hat{v}) \quad \forall\hat{u}, \hat{v} \in \hat{V}.$$

Matrices have the following structure:

$$A_{\mathrm{I}} = \begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{21} & A_{20} & 0 \\ 0 & 0 & 0 \end{bmatrix},\ A_{\mathrm{II}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & A_{02} & A_{23} \\ 0 & A_{32} & A_{33} \end{bmatrix}.$$

We define extended matrix, as the sum of the first matrix and the second matrix multiplied by a positive parameter

$$C = A_{\rm I} + \gamma A_{\rm II},\ \begin{bmatrix} C_{11} & C_{12} & 0 \\ C_{21} & C_{22} & C_{23} \\ 0 & C_{32} & C_{33} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{21} & A_{20} & 0 \\ 0 & 0 & 0 \end{bmatrix} + \gamma \begin{bmatrix} 0 & 0 & 0 \\ 0 & A_{02} & A_{23} \\ 0 & A_{32} & A_{33} \end{bmatrix},\ \gamma \in (0; +\infty).$$

We calculate the elements of this matrix using the formulas

$$c_{i,j} = h_1^{-1} h_2^{-1} C(\Phi_i, \Phi_j),\ i, j = 1, 2, \dots, N.$$

We write out the iterative process in the following form

$$\overline{u}^1 \in \mathbb{R}^N : C\overline{u}^1 = \overline{f},$$

$$\overline{u}^k \in \mathbb{R}^N : C(\overline{u}^k - \overline{u}^{k-1}) = -\tau_{k-1} A_{\rm II} \overline{u}^{k-1},\ \tau_{k-1} = (\overline{r}^{k-1}, \overline{\eta}^{k-1})/(\overline{\eta}^{k-1}, \overline{\eta}^{k-1}),\ k \in \mathbb{N}\backslash\{1\}.$$

Let us present an algorithm that implements the method of iterative extensions for solving the problem of continued harmonic system on a square.

1. Calculate the value of the squared norm of the initial absolute error, which preserves throughout all calculations
$$E_0 = (\overline{f}, \overline{f})h^2.$$
2. Find the first approximation
$$\overline{u}^1 : C\overline{u}^1 = \overline{f},$$
$$\begin{bmatrix} \overline{u}_1^1 \\ \overline{u}_2^1 \\ \overline{u}_3^1 \end{bmatrix} \in \overline{V}_{\rm I} : \begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{21} & A_{20} + \gamma A_{02} & \gamma A_{23} \\ 0 & \gamma A_{32} & \gamma A_{33} \end{bmatrix} \begin{bmatrix} \overline{u}_1^1 \\ \overline{u}_2^1 \\ \overline{u}_3^1 \end{bmatrix} = \begin{bmatrix} \overline{f}_1 \\ \overline{0} \\ \overline{0} \end{bmatrix},\ \gamma \in (0; +\infty).$$
3. Calculate residual
$$\overline{r}^{k-1} = B\overline{u}^{k-1} - \overline{f} = A_{\rm II}\overline{u}^{k-1},\ k \in \mathbb{N}\backslash\{1\},$$
$$\begin{bmatrix} \overline{r}_1^{k-1} \\ \overline{r}_2^{k-1} \\ \overline{r}_3^{k-1} \end{bmatrix} = \begin{bmatrix} \overline{0} \\ A_{02}\overline{u}_2^{k-1} + A_{23}\overline{u}_3^{k-1} \\ \overline{0} \end{bmatrix},\ k \in \mathbb{N}\backslash\{1\}.$$
4. Calculate the next value of the squared norm of the absolute error
$$E_{k-1} = (\overline{r}^{k-1}, \overline{r}^{k-1})h^2,\ k \in \mathbb{N}\backslash\{1\}.$$
5. Calculate correction
$$\overline{w}^{k-1} : C\overline{w}^{k-1} = \overline{r}^{k-1},\ k \in \mathbb{N}\backslash\{1\},$$
$$\begin{bmatrix} \overline{w}_1^{k-1} \\ \overline{w}_2^{k-1} \\ \overline{w}_3^{k-1} \end{bmatrix} \in \overline{V}_2 : \begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{21} & A_{20} + \gamma A_{02} & \gamma A_{23} \\ 0 & \gamma A_{32} & \gamma A_{33} \end{bmatrix} \begin{bmatrix} \overline{w}_1^{k-1} \\ \overline{w}_2^{k-1} \\ \overline{w}_3^{k-1} \end{bmatrix} = \begin{bmatrix} \overline{0} \\ \overline{r}_2^{k-1} \\ \overline{0} \end{bmatrix},\ \gamma \in (0; +\infty),\ k \in \mathbb{N}\backslash\{1\}.$$

6. Calculate equivalent residual

$$\overline{\eta}^{k-1} = B\overline{w}^{k-1} = A_{\mathrm{II}}\overline{w}^{k-1},\ k \in \mathbb{N}\backslash\{1\},$$

$$\begin{bmatrix} \overline{\eta}_1^{k-1} \\ \overline{\eta}_2^{k-1} \\ \overline{\eta}_3^{k-1} \end{bmatrix} = \begin{bmatrix} \overline{0} \\ A_{02}\overline{w}_2^{k-1} + A_{23}\overline{w}_3^{k-1} \\ \overline{0} \end{bmatrix},\ k \in \mathbb{N}\backslash\{1\}.$$

7. Calculate the iterative parameter

$$\tau_{k-1} = (\overline{r}^{k-1}, \overline{\eta}^{k-1})/(\overline{\eta}^{k-1}, \overline{\eta}^{k-1}),\ k \in \mathbb{N}\backslash\{1\}.$$

8. Find the new approximation

$$\overline{u}^k = \overline{u}^{k-1} - \tau_{k-1}\overline{w}^{k-1},\ k \in \mathbb{N}\backslash\{1\},$$

$$\begin{bmatrix} \overline{u}_1^k \\ \overline{u}_2^k \\ \overline{u}_3^k \end{bmatrix} = \begin{bmatrix} \overline{u}_1^{k-1} \\ \overline{u}_2^{k-1} \\ \overline{u}_3^{k-1} \end{bmatrix} - \tau_{k-1} \begin{bmatrix} \overline{w}_1^{k-1} \\ \overline{w}_2^{k-1} \\ \overline{w}_3^{k-1} \end{bmatrix} \in \overline{V}_{\mathrm{I}},\ k \in \mathbb{N}\backslash\{1\}.$$

9. Check the iteration stop criterion

$$E_{k-1} \leq E^2 E_0,\ k \in \mathbb{N}\backslash\{1\},\ E \in (0;1).$$

If the criterion did not reach, then we repeat everything from step 3.

With a parameter $\gamma = 27/2$, we present an algorithm, based on iterative factorization method, for solving problems arising on steps 2 and 5. At each step of iterative process we write out given system of linear algebraic equations, that we need to solve, in matrix form

$$\overline{v} \in \mathbb{R}^N \colon C\overline{v} = \overline{g},\ \overline{g} \in \mathbb{R}^N,\ N = (n-2)(n-2).$$

Let us formulate the method of iterative factorization, an iterative process for solving a system with a matrix C:

$$\overline{v}^l \in \mathbb{R}^N \colon LL'(\overline{v}^l - \overline{v}^{l-1}) = -\tau_{l-1}(C\overline{v}^{l-1} - \overline{g}),$$

$$\tau_{l-1} = (\overline{r}^{l-1}, \overline{w}^{l-1})/(\overline{w}^{l-1}, \overline{\eta}^{l-1}), \quad \overline{v}^0 = \overline{0} \in \mathbb{R}^N,$$

$$\overline{r}^{l-1} = C\overline{v}^{l-1} - \overline{g}, \quad \overline{w}^{l-1} = (LL')^{-1}\overline{r}^{l-1}, \quad \overline{\eta}^{l-1} = C\overline{w}^{l-1},$$

where

$$C = A_{\mathrm{I}} + \gamma A_{\mathrm{II}},\ L' = \nabla_x + \nabla_y + \kappa E,\ \kappa = \sqrt{\kappa_{\mathrm{II}}},$$

$$\langle \nabla_x, \overline{u}, \overline{v} \rangle = \sum_{i=2}^{n-1}\sum_{j=2}^{n-1}(-(u_{i+1,j} - u_{i,j})h^{-1})v_{i,j}h^2,\ u_{n,j} = v_{n,j} = 0,\ j = 2, \ldots, n-1,$$

$$\langle \nabla_y, \overline{u}, \overline{v} \rangle = \sum_{i=2}^{n-1}\sum_{j=2}^{n-1}(-(u_{i,+1j} - u_{i,j})h^{-1})v_{i,j}h^2,\ u_{i,n} = v_{i,n} = 0,\ i = 2, \ldots, n-1.$$

We write out method formulated above in the form of algorithm:

1. Start with the zero initial approximation

$$\overline{v}^{l-1} = \overline{0} \in \mathbb{R}^N,\ l = 1.$$

2. Calculate inital residual

$$\overline{r}^{l-1} = -\overline{g},\ l = 1.$$

3. Calculate next residual

$$\overline{r}^{l-1} = C\overline{v}^{l-1} - \overline{g},\ l \in \mathbb{N}\backslash\{1\}.$$

4. Calculate the value of the squared norm of the initial absolute error, which preserves throughout all calculations

$$e_{l-1} = (\overline{r}^{l-1}, \overline{r}^{l-1})h^2,\ l \in \mathbb{N}.$$

5. Calculate correction

$$\overline{w}^{l-1} \in \mathbb{R}^N : LL'\overline{w}^{l-1} = \overline{r}^{l-1},\ l \in \mathbb{N}.$$

6. Calculate equivalent residual

$$\overline{\eta}^{l-1} = C\overline{w}^{l-1},\ l \in \mathbb{N}.$$

7. Calculate the iterative parameter

$$\tau_{l-1} = (\overline{r}^{l-1}, \overline{w}^{l-1})/(\overline{w}^{l-1}, \overline{\eta}^{l-1}),\ l \in \mathbb{N}.$$

8. Find the new approximation

$$\overline{v}^{l} = \overline{v}^{l-1} - \tau_{l-1}\overline{w}^{l-1},\ l \in \mathbb{N}.$$

9. Check the iteration stop criterion

$$e_{l-1} \leq e^2 e_0,\ l \in \mathbb{N},\ e = 0.001 \in (0; 1).$$

If the criterion did not reach, then we repeat everything from step 3.

Let us consider an L-shaped membrane with a very small thickness $\check{h}$ relative to its size, located horizontally on an elastic base, fixed at the boundary except for two large and adjacent sides. We assume that the points of the membrane belong to the set in a rectangular coordinate system

$$\big([0; 2.5] \times [0; 2.5]\big\backslash(1.5; 2.5] \times (1.5; 2.5]\big) \times [-\check{h}/2; \check{h}/2],$$

i. g. we consider the problem:

$$-\Delta\check{u}_1 + \check{u}_1 = \check{f}_1,\ \check{u}_1, \check{f} \in [0; 2.5] \times [0; 2.5]\big\backslash(1.5; 2.5] \times (1.5; 2.5],$$

$$\check{u}_1 = 0,\ \check{u}_1 \in \{2.5\} \times (0;1.5) \cup (0;1.5) \times \{2.5\} \cup \{1.5\} \times (1.5;2.5) \cup (1.5;2.5) \times \{1.5\},$$

$$\frac{\partial \check{u}_1}{\partial n} = 0,\ \check{u}_1 \in \{0\} \times (0;2.5) \cup (0;2.5) \times \{0\}.$$

Here, $\check{u}_1$ is function of point movement of L-shaped membrane, located horizontally on elastic base, under the vertical pressure, determined by right side of equation $\check{f}_1$.

Let us find the median surface of the membrane, on which vertical pressure $\check{P}_1$ acts, as a numerical solution to the problem of a screened harmonic system in the Sobolev space on an L-shaped domain, when the tension coefficient $\check{T}_1 = 1.5$, stiffness coefficient of the elastic base $\check{K}_1 = 1.5$ and pressure

$$\check{P}_1 = 1.5\ ((392 - 384x)(64y^3 - 169y^2 + 225) + (64x^3 - 196x^2 + 225)(392 - 384y))/184^2 +$$

$$+1.5\ (64x^3 - 196x^2 + 225)(64y^3 - 196y^2 + 225)/184^2.$$

Let us find an approximation to the solution to this problem, when we specify $n = 254$ and choose the zero initial approximation. Iterative process of the method of iterative extensions stops in several iterations, if we specify evaluation criterion $E = 0.001$ for the relative error in a norm stronger than the energy norm. On Fig. 1 we display the last approximation for $n = 254$ and the solution.

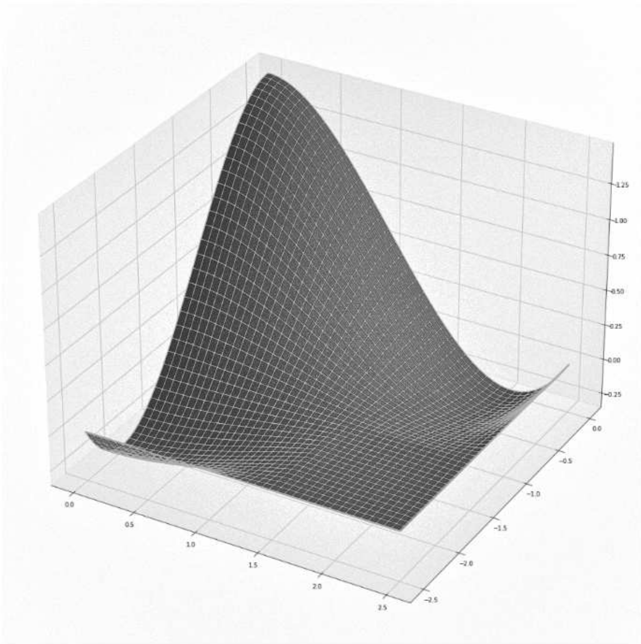

Fig. 1. Solution and last approximation

We calculate the value of the maximum error on the finest grid for $n = 502$.

$$\varepsilon = \frac{\max\limits_{2 \le i,j \le n-1} \|u^k_{i,j} - \check{u}_{i,j}\|}{\max\limits_{2 \le i,j \le n-1} \|\check{u}_{i,j}\|} = 0.00021.$$

On Fig. 2 we display the graph of the function of the number of iterations in the iterative process as the function of number of nodes in the directions of the axes.

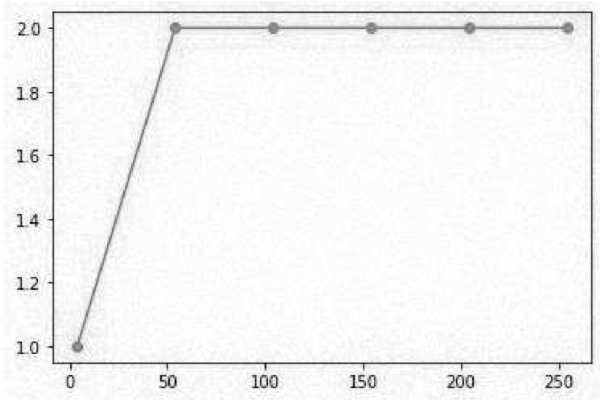

Fig. 2. Number of iterations as the function of number of nodes

We solved problem and developed method of iterative extensions, asymptotically optimal in terms of the number of operations, with automation of control over the optimal choice of iterative parameters and with a stop criterion, when a given accuracy reached, for the analysis of screened harmonic systems in geometrically complex domains. Special mathematical and algorithmic software implement this method and make it possible to solve model problems of screened harmonic systems and obtain graphical representations of solutions.

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МЕТОД ИТЕРАЦИОННЫХ РАСШИРЕНИЙ ПРИ АНАЛИЗЕ ЭКРАНИРОВАННЫХ ГАРМОНИЧЕСКИХ СИСТЕМ

М. П. Еремчук, А. Л. Ушаков

Описывается асимптотически оптимальный по количеству операций итерационный метод решения проблемных краевых задач для экранированного уравнения Пуассона в геометрически сложных областях как анализ экранированных гармонических систем, описывающих соответствующие стационарные физические системы. В природе и технике, например, в механике множество стационарных физических систем описывается краевыми задачами для экранированного уравнения Пуассона в геометрически сложных областях.

Ключевые слова: метод фиктивных компонент; метод итерационных расширений; экранированные гармонические системы; экранированное уравнение Пуассона.

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