REVIEW

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METHODS OF AUTOMATIC AND OPTIMAL CONTROL IN DYNAMIC MEASUREMENTS

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The article reviews the results of solving dynamic measurement problems of two scientific schools of South Ural State University. The dynamic properties of the measurement system are critical factors affecting the dynamic measurement error, while the structures of dynamic measurement system and automatic control system have common principles of construction. Thus, the methods of automatic control theory were implemented in the study of dynamic measuring systems. However, dynamic measuring systems are characterized by the absence of feedback, which required the development of new methods when using the ideas of automatic control theory. These include the method of modal control of dynamic characteristics of measuring systems. It led to the development and application of other methods: iterative principle of measuring systems, method of sliding modes, parametric adaptation of systems, neural network technologies, numerical methods for solving inverse problems. The first section of the article is devoted to these studies. The second section presents the results of the theory of optimal dynamic measurements. The problem of restoring a dynamically distorted signal is solved here using the methods of optimal control theory, and the measuring device is simulated by a Leontief-type system. The reduction of the solution of the inverse problem of dynamic measurements to a direct mathematical problem allowed us to effectively apply the existing mathematical apparatus of the theory of Sobolev equations in the case of taking into account the inertia of the measuring system. Analytical and then numerical studies were initiated to investigate the problem of restoring a dynamically distorted signal in the presence of «noise», which led to the creation of the theory of stochastic equations of Sobolev and Leontief types and the development of numerical methods. The review focuses on numerical methods based on the idea of extracting a useful output signal from a known noisy observation and then applying a numerical method to recover the input signal. In addition, the algorithm of a new numerical method based on the use of the counting theorem and simple averaging is briefly presented. The bibliographic review is based on the obtained results, though it is far from being exhaustive.

Keywords: dynamic measurements; automatic control; optimal control; Leontief-type systems; optimal dynamic measurement.

Dedicated to the 80th anniversary of South Ural State University

Introduction

When implementing projects of various innovative novelty, it is necessary to conduct tests at changes in process speed and (or) spatial position in an extremely small time interval.

In these cases, the input signal is distorted by the measuring instrument and the output signal is significantly different from the input signal. In addition, during the measurement there are disturbances of different nature caused by environmental factors and internal electrical noise of the measuring instrument. The task of input signal recovery at known parameters of the measuring system and output signal is the second inverse task of dynamic measurements. It is considered to be the most difficult task of dynamic measurements [11].

Historically, one of the first works on dynamic measurements is D.I. Mendeleev's study 1897, devoted to accurate weighing on laboratory scales. The concept of inverse and incorrect problems and regularization methods created by A.N. Tikhonov [44] gave an impetus to the development of the theory of dynamic measurements. In [23] a small historical sketch highlighted the works by G.I. Vasilenko, G.N. Solopchenko, V.A. Granovsky, F.L. Chernousko, A.B. Kurzhansky and others and their role in the development of the theory of dynamic measurements.

The creation and development of the research direction on the application of automatic control methods in dynamic systems is associated with the works of A.L. Shestakov and his students [35]. Automatic control systems are structurally different from dynamic measurement systems in that the input signal is not available either for direct measurement or for correction. A.L. Shestakov adapted the method of modal control for application in measuring systems by constructing for it a special structure of a correcting device in the form of a sensor model. The use of a sensor model having model input and output signals allows obtaining an estimate of its dynamic error [24]. On the basis of this method, we proposed a method for calculating the optimal correction device for the RMS error of a dynamic system with a single input, and methods for analyzing the dynamic error depending on the parameters of the measuring system. It should be noted that A.L. Shestakov received several certificates of authorship based on the results of these studies.

One of the directions of research was the study of properties and features of dynamic measuring systems with iterative principle of restoration of the measured signal [25]. It showed high noise immunity of such systems in the presence of noise at the sensor output.

The method of modal control of dynamic properties of measuring systems in the case of an observable state coordinate vector of the primary transducer was developed by D.Y. Iosifov and A.L. Shestakov [26]. For such systems, a method of optimal tuning of dynamic parameters by the criterion of the minimum variance of the measurement system error [9] was obtained.

The next direction in the development of automatic control methods in measuring systems is the study of sliding modes. It is known that such modes have a number of attractive properties: high dynamic accuracy, invariance to perturbing influences and to variations of dynamic properties of the object, reduced order of the system. M. Bizyaev and A.L. Shestakov studied the models of sliding measuring systems with one measuring coordinate of the primary measuring transducer and the measured vector of its state [3]. They proposed cascade structures of measuring systems in sliding mode, which ensure the absence of auto oscillations.

A special place is occupied by the direction related to the application of the neural network method for the realization of dynamic measurement systems on its basis. A.S. Volosnikov and A.L. Shestakov considered various neural network models of the primary measuring transducer and developed various algorithms for their training [45, 46].

In addition, numerical methods for solving inverse problems, in which the measuring system is simulated by a boundary value problem for differential equations [1, 47], have been successfully applied.

The theory of optimal dynamic measurements emerged as a result of successful interaction between two scientific schools headed by A.L. Shestakov and G.A. Sviridyuk. It is based on the application of the methods of Sobolev-type equations theory and optimal control to the solution of the problem of restoration of a dynamically distorted signal [27], [32]. The mathematical model of a complex measurement system is based on a Leontief-type system (a finite-dimensional analog of the Sobolev equation) [17], and the input signal is found as a solution to the problem of optimal control of solutions of this system, which determined the name of the problem solution – optimal dynamic measurement [36].

Since analytical and numerical methods for solving various optimal control problems for Leontief-type systems have already been developed by A.V. Keller [12, 43], it led to the formation of various approaches to mathematical modeling of the input signal recovery problem, for example, taking into account resonant interference [28], degradation of the measuring device [16]. At the same time, A.V. Keller, E.I. Nazarova, Y.V. Khudyakov, and M.A. Sagadeeva began to develop numerical methods for solving various problems of optimal dynamic measurement [15, 29, 37]. However, all these algorithms had a large time complexity, which is not always acceptable for engineering practice, so a spline method for solving the problem of optimal dynamic measurement [33] was developed and its convergence was shown [14].

In order to solve the problem of optimal dynamic measurement in the presence of random noise of the «white noise» type, the research began to be conducted in two directions. The first direction uses the results of the theory of stochastic equations of the Sobolev type developed by A. Favini, G.A. Sviridyuk, A.A. Zamyshlyayev. Favini, G.A. Sviridyuk, A.A. Zamyshlyaeva, N.A. Manakova, S.A. Zagrebina, and M.A. Sagadeeva [7, 8]. In these studies, «white noise» is understood in the sense of symmetric derivative in the mean or Nelson – Glicklich derivative [10, 18]. This approach is used to solve the simplest optimal dynamic measurement problem with «white noise» [49]. In the second direction, white noise is understood in the classical sense, and it uses various numerical methods for preprocessing the observed signal [13, 20, 37, 39, 40] and extracting a useful output signal. Besides, A.V. Keller, A.A. Zamyshlyaeva and N.A. Manakova proposed a new numerical method combining the spline method, the theory of samples statistical methods [38]. In the first computational experiments it showed high efficiency.

Note that review articles were traditional for the scientific team conducting research in the framework of the theory of optimal dynamic measurements. They summarized the results of work of a long period, for example, [4]. This article is of a review nature. The first section is devoted to the application of automatic control methods in the studies of the inverse problem of dynamic measurements. The second section presents the results of the theory of optimal dynamic measurements.

1. Methods of Automatic Control in Dynamic Measurements

Methods of increasing the accuracy of dynamic measurements based on the methods of automatic control theory are presented in detail in A.L. Shestakov's monograph [23].

1.1. Method of Modal Control of Dynamic Characteristics of Measuring Systems

The similarity of the structures of the automatic control system (Figure 1) and the dynamic measurement system (Figure 2) became the starting point for applying the methods of automatic control theory to correct the dynamic measurement error [35]. In Fig. 1 u is an input signal of the system; x is a state vector of the system; y is an observation vector; A, B, C, D are corresponding matrices. The required dynamic properties are provided by setting the coefficients of the feedback matrix K.

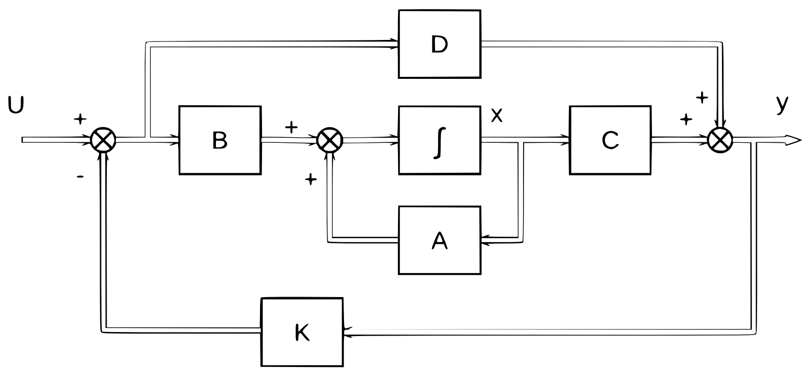

Fig. 1. Structural diagram of the automatic control system

As already mentioned, the absence of a measurable input influence leads to the impossibility of establishing feedback from the output to the input. Therefore, in dynamic measurement systems, the reduction of dynamic measurement errors is realized by means of a filter at the output of the primary measurement transducer (Figure 2).

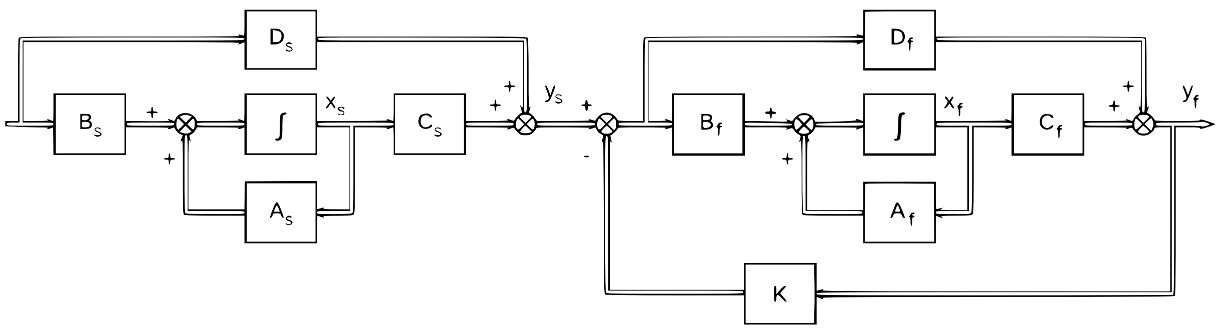

Fig. 2. Structural diagram of the dynamic measuring system

Here u is an input measured vector of the system; x_s, y_s are state and observation vectors; A_s, B_s, C_s, D_s are corresponding matrices of the sensing unit; x_f, y_f are vectors of the state and observation of the filter; A_f, B_f, C_f, D_f are filter matrices of the corresponding dimension; K is a matrix for setting the coefficients to provide the required dynamic error.

A.L. Shestakov developed methods of dynamic error correction based on the similarity of the structures of the sensing unit and filter. Figure 3 shows the structure of a dynamic measurement system with modal control of dynamic characteristics.

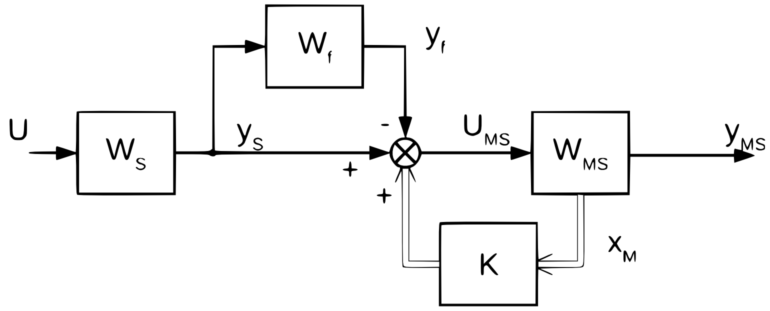

Fig. 3. Structural diagram of a dynamic measuring system with modal control of dynamic characteristics

The transfer functions of the sensor W_S and the sensor model W_{MS} are identical, which leads to their modeling by either one differential equation or one system of differential equations. Let us formulate the basic idea of using the sensor model to correct the dynamic measurement error: if the output signals y_S and y_{MS} are close to each other, then the input signals U и U_{MS} will also differ little from each other. Consequently, the input signal of the U_{MS} model allows the estimation of the sensor input signal U. In engineering practice, there are tasks of indirect measurement of signals by integrating the output signal of the primary measuring transducer. For example, it is necessary to obtain its velocity or displacement from the acceleration data of an object. In this case, the structure of the measurement system contains integrators of the output coordinate of the sensor, and the signals at the outputs of the integrators will be part of the state coordinate vector of the measuring transducer (Figure 4).

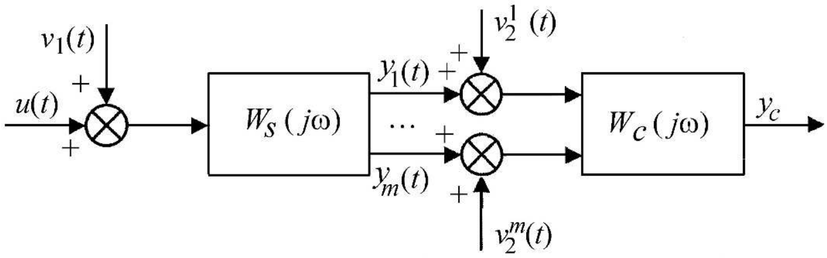

Fig. 4. Structural diagram of a measuring system with the observed state coordinate vector

For this case D.Y. Iosifov and A.L. Shestakov [9, 26] carry out a generalization of the method of modal control of dynamic properties of measuring systems. At known spectral densities of the measuring signal $S_u(\omega)$ and noise $S_{\nu_2^1}(\omega), ..., S_{\nu_2^m}(\omega)$ in the coordinate channels of the observed state, the optimal individual characteristics of the correcting device reducing the measurement error are obtained.

1.2. Iterative Dynamic Measurement Systems

The iterative principle of building automatic control systems allows designing systems of high dynamic accuracy, however, they are not widely used in control systems due to the difficulties of realization. In measurement systems, the iterative principle is realized more simply in the form of an additional data processing channel [25]. The structure diagram of the iterative measuring device is presented in Figure 5.

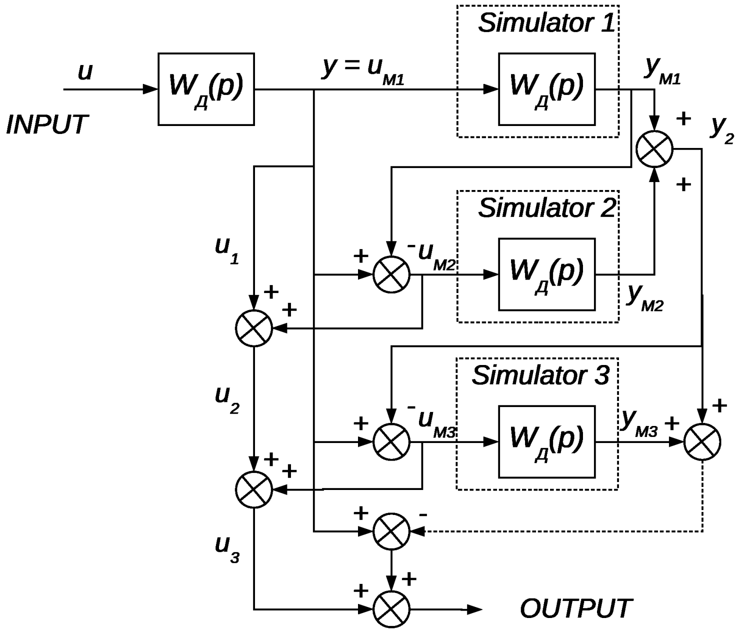

Fig. 5. Structural diagram of a dynamic measuring system with iterative principle

The idea of dynamic error correction in such a system is as follows. The output signal of the sensor y has some dynamic error of reproduction of the input signal of the sensor u. At the output of the sensor there is its model, which reproduces the signal y, which is dynamically distorted with respect to the measured signal u. By feeding the difference of signals $y{-}y_{M1}$ to the input of the second model, we reproduce this difference at the output of this model. The sum of signals $y_{M1}{+}y_{M2}{=}y_2$ reproduces the signal y with higher accuracy. Consequently, the sum of their input signals $u_{M1}{+}u_{M1}{=}u_2$ reproduces the signal u with higher accuracy, since the transfer function of the model and the sensor are the same. Then the difference of signals $y{-}y_2$ is fed to the third model. The total output signal of the first three models y_3 more accurately reproduces the output signal of the sensor y. Consequently, the total input signal of the three models $u_{M1}{+}u_{M2}{+}u_{M3}{=}u_3$ corresponds more closely to the measured signal u.

As a result of the study of frequency properties of iterative dynamic systems, the condition of error reduction at all frequencies and the limiting value of the damping coefficient were obtained.

The study of iterative measurement systems in the presence of noise at the sensor output showed that such systems insignificantly amplify the noise component of the dynamic error and are effective if the spectrum of the useful signal is located mainly in the frequency band.

1.3. Sliding Mode Method in Dynamic Measurement Systems

The introduction of the sensor model into the structural scheme of the measuring system made it possible to apply there, using a nonlinear element and an amplifying link, the theory of sliding modes, allowing potentially obtaining high dynamic accuracy and having a low sensitivity to perturbations [3]. The structural diagram of the sliding mode measuring device is shown in Figure 6. In it, a relay with $y - y_m$. as input signal is used

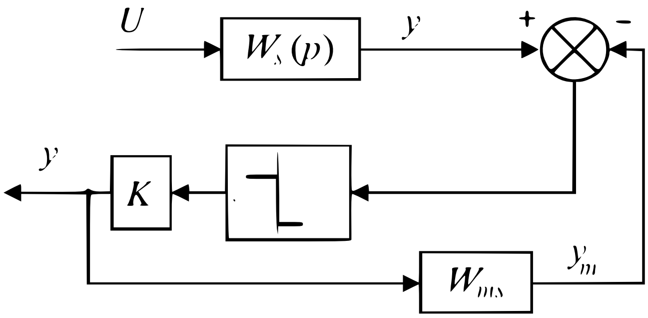

Fig. 6. Structural diagram of a dynamic measuring system in a sliding mode

as a nonlinear element for the occurrence of sliding mode. After the nonlinear element, a gain factor K is introduced, which affects both the amplitude of the signal at the output of the relay element and the switching frequency of the relay. As the gain K increases, the switching frequency of the relay element also increases. Therefore K should be chosen so that the spectrum of high-frequency oscillations lies outside the spectrum of the main signal. It should be noted that high-frequency components distorting the measured signal occur at the output of the relay element. Therefore, for qualitative restoration of the input signal of the measuring transducer it is necessary to install a low-pass filter after the relay element and gain K.

The presence of auto oscillations in the closed loop of the measurement system can lead to sliding mode failure for a system with a sensor model above the second order. Methods for eliminating auto oscillations have been proposed. The first of them was the method providing model reduction by means of order reduction. M.N. Bizyaev and A.L. Shestakov [3] showed that this method is effective only when the system order is reduced by one or two orders. The method of structural transformations, which consists in constructing a cascade structure of the measuring system, turned out to be more effective. Each subsequent cascade more accurately reconstructs the measurement signal, and the cascade partitioning solves the problem of auto oscillations and allows reducing the error of dynamic measurements.

1.4. Neural Network Method in Dynamic Measurement Systems

First of all, we note that several neural network dynamic models of measuring devices and algorithms for their training have been developed and implemented: with the purpose of determining the system parameters, according to the criterion of minimum RMS error, in the presence of noise at the sensor output, with an inverse sensor model in the form of sequential sections of the first and second order, ensuring the stability of the measuring system.

The sequential approach to synthesizing a neural network dynamic measuring system consists in determining the parameters of the transfer function of the sensor model and recovering discrete values of the input signal of the corrected model on the basis of some number of sequentially connected identical neural network measuring systems of the first order approximating the inverse transfer function of the aperiodic link. With an appropriate method of forming the input and target training sequences, reflecting the relationship between the input and output of the discrete sensor model, the parameters of the neural network model can be adjusted in the training process so that, at a given level of accuracy, the samples of the output signal of the neural network model will be equal to the corresponding discrete samples of the sensor output signal (Figure 7).

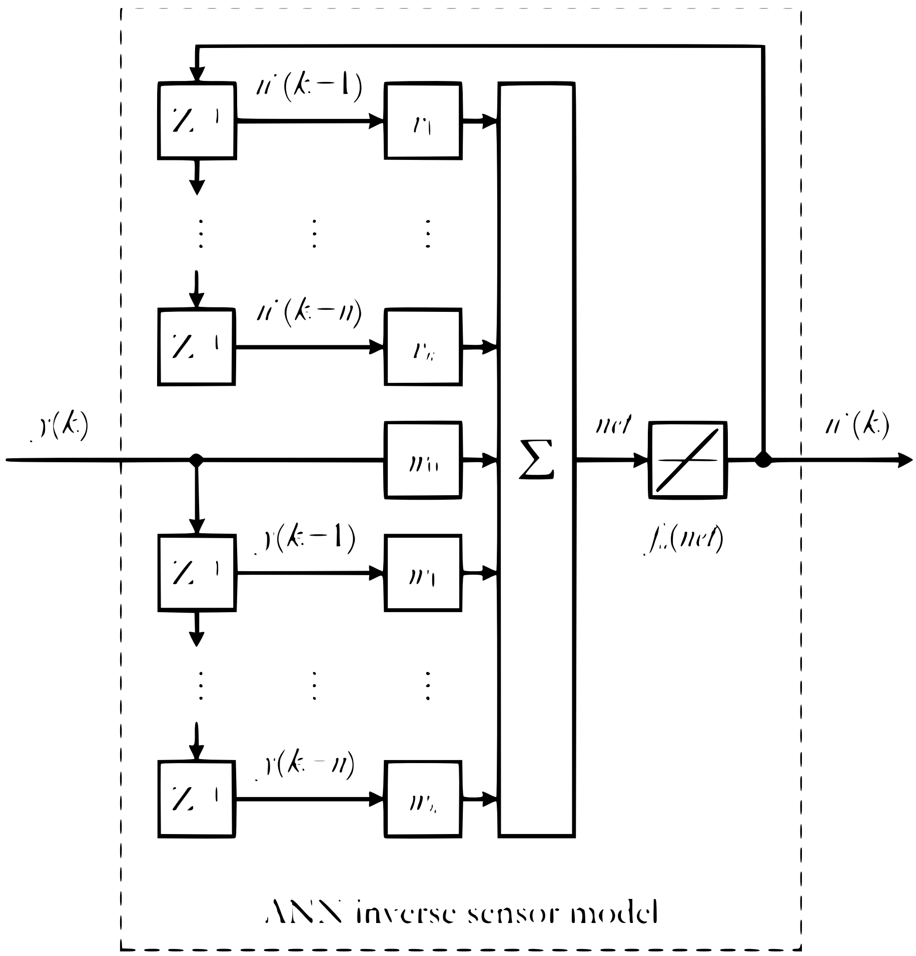

Fig. 7. Block diagram of the artificial neural network (ANN) direct sensor model

Therefore, minimization of the error function between the simulated and real output of the measurement system is chosen as a criterion for training the neural network model. The formation of the required type of the transfer function of the sensor model is based on the transformation of the measured sensor output signal by means of a correction filter and is a sequential connection of identical aperiodic links.

Based on the direct model and its training scheme, a neural network inverse dynamic sensor model and its tuning scheme are developed. The inverse model should be able to recover the dynamically distorted sensor input signal. However, when the neural network inverse model is implemented, there will be a high-frequency noise component in the recovered sensor input signal. To filter out the noise, an additional element corresponding to a filter is introduced into the training scheme, which can be a moving average filter or a recurrent filter. After training is completed, the neural network inverse model can function in dynamic mode.

As is known, dynamic error can be determined by two components: inertia of the measuring transducer (manifested by changes in amplitude and phase of the signal) and additive noise, which are superimposed on the output signal. Consequently, when one component of dynamic error is corrected, the other component grows. Applying the algorithm of dynamic error correction, the optimal filter parameters are selected to ensure the minimum square of the error between the real and simulated observation. Numerous computational experiments have shown high efficiency of neural network correction of dynamic error of measurements.

1.5. Regularization Methods in Solving the Problem of Dynamically Distorted Signal Recovery

The interdisciplinary ties developed at South Ural State University have led to the emergence of works on the study of the solution of the inverse problem of dynamic measurements as an inverse boundary value problem for an ordinary differential equation.

V.I. Zalyapin, Ye.V. Kharitonova, Yu.S. Popenko (Asfandiyarova) [1, 48] study the following mathematical model of dynamic measurements.

$$\begin{cases} L[x] = u(t), \\ \quad F_j = \alpha_j,\ j = 1, 2, ..., n, \end{cases} \tag{1}$$

where $L[x] = x^{(n)} + p_{n-1}x^{(n-1)} + \Delta\Delta\Delta + p_1 x' + p_0 x$, $u(t)$ is the input signal, coefficients $p_i(t)$ are continuous functions on $[a, b]$, $F_j(x)$ are linear, linearly independent functionals, α_j are constants. The problem of finding the solution $f(t)$ of equation (1) with the experimentally known output signal $x(t)$ and given boundary conditions is called an inverse problem.

The linear boundary value problem (1) can be equivalently replaced [48] by some problem with homogeneous boundary conditions ($\alpha_j = 0$ for $j = 1, 2, ..., n$), which we will call semihomogeneous:

$$\begin{cases} L[x] = u(t), \\ \quad F_j = 0,\ j = 1, 2, ..., n, \end{cases} \tag{2}$$

If $F(x)$ is a linear functional in $C^{n-1}_{[a;b]}$, then the numbers c_i and the function of bounded variation $\sigma(t)$ exist, and $F(x)$ can be represented in the following form

$$F(x) = \sum_{i=1}^{n} c_i x^{(i-1)}(a) + \int_a^b x^{(n-1)}(t) d\sigma(t). \tag{3}$$

The representation (3) is associated with the classical initial Cauchy problem. It is easy to show that for functionals $F(x)$ another representation associated with the simple Vallee-Poussin problem can be obtained

$$F(x) = \sum_{i=1}^{n} c_i x(t_i) + \int_a^b x^{(n-1)}(t) d\sigma(t). \tag{4}$$

where t_i are points from the interval $[a, b]$, such that $a \leq t_1 < t_2 < ... < t_n \leq b$.

The proposed methods for solving the investigated problems use the Green's function $G(t, \tau)$ and are based on the well-known relationship, which gives the solution of the semihomogeneous boundary value problem (2) in the integral form. The solution of the inverse problem represents the considering an integral equation with unknown function $u(t)$ for a given function $x(t)$. It is a Fredholm equation of the first kind and, as is well known, the problem of solving such equation is unstable, so, this requires a special procedure of regularization to ensure a satisfactory for applications accuracy of the obtained solution.

The works of D.D. Yaparov and A.L. Shestakov propose an algorithm for processing the data obtained during dynamic measurements based on the finite-difference approach [47]. The restoration of the input signal is carried out using the transfer function of the sensor. The transfer function of the sensor is presented in the form of a differential equation. This equation describes the state of a dynamic system in real time. The proposed computational scheme of the method is based on finite–difference analogs of partial derivatives and the Tikhonov regularization method was used to construct a numerical model of the sensor. The problem of stability of the method for solving high–order differential equations is also one of the central problems of data processing in automatic control systems. Based on the approach of the generalized quasi–optimal choice of the regularization parameter in the Lavrent'ev method, the dependence of the regularization parameter, the parameters of the dynamic measuring system, the noise index and the required level of accuracy was found.

2. Theory of Optimal Dynamic Measurements

2.1. Mathematical Model of the Measuring System as a Leontief-Type System

The mathematical model of the measuring system assumes the presence of several measuring transducers in it. Differential equations in the system reflect a set of dynamic elements of the system (transducers), and algebraic equations reflect connections between dynamic elements. Theoretical aspects of building a mathematical model of a measuring system are based on the ideas of the theory of descriptor systems [2]. As an example, let us take a model with the iterative principle of restoring a dynamically distorted signal (Figure 5). This model is represented by the system

$$\begin{cases} \widehat{u}_{M1} = u_{M1}, \\ \dot{z}_1 = \widehat{A}_1 z_1 + \widehat{B}_1 \widehat{u}_{M1}, \\ \widehat{y}_{M1} = \widehat{C}_1 z_1 + \widehat{D}_1 \widehat{\eta}_1, \\ \widehat{u}_{M2} = u_{M1} - \widehat{y}_{M1}, \\ \dot{z}_2 = \widehat{A}_2 z_2 + \widehat{B}_2 \widehat{u}_{M2}, \\ \widehat{y}_{M2} = \widehat{C}_2 z_2 + \widehat{D}_2 \widehat{\eta}_2, \\ \widehat{u}_{M3} = u_{M1} - \widehat{y}_{M2}, \\ \dot{z}_3 = \widehat{A}_3 z_3 + \widehat{B}_3 \widehat{u}_{M3}, \\ \widehat{y}_{M3} = \widehat{C}_3 z_3 + \widehat{D}_3 \widehat{\eta}_3, \\ \widehat{u}_{M4} = u_{M1} - \widehat{y}_{M3}, \\ \ldots \\ \dot{z}_N = \widehat{A}_N z_N + \widehat{B}_N \widehat{u}_{MN}, \\ \widehat{y}_{MN} = \widehat{C}_N z_N + \widehat{D}_N \widehat{\eta}_N, \\ y_M = \widehat{u}_{M1} + \widehat{u}_{M2} + \widehat{u}_{M3} + \ldots + \widehat{u}_{MN} + u_{M1} - \widehat{y}_{M1} - \widehat{y}_{M2} - \widehat{y}_{M3} - \ldots - \widehat{y}_{MN}. \end{cases}$$

The view of this system reflects the idea of dynamic error correction with the iterative principle, which consists in the sequential use of any number of measuring transducers, the output of the system is the sum of the observed signal and the errors simlulated by the iterative links (see Section 1.2).

Let us write down the mathematical model of the measuring device, which is presented in Figure 5, as a matrix equation for the case of two measuring transducers

$$\begin{pmatrix} I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{\widehat{u}}_{M1} \\ \dot{\widehat{u}}_{M2} \\ \dot{\widehat{y}}_{M1} \\ \dot{\widehat{y}}_{M2} \\ \dot{y}_M \end{pmatrix} =$$

$$= \begin{pmatrix} \widehat{A}_1 & 0 & \widehat{B}_1 & 0 & 0 & 0 & 0 \\ 0 & \widehat{A}_2 & 0 & \widehat{B}_2 & 0 & 0 & 0 \\ \widehat{C}_1 & 0 & 0 & 0 & -I & 0 & 0 \\ 0 & \widehat{C}_2 & 0 & 0 & 0 & -I & 0 \\ 0 & 0 & -I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -I & -I & 0 & 0 \\ 0 & 0 & I & I & -I & -I & -I \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ \widehat{u}_{M1} \\ \widehat{u}_{M2} \\ \widehat{y}_{M1} \\ \widehat{y}_{M2} \\ y_M \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \widehat{D}_1 & 0 \\ 0 & 0 & 0 & \widehat{D}_2 \\ I & 0 & 0 & 0 \\ I & 0 & 0 & 0 \\ I & 0 & D_1 & D_2 \end{pmatrix} \begin{pmatrix} u_1 \\ 0 \\ \widehat{\eta}_1 \\ \widehat{\eta}_2 \end{pmatrix}.$$

Thus, the mathematical model of the measuring system is represented as a Leontief-type system

$$L\dot{x}(t) = Mx(t) + Du(t), \quad y(t) = Nx(t), \tag{5}$$

where $x = x(t)$ is a vector-function of the measuring system state, $u = u(t)$ and $y = y(t)$ are vector-functions of input (measurement) and output (observation), L, M, D and N are matrices characterizing the design of the measuring system [17].

Since signal distortion occurs not only due to the inertia of the measuring system, but also, possibly, by other processes leading to the degradation of the measuring system, the system was proposed as a model

$$L\dot{x}(t) = a(t)Mx(t) + Du(t), \quad y(t) = b(t)Nx(t) + Fu(t), \tag{6}$$

where $a = a(t)$ and $b = b(t)$ are vector-functions of device degradation [21].

2.2. Formulation of Optimal Dynamic Measurement Problems

The first mathematical model of the theory of optimal dynamic measurements was proposed by A.L. Shestakov and G.A. Sviridyuk in [27].

Let us introduce the state spaces of the measuring device $\aleph = \{x \in L_2((0,\tau), R^n) : \dot{x} \in L_2((0,\tau), R^n)\}$, observations $\Upsilon = C[\aleph]$ and measurements $\mathfrak{A} = \{u \in L_2((0,\tau), R^n) : u^{(p+1)} \in L_2((0,\tau), R^n)\}$. Modeling the dynamic measuring system in the absence of intereference, we will consider

$$\begin{cases} L\dot{x} = Ax + Bu, \\ y = Cx \end{cases} \tag{7}$$

with the Showalter – Sidorov initial condition

$$\left[(\alpha L - A)^{-1} L\right]^{p+1} (x(0) - x_0) = 0 \tag{8}$$

In $\mathfrak{A}$ let us distinguish a closed convex set of admissible measurements $\mathfrak{A}_\partial \subset \mathfrak{A}$ of the type

$$\mathfrak{A}_\partial = \left\{ u \in \mathfrak{A} : \sum_{q=0}^{\theta} \int_0^\tau \left\| u^{(q)}(t) \right\|^2 dt \le d \right\}. \tag{9}$$

Parameter d is defined based on the physical properties of the measured process. We need to find such $v \in \mathfrak{A}_\partial$ – the optimal dynamic dimension – that achieves a minimum value of

$$J(v) = \min_{u \in \mathfrak{A}_\partial} J(u) \tag{10}$$

the penalty function

$$J(u) = \sum_{q=0}^{1} \int_0^\tau \left\| y^{(q)}(u,t) - y_0^{(q)}(t) \right\|^2 dt \tag{11}$$

where $y_0(t)$, $t \in [0, \tau]$ is a continuous-differentiable function (we will consider it as «real observation»), plotted on the basis of observed values Y_{0i} at the output of the measuring system. The considered problem (7) – (11)will be called the main problem of optimal dynamic measurements. Note that in the absence of interference, the distortion of the input signal is caused only by the inertia of the measuring device. The following is true

Theorem 1. *[12]. Let L and A be square matrices of order n, matrix A be $(L;p)$ regular, $detA \neq 0$. Then for any $x_0 \in R^n$ there exists the only solution $v \in \mathfrak{A}_\partial$ of the problem* (7) *–* (11)*, where $x\,(v) \in \aleph$ satisfies the system* (7)*, condition* (8) *and is defined by the formula*

$$x\,(t) = \lim_{k\to\infty} x_k(t) = \lim_{k\to\infty} \left[\sum_{q=0}^{p} \left(A^{-1} \left(\left(kL_k^L\,(A)\right)^{p+1} - \mathbb{I}_n \right) L \right) \times \right.$$
$$\times A^{-1} \left(\mathbb{I}_n - \left(kL_k^L\,(A)\right)^{p+1} \right) (Bu)^{(q)} + \left(\left(L - \tfrac{t}{k}A\right)^{-1} L \right)^k x_0 + \tag{12}$$
$$\left. + \int_0^t \left(\left(L - \tfrac{t-s}{k}A\right)^{-1} L \right)^k \left(L - \tfrac{t-s}{k}A\right)^{-1} \times \left(kL_k^L\,(A)\right)^{p+1} Bu\,(s)\,ds \right],$$

where $\lim\limits_{k\to\infty} \left(kL_k^L\,(A)\right)^{p+1}$ is a projector, $L^L\,(A)$ is a left resolvent of A.

For the model of the measuring device (6) approximate solutions are constructed as follows

$$x_k(t) = \int_0^t X(t,s) L_1^{-1} Q u_k(s) ds + \sum_{q=0}^{p} H^q M^{-1} (Q - \mathbb{I}_n) \left(\frac{1}{a(t)} \frac{d}{dt} \right)^q \frac{Du_k(t)}{a(t)}, \tag{13}$$

$$y_k(t) = b(t) N x_k(t) + F u_k(t). \tag{14}$$

Another powerful impetus for the development of the theory of optimal dynamic measurements was the development of the introduction of the concept of «white noise» as the Nelson – Glicklich derivative of the Wiener process [10, 18] for white noise measurements [7, 8, 30, 31, 34].

Let $\Omega \equiv (\Omega, \mathcal{A}, \mathbf{P})$ is complete probability space. *A random variable* will be called a measurable mapping $\xi : \Omega \to \mathbb{R}$. Random variables with zero mathematical expectation and finite variance form the Hilbert space $\mathbf{L}_2$ with scalar product $(\xi_1, \xi_2) = \mathbf{E}\xi_1\xi_2$. The fact that random variables $\xi \in \mathbf{L}_2$ have normal (Gaussian) distribution, we will denote by $\xi \sim N(0, \sigma^2)$, where $\mathbf{E}\xi = 0$ and $\mathbf{D}\xi = \sigma^2$.

The mapping $\eta : \mathfrak{I} \subset \mathbb{R} \times \Omega \to \mathbb{R}$ will be called (*one-dimensional*) *stochastic process.* The variable of the stochastic process $\eta = \eta(t, \cdot)$ at each fixed $t \in \mathfrak{I}$ is a random variable, i.e. $\eta(t, \cdot) \in \mathbf{L}_2$, which will be called *stochastic cross section*, whereas the variable of the stochastoc process $\eta = \eta(\cdot, \omega)$ at each fixed $\omega \in \Omega$ is called (*selective) trajectory.* A continuous stochastic process whose (independent) cross sections are Gaussian is called a Gaussian process.

The most important example of a continuous Gaussian stochastic process is the Wiener process $\beta = \beta(t)$, modeling Brownian motion on a straight line in Einstein – Smoluchowski theory [6] presented by the formula

$$\beta(t) = \sum_{k=0}^{\infty} \xi_k \sin \frac{\pi}{2}(2k+1)t, \tag{15}$$

where $\xi_k \sim N(0, [\frac{\pi}{2}(2k+1)]^{-2})$are independent normally distributed variables. Sections of a stochastic process β are normally distributed random variables with $\mathbf{E}\beta(t) = 0$ and $\mathbf{D}\beta(t) = \sigma^2 t$ at some $\sigma > 0$. The stochastic process β presented by means of (15) will be called *one-dimensional Brownian motion.*

Sections of a stochastic process $\overset{\circ}{\beta}$ are distributed according to the normal law with parameters $(0, \frac{\sigma^2}{4t})$, i.e. $\overset{\circ}{\beta}(t) \sim N(0, \frac{\sigma^2}{4t})$. That is why the Nelson – Glicklich derivative $\overset{\circ}{\beta}$ of the Brownian motion β from (15) will be called *one-dimensional «white noise».*

Let us consider the following stochastic model of the measuring system:

$$\begin{cases} L\dot{\xi} = A\xi + B\left(u + \varphi\right), \\ \eta = C\xi + v, \\ \left[(\alpha L - A)^{-1} L\right]^{p+1} \left(x\left(0\right) - x_0\right) = 0 \end{cases} \tag{16}$$

Here, matrices L, A, B, C have the same meaning as in (5). Random processes φ and v determine noise in circuits and at the output of the measuring device, respectively.

Consider the first equation of the system (16), which is a stochastic equation of Leontief type:

$$L \overset{\circ}{\xi} = A\xi + \ B(u + \varphi), \tag{17}$$

where $u : I \longrightarrow \mathbb{R}^n$ is a vector function of the useful input signal, φ is a stochastic process modeling noise, where the frequencies of the useful signal are different from the noise frequencies. Let matrix A be (L, p)-regular, $p \in \{0\} \cup \mathbb{N}$, and initial states (17) are described by the Schoulter-Sidorov initial condition:

$$\left[(\alpha L - A)^{-1} L\right]^{p+1} \left(\xi\left(0\right) - \xi_0\right) = 0 \tag{18}$$

where $\xi_0 = \sum_{k=0}^{n} \xi_{0,k} e_k$, $\xi_{0,k}$ are pairwise independent Gaussian random variables, and $\{e_k\}_{k=1}^{n}$ is an orthonormalized basis in R^n.

Theorem 2. *For any vector function $u \in C^{p+1}(I,\mathbb{R}^n)$, and initial values ξ_0 and stochastic process $\varphi \in C^{p+1}\boldsymbol{L_2}(I,\mathbb{R}^n)$, independent for any $t \in I$, there exists the unique solution ξ of the problem* (5), (6)*, given by the formula*

$$\xi(t) = \xi_u(t) + \xi_\varphi(t), \xi_u \in C^1(I,\mathbb{R}^n), \ \xi_u \in C^1\boldsymbol{L_2}(I,\mathbb{R}^n) \tag{19}$$

where ξ_u is a deterministic, and ξ_φ is a stochastic part of the solution

$$\begin{aligned} \xi_u(t) &= \int_0^t U^{t-s}L_1^{-1}Qu(s)\,ds + \sum_{q=0}^{p}\left(M^{-1}(I_n - Q)L\right)^q M^{-1}(Q - I_n)u^{(q)}(t), \\ \xi_\varphi(t) &= U^t\xi_0 \int_0^t U^{t-s}L_1^{-1}Q\varphi(s)\,ds + \sum_{q=0}^{p}\left(M^{-1}(I_n - Q)L\right)^q M^{-1}(Q - I_n)\overset{\circ}{\varphi}{}^{(q)}(t). \end{aligned} \tag{20}$$

Here $U^t = \lim\limits_{r\to\infty}\left(\left(L - \frac{t}{r}M\right)^{-1}L\right)^r$, $Q = \lim\limits_{r\to\infty}\left(rL_r^L(M)\right)^p$, $L_r^L(M) = L(L - \frac{1}{r}M)^{-1}$, *and* I_n *is a unit matrix of order* n.

Dividing the problem into deterministic and stochastic ones, we show the existence of a single solution of a stochastic Leontief-type system.

Similarly to the deterministic case in the study of the problem of restoration of dynamically distorted signal by random noise in the circuits and at the output of MT, consider the control problem (10), where the quality functional

$$J(u) = J(\eta(u)) = \sum_{k=0}^{1}\int_0^\tau E\left\|\overset{\circ}{\eta}{}^{(k)}(t) - \eta_0^{(k)}(t)\right\|^2 dt \tag{21}$$

reflects the closeness of the real observation $\eta_0(t)$ and virtual observation $\eta(t)$,obtained on the basis of the mathematical model of the measuring device.

The minimum point $v(t)$ of the functional (21) on the set U_∂, which is the solution to the optimal control problem (10), is optimal dynamic measurement. In practice there is only indirect information about $v(t)$.

Since the input signal is subject to noise in the circuits and at the output of the measuring device, the virtual observation $\eta(t)$ is a stochastic process, the real observation $\overline{\eta_0}(t)$ at each point $t \in I$. Let us denote by $\widetilde{\eta_0}(t)$ a stochastic process $\eta_0(t) - \overline{\eta_0}(t)$ with a zero mathematical expectation. Let us transform the quality functional:

$$J(u) = \sum_{k=0}^{1}\int_0^\tau E\left\|\overset{\circ}{\eta}{}^{(k)}(t) - \eta_0^{(k)}(t)\right\|^2 dt = \sum_{k=0}^{1}\int_0^\tau E\left\|C\overset{\circ}{\xi}{}^{(k)}(t) + \overset{\circ}{v}{}^{(k)} - (\overline{\eta_0}^{(k)}(t) + \overset{\circ}{\widetilde{\eta}}{}_0^{(k)}(t))\right\|^2 =$$

$$= \sum_{k=0}^{1}\int_0^\tau \left\|C\xi_u^{(k)}(t) - \overline{\eta_0}^{(k)}(t)\right\|^2 dt + \sum_{k=0}^{1}\int_0^\tau E\left\|C\overset{\circ}{\xi}{}_\varphi^{(k)}(t) + \overset{\circ}{v}{}^{(k)} - \overset{\circ}{\widetilde{\eta}}{}_0^{(k)}(t)\right\|^2 dt$$

Thus, noise and random initial conditions do not affect the optimal dynamic measurement as the minimum point of the quality functional. They only affect the value of the optimality criterion, namely increase the value.

2.3. Spline Method for Solving the Problem of Optimal Dynamic Measurement Taking Into Account the Inertia of the Measuring Device

Let us describe the spline method for solving the problem of optimal dynamic measurement (7) – (11).

Suppose that the following components are given: the matrices included in system (7); the initial value $x_0 \in R^n$; the array of observed values Y_{0i} at the nodal points $t_i = 0, 1, \dots, n$ of the output signal, and $t_{i+1} - t_i = \delta$, $t_0 = 0$, $t_n = \tau$.

Divide the interval $[0, \tau]$into M intervals $[\tau_{m-1}, \tau_m]$, where $m = 1, 2, \dots, M$, and $t_0 = \tau_0 = 0$, $t_n = \tau_M$.

At each interval $[\tau_{m-1}, \tau_m]$, construct the interpolation function $y^{\ell}_{0m}(t)$ in the form of a polynomial of the degree $\ell \leq (n-1)/M$.

For $m = 1, 2, \dots, M$ at $[\tau_{m-1}, \tau_m]$, consecutively solve the optimal dynamic measurement problem (7)-(11) for $u \in \mathfrak{A}_{\partial m}$, where $\mathfrak{A}_{\partial m} \subset \mathfrak{A}_{\partial}$ is a closed convex subset of $\mathfrak{A}_{\partial}$, by the method described in [33]. We find the approximate value of the optimal measurement $v^{\ell}_{km}(t)$ in the form of a polynomial of the degree ℓ imposing the continuity condition

$$v^{\ell}_{km}(\tau_m) = v^{\ell}_{k,m+1}(\tau_m). \tag{22}$$

As a result, we get a spline function $\tilde{v}^{\ell}_{k}(t) = \bigcup_m v^{\ell}_{km}(t)$ continuous on $[0, \tau]$.

2.4. Methods of Extracting Useful Output Signal

In the course of dynamic measurements in the presence of noise of various nature, the output signal is noisy. The useful output signal is the output signal that is distorted only by the inertia of the measuring device. Thus, the real observation is represented as a sum of two components – useful output signal and noise component of the output signal. For «filtering the observation» and selecting the useful output signal, it was proposed to use the Savitsky–Golei digital filter [13], the moving average digital filter [40],the one-dimensional Kalman filter [39] and the Pyt'ev–Chulichkov method [20]. After applying the filters and obtaining an approximate useful output signal, we then proceed to implement a spline method for solving the deterministic (7) – (11) dynamic measurement problem.

Suppose that the following components are given: matrices included in the system (16), initial value $x_0 \in \mathbb{R}^n$, array of observed values Y_{0i} in nodal points $t_i = 0, 1, \dots, n$ of the output signal and $t_{i+1} - t_i = \delta$, $t_0 = 0$, $t_n = \tau$. In order to obtain a useful signal output in [13] a digital Savitzky – Goley filter [22], which is a noise filtering method based on the least squares method, is proposed. The idea is to construct a polynomial of the s degree, approaching $2\mu + 1$ equidistant points and use the value of the polynomial in the $\mu + 1$ th point as a value of useful output signal for this purpose it is necessary to determine the parameters μ and s of the digital filter Savitsky – Golay and apply the filter to the array of values Y_{0i}. As a result, the values of useful output signal y_{0i}, $i = 0, 1, \dots, n$ will be obtained. The peculiarities of selection of parameters of this filter are discussed in [13], the results of computational experiments are given.

In [39] a discrete one-dimensional Kalman filter was proposed to extract the useful output signal. We assume that Y_{0i} and Y_i are related as follows:

$$Y_{0i} = Y_i + \xi_i,$$

where $\xi_i\tilde{N}(0,\sigma_\xi^2)$ are normally distributed random variables. Using the Kalman filter, the optimal estimate at time t is computed in two steps: prediction from the process model and correction from the observational data. Let us denote by $\hat{Y}_i$ the prediction of the output at time t_i from the estimate at time t_{i+1}. For the first $N+1$ observations, we assume that $Y_i = Y_{0i}$, $i = 0, 1, \ldots, N$.

All subsequent considerations will be carried out for $i = N+1, N+2, \ldots, n$. Let us assume that we consider the output signal outside its physical model, in this case the forecast $\hat{Y}_i$ is given by the equation

$$\hat{Y}_i = Yi - 1. \tag{23}$$

To obtain the best approximation to the desired value of Y_i, a weighted average between the observation Y_{0i} and the prediction $\hat{Y}_i$i at time t_i is found, where the weights are the values of K_i and $1 - K_{i,i}$ where K_i is the Kalman coefficient.

To obtain the best approximation to the value sought, a weighted average is found between the observation at the time, where the weights are the values of and

$$Y_i = K_iY_{0i} + (1-K_i)\hat{Y}_i \quad \text{или} \quad Y_i = \hat{Y}_i + K_i(Y_{0i} - \hat{Y}_i).$$

The Kalman coefficient is calculated by the formula:

$$K_i = \frac{\hat{P}_i}{\hat{P}_i + \sigma_\xi^2},$$

where $\hat{P}_i$ is the estimation of forecast error variance, $\hat{P}_i = P_{i-1} + w_i, \quad P_N = const,$

$$w_i = \frac{1}{N-1}\sum_{m=0}^{N-1}\left((Y_{0,i-m} - Y_{i-m-1}) - \frac{1}{N}\sum_{m=0}^{N-1}(Y_{0,i-m} - Y_{i-m-1})\right)^2.$$

The value of the parameter N is selected experimentally. For the next forecast, the estimate of the forecast error variance should be recalculated by the formula

$$P_i = (1-K_i)\hat{P}_i.$$

As a result, we obtain the values of the useful observable signal. Now let us present the results of adapting the Pyt'ev – Chulichkov method [19] to the solution of optimal dynamic measurement problems [20, 21].

In general, we assume that an observation is an n-dimensional vector $\{\eta^1(t), \eta^2(t), \ldots, \eta^n(t)\}$, whose values are known at time instants $\{t_j : j \in \mathcal{I}\}$, $\mathcal{I} = \{0, 1, \ldots, N\}$, i.e. $\eta^i(t_j)$ ($i = \overline{1,n}$, $j = \overline{0,N}$) are obtained. Additionally, a priori information is known about the extremum and the nature of the convexity of the utility part of each of the observed quantities $\eta^i(t)$. In connection with this and the linearity of the model of the measurement system, the assumption is made that for each component of the observation there is a representation of the

$$\eta^i(t) = \widetilde{y}_i(t) + \overset{\circ}{\beta}_i(t), \qquad i = \overline{1,n}.$$

Here, $\widetilde{y}_i(t)$ is the useful part corresponding to the i- observation coordinate, and $\overset{\circ}{\beta}_i(t)$ is part contributing interference in the corresponding coordinate – «white noise», whose cross sections have the distribution $N(0, \frac{\sigma^2}{4t})$.

To describe the statistical criterion for determining the useful part of an observation, we introduce a class of V_k convex upward functions with a single maximum point t_k on a uniform grid $\{t_j : j \in \mathcal{I}\}$, $\mathcal{I} = \{0, 1, \ldots, N\}$.

Let us fix the observation coordinate $i \in \mathbb{N} : 1 \leq i \leq n$. Let the useful component part of the signal $\tilde{y}_i(t)$ have a maximum at a point $k_0 \in \mathcal{I}$ _0 of a uniform grid, i.e. $\tilde{y}_i \in V_{k_0}$. Let us estimate parameter k_0 by the values $\{\eta^i(t_j)\}_{j=0}^N$ with a given probability γ

$$\eta^i(t) = \widetilde{y}_i(t) + \overset{\circ}{\beta}_i(t), \quad \widetilde{y}_i \in V_{k_0}, \quad \overset{\circ}{\beta}_i(t) \sim N(0, \frac{\sigma^2}{4t}).$$

Based on the results [5, 19] in order to estimate the parameter $k_0 \in \mathcal{I}$ for i coordinated of observation, the following statistics is applied

$$\tau_k(i) = \frac{\sum_{j=0}^{N} (\eta^i(t_j)\sqrt{t_j} - P_k(\eta^i(t_j)\sqrt{t_j}))^2}{\sum_{j=0}^{N} (\overline{\eta^i} - P_k(\eta^i(t_j)\sqrt{t_j}))^2},$$

where $\overline{\eta^i} = \frac{1}{N+1} \sum_{j=0}^{N} \eta^i(t_j)\sqrt{t_j}$, and $P_k(\eta^i(t)\sqrt{t})$ is a projection of $\eta^i(t)\sqrt{t}$ onto the set V_k, whose existence is shown in [5], and its construction is described in [20].The value of the constructed statistics is used to find the value of the parameter k, at which the useful part of the signal $\eta^i(t)\sqrt{t}$ is closest in shape to $P_k(\eta^i(t)\sqrt{t})$.

The problem of constructing values of a single coordinate of observations on a uniform grid $\{t_j\}_{j=0}^N$ is viewed as the problem of the best approximation of $\widetilde{y}_i(t)\sqrt{t}$ by elements of the set V_k, that is, finding a function $P_k(\eta^i(t)\sqrt{t}) \in V_k$, such that $\|P_k(\eta^i(t)\sqrt{t}) - (\eta^i(t)\sqrt{t})\|^2 = \inf_{f_i \in V_k} \|f_i - \eta^i(t)\sqrt{t}\|^2$. An algorithm for constructing the utility values of a single observation coordinate $P_k(\eta^i(t)\sqrt{t})$ is given in [21].

Applying to all coordinates the algorithm for constructing the useful part of one coordinate of the observation distorted by white noise, under the additional assumption of the singularity of its extremum point and upward convexity, we obtain the values of the smoothed vector function of the observation $\tilde{y}(t)$.

2.5. Numerical Method for Restoration of Dynamically Distorted Signal Based on the Sampling Theorem With Simple Averaging

As initial data, we know the elements of the system matrices and initial conditions (16), the quality functional is defined as (16), an array of values Y_{0i} of the observed signal at time t_i with interva δ, $i = 1, 2, ..., N$.

From the set of Y_{0i} values, a subset is formed, the elements of which are selected through an equal interval, which we will call the discretization interval of the algorithm Δ in time. Note that then $\Delta = K \cdot \delta$, $K \in \mathbf{Z}$. The choice of the discretization interval is an important independent problem, which is described in detail in [38]. he choice of Δ given a known δ determines the value of K, which in turn determines the number K of basic cycles of the algorithm. We denote the cycle number by the letter ℓ, so $\ell = 1, 2, ..., K$.

Basic calculations for each of the main cycles of finding an approximate dynamic measurement of the input signal $v^\ell(t)$, $\ell = 1, 2, .., K$ are performed.

The initial calculation point t_0^ℓ and the state of the system are determined as x_0^ℓ: t_0^ℓ: $t_0^1 = 0$, $t_0^\ell = t_0^1 + (\ell - 1)\delta$, $\ell = 2, .., K$. x_0^ℓ: $x_0^1 = 0$, $x_0^\ell = C^{-1}Y(t_0^1 + (\ell - 1)\delta)$, $\ell = 2, .., K$.

The points for the basic calculation are selected as $T_k^\ell = t_0^\ell + (k - 1)\Delta$, $k = 1, 2, ...R$, where $R = \left[\frac{N}{K}\right]$. These points are grouped into four $(T_1^\ell, T_2^\ell, T_3^\ell, T_4^\ell), (T_4^\ell, T_5^\ell, T_6^\ell, T_7^\ell)$ etc. These groups will be called «arrays for T». Each array for T allows the formation of array for Y: $(Y_1^\ell, Y_2^\ell, Y_3^\ell, Y_4^\ell), (Y_4^\ell, Y_5^\ell, Y_6^\ell, Y_7^\ell)$ etc. For each j array of ℓ main cycle at the moment $[T_{3j+1}^\ell, T_{3j+4}^\ell]$, $j = 0, ..., [(R - 1)/3]$: 1) by means of Y_k^ℓ interpolation determines the observation function $y_j^\ell(t)$; 2) the optimal dynamic measurement problem is solved using the constructed observation function.

The approximate measurement is sought in the form of a polynomial of a given degree, therefore the main procedure is reduced to the search of such an array of its coefficients at which at which the minimum of the functional is achieved. The algorithm implements for this purpose multistep iterative method proposed in [12]. It utilizes the ideas of a multi-step multi-coordinate descent with memory; when selecting a step, the results of the preceding iteration are used with the verification of the constraint conditions for belonging to the set of admissible measurements. The procedure of finding the minimum of the quality functional is completed when the absolute value of the difference between the functional values of the last and penultimate iteration of the cycle reaches a value smaller than the specified error. The main stages of the algorithm for solving the problem of optimal dynamic measurement are outlined in [38]. The peculiarity of the algorithm proposed here is the condition of equality of the values of $u_j^\ell(t)$ at the boundary points of the sets $[u_j^\ell(T_{3j+4}) = u_{j+1}^\ell(T_{3(j+1)+1})]$, $j = 0, ..., [(R - 1)/3]$. As a result, we obtain an approximate optimal dynamic measurement $v_j^\ell(t)$ for each j-th set of ℓ-th main cycle on the time interval $[T_{3j+1}^\ell, T_{3j+4}^\ell]$, $j = 0, ..., [(R - 1)/3]$.

We calculate the values $v_i^\ell = v_j^\ell(t_i)$, $\ell = 1, 2, .., K$, $i = 1, 2, ..., N$, $j = 1, ..., [(R - 1)/3]$.

Using the obtained K of values v_i^ℓ, $\ell = 1, .., K$ at each point t_i, we calculate the average values of $\overline{v_i}$ at each point t_i. The simulated optimal dynamic measurement $v(t)$ is obtained by the interpolation of average values $\overline{v_i}$. Computational experiments have shown the high efficiency of this algorithm.

This article is written in connection with the 80th anniversary of South Ural State University as a sign of the deepest respect for the leaders of the two scientific schools – professors A.L. Shestakov and G.A. Sviridyuk, whose tireless scientific work has become both an example for their students and an inspiration for their colleagues.

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МЕТОДЫ АВТОМАТИЧЕСКОГО И ОПТИМАЛЬНОГО УПРАВЛЕНИЯ В ДИНАМИЧЕСКИХ ИЗМЕРЕНИЯХ

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Статья представляет собой обзор результатов решения задач динамических измерений двух научных школ, работающих в Южно–Уральском государственном университете. Статья представляет собой обзор результатов решения задач динамических измерений двух научных школ, работающих в Южно–Уральском государственном университете. Динамические свойства измерительной системы являются критически важными факторами, влияющими на динамическиую погрешность измерений, при этом структуры динамической измерительной системы и системы автоматического управления имеют общие принципы построения. Это позволило для исследовании динамических измерительных систем применить методы теории автоматического управления. Однако, динамические измерительные системы отличает отсутствие обратной связи, что потребовало при использовании идей теории автоматического управления разработки новых методов. К ним относится метод модального управления динамическими характеристиками измерительных систем. Его развитие привело к разработке и применению других методов: итерационного принципа измерительных систем, метода скользящих режимов, параметрической адаптации систем, нейросетевые технологии, численные методы решения обратных задач. Этим исследованиям в статье посвящен первый раздел. Во втором разделе представлены результаты теории оптимальных динамических измерений, в рамках которой задача восстановления динамически искаженного сигнала решается с использованием методов теории оптимального управления, а измерительное устройство моделируется системой леонтьевского типа. Сведение решения обратной задачи динамических измерений к прямой математической задаче позволило результативно применить существующий математический аппарат теории уравнений соболевского типа в случае учета инерционности измерительной системы. Для исследования задачи восстановления динамически искаженного сигнала при наличии «шумов» были начаты аналитические, а затем и численные исследования, которые привели к созданию теории стохастических уравнений соболевского и леонтьевского типа и развитию численных методов. В обзоре особое внимание уделено численным методам, построенных на идее выделения полезного выходного сигнала по известному зашумленному наблюдению с последующим применением численного метода восстановления входного сигнала. Кроме того кратко представлен алгоритм нового численного метода, основанным на использовании теоремы отсчетов и простого усреднения. Библиографический обзор составлен на основе излагаемых результатов и, безусловно, не является исчерпывающим.

Ключевые слова: динамические измерения; автоматическое управление; опитмальное управление; системы леонтьевского типа; оптимальное динамическое измерение.

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