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ASYMPTOTIC PROPERTIES OF SOME NEUTRAL TYPE SYSTEMS WITH LINEAR DELAY

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The paper studies differential equations with linear delay of neutral type. Equations with linear delay occur in problems of mechanics, biology, and economics. A feature of such equations is that the delay is unbounded, which significantly reduces the applicability of traditional methods for studying stability problems for similar systems. One of the approaches to studying asymptotic properties is to replace the argument. The system is reduced to a system with a constant delay, but in this case an exponential factor appears on the right side of the system obtained, and the right side of the resulting system becomes unbounded as $t > \infty$. The asymptotic properties of systems without neutral terms were studied by the authors earlier. Taking into account the asymptotic properties of these systems (without neutral terms on the right side), an analysis of the asymptotic properties (boundedness, stability and asymptotic stability) of some systems of a neutral type is carried out. Since the property of stability is a more subtle property than the property of asymptotic stability, we study a system of neutral type with perturbations, which is simply stable when unperturbed.

Keywords: stability; asymptotic stability; Lyapunov-Krasovsky functionals.

Introduction

A large list of systems with linear or proportional delay occurring in problems of physics, mechanics, biology, economics, queuing theory, etc. is given in [1]. The papers [2], [3], and [4] studied the systems without neutral terms and those reducible to them. In particular, the problem of vertical oscillations of a locomotive pantograph runner passing through the elastic support was considered in [4]. When moving away from the support, it is necessary to consider equations of neutral type. Among the works devoted to the study of neutral type systems, we note [5] and [6]. We consider a linear normalized space R^m, in which we define the norm of the vector $\|w\| = \sum\limits_{j=1}^{m} |w_j|$ $w = {w_j}^{\top}$, where w_j $(j = 1, 1, ...m)$ are the components of the vector w and $\top$ is the transpose sign, for example, by equality $\|w\| = \sum\limits_{j=1}^{m} |w_j|$. We define the norm of the matrix $D = d_ij(i, j = 1, ..., m)$ in accordance with the norm of the vector [7, p.12]: $\|D\| = \max\limits_{j} \sum\limits_{i} |d_{ij}|)$. This choice is due to the fact that the norm of a matrix is defined in almost the same way as the norm of a vector. We also consider the norm of the vector function $w(\tau)$ on the interval $\tau \in [0, \sigma]$:

$$\|w(\tau)\|_{\sigma} = \sup_{0 \leq \tau \leq \sigma} \|w(\tau)\|, \ \ \sigma = -\ln(\mu). \tag{0.1}$$

We will need the norm on the segment in the future, since we are interested in limited solutions and intend to apply the apparatus of the theory of variable systems to study their properties [8, p. 23]. Accordingly, in the future replacement of the argument $\tau = ln(t/t_0)$, the initial system with linear delay is led to a system with constant delay. We give some definitions necessary in the future [9, p.161]

Definition 1. A continuous solution to a neutral type system with delay σ = const $\sigma > 0$

$$dx(\tau)/d\tau = A(\tau)x(\tau) + B(\tau)x(\tau-\sigma) + R(\tau)dx(\tau-\sigma)/dt, \ \ \tau \geq \tau_0, \qquad (0.2)$$

determined by a continuously differentiated initial vector-function $\phi(\xi)$: $\tau_0 - \sigma \leq \xi \leq \tau_0$, is called bounded if there is a constant $C > 0$, such that boundedness of the solution $x(t, \phi(\xi), \phi'(\varepsilon))$ follows from the inequality $\|x(\tau)\| + \|x'(\tau)\| < C$.

Definition 2. A solution to a neutral type system with delay σ = const, $\sigma > 0$ $x(\tau, \phi(\xi), \phi'(\xi))$ is called stable if, for any $\varepsilon > 0$, there is $\delta(\varepsilon) > 0$ such that $\|x(\tau, \phi(\xi), \phi'(\xi))\| < \varepsilon$, $\tau \geq \tau_0$ as soon as $\|\phi(\xi)\| + \|\phi'(\xi)\| < \delta$.

Definition 3. If $x(\tau, \phi(\xi), \phi'(\xi)$, being stable, has also the property of $\lim_{\tau\to\infty} x(\tau, \phi(\xi), \phi'(\xi)) = 0$, then the solution is asymptotically stable.

If we now take into account the norm of the vector function defined by equality (0.1), then with this normalization, the linear space of continuous vector functions will be a Banach space [8, p.162]. We denote it as C^0. In the future, we will use some properties of this space. In addition, taking into account the introduced definitions of stability and asymptotic stability, we will consider the space C^1 which is the space of continuous vector functions that have a continuous derivative [9, p.159]. Since for linear systems, the uniform boundedness of the solution implies Lyapunov stability [5, p.194], and due to the fact that the authors have previously obtained asymptotic estimates for some of the systems under study using similar uniform estimates which we will use in the future, we will often prove the uniform boundedness of linear time delay systems.

Consider a linear homogeneous system of neutral type with a delay linearly dependent on time (argument)

$$dx(t)/dt = Ax(t) + Bx(\mu t) + \hat{R}(t)dx(\mu t)/dt, \ \ t \geq t_0 > 0, \ \ \mu = \text{const}, \ \ 0 < \mu < 1. \quad (0.3)$$

The solution to the system (0.3) is defined on the initial set $s \in [\mu t_0, t_0]$ by the vector function $\phi(s) \in C^1$. A and B are constant matrices $m \times m$, $R(t)$ is a continuously differentiable $m \times m$ matrix, and $x(t) \in R^m$. By replacing the argument $\tau = ln(t/t_0)$, we obtain a system with constant delay $\sigma = -ln(\mu)$, $\sigma > 0$.

$$dz(\tau)/d\tau = t_0 e^{\tau}[Az(\tau) + Bz(\tau-\sigma)] + \mu R(\tau)dz(\tau-\sigma)/d\tau, \ \ \tau \geq 0. \qquad (0.4)$$

$\sigma = -\ln(\mu)$, $\sigma > 0$. Here $R(\tau) = \hat{R}(t_0 e^{\tau})$. We will study the question of the system (0.4) stability depending on the properties of the matrices A, B, and $R(\tau)$.

1. Boundedness of the Solution of a System Without Neutral Members

Let us first consider the asymptotic behavior of the system without neutral terms. Let

$$Re(\lambda_j) < 0, \ \ j = 1, ...m, \qquad (1.1)$$

where λ_j are the eigenvalues of matrix A. Therefore, there is a value $\beta_0 > 0$: $-\beta_0 = \max_j(Re(\lambda_j)) + \varepsilon$, $Re(\lambda_j) < -\beta_0$. Then, along with this, we assume that the eigenvalues

ρ of the matrix $-A^(-1)B$ satisfy the inequality $|\rho| \leq 1$, while all eigenvalues $|\rho_k| = 1$ correspond to Jordan blocks of size 1 [8, p.21] .

Theorem 1. When the conditions imposed on the eigenvalues λ given at the (1.1) and $|\rho| \leq 1$, (while all eigenvalues $|\rho_k| = 1$ correspond to Jordan blocks of size) the solution of the system

$$dy(t)/dt = Ay(t) + By(\mu t) \tag{1.2}$$

uniformly bounded.

Proof. Having differentiated both parts of system (1.2), we obtain a system of the form

$$(dy'(t))/dt = Ay'(t) + \mu By'(\mu t), \;\; t > t_0/\mu, \;\; y'(t) = dy(t)/dt. \tag{1.3}$$

It is known [9, p.20] that for $t > t_0/\mu$, a solution to system (1.3) exists. Then, due to the fact that the inequality $|\rho_j| < 1$ holds for the eigenvalues $\bar{\rho}_j$ of the matrix $-\mu A^{-1}B$ (taking into account (1.2)), the estimate follows from the results of work [2]

$$\|y'(t)\| \leq M(t/t_0)^{-\beta} \sup_{\tau \in (\mu t_0 \leq s \leq \mu^(-1)t_0)} \|y'(s)\|, \;\; M, \beta = \text{const}, \;\; M > 1, \;\; \beta > 0. \tag{1.4}$$

The constants M and β are the same for any $t_0 \geq t_0^*$, $t_0^* = \text{const}$, t_0^* is fixed, and $t_0^* > 0$). Now, from equality (1.2) we obtain the inhomogeneous difference matrix equation

$$y(t) = -A^{-1}By(\mu t) + A^{-1}y'(t). \tag{1.5}$$

Assuming that the value y'(t) is inhomogeneity, we write the solution to system (1.5) using the formula for variation of constants [8]:

$$y(t) = (-A^{-1}B)^n \phi(\mu^{n+1}t) + \sum_{j=0}^{n} \left(\mu^(- A^{-1}B)\right)^j A^{-1}y^{'}(\mu^j t). \tag{1.6}$$

In view of the assumptions regarding the eigenvalues of the matrix $-A^{-1}B$, the following inequality holds:

$$\|(-A^{-1}B)^n\| \leq M_1, \;\; M_1 = \text{const}, \;\; M_1 > 1, \;\; n = 1, 2, ... \tag{1.7}$$

Note that if we do not take into account the properties of the eigenvalues of the matrix $A^{-1}B$, but only taking into account the estimate

$$\|expA(t-s)\| < M_0 exp{-}\beta_0(t-s), \;\; M_0 = \text{const}, \;\; M_0 > 1, \tag{1.8}$$

when writing the solution to system (1.2) in integral form (considering the terms with delay to be inhomogeneous), then, giventhis integral solution to system (1.2), we have the inequality

$$\sup_{t \in [t_0, \mu^{-1}t_0]} \|y'(t)\| \leq [\|A\|K_0 + \|B\|] \sup_{t \in [\mu t_0, t_0]} y(t), \;\; K_0 = M_0[1 + \frac{\|B\|}{\beta_0}] \tag{1.9}$$

Hence from (1.4), (1.6), (1.9) we have the estimate

$$y_n \leq M_1 \left[\sup_t \|y_0(t)\| + \|A^{-1}\| M_1 \sum_{j=0}^{n-1} \|y_j'\| \right] < M_1 \left\{ 1 + \|A^{-1}\| \left[\frac{\|A\|K_0 + \|B\|}{\mu(1-\mu^\beta)} \right] \right\} \sup_t \|y_0\|,$$

$$y_n = \sup_{t\in(\mu^{1-n},\mu^{-n}]} \|y(t)\|. \tag{1.10}$$

From estimate (1.10) it follows that the solution of system (1.2) is uniformly bounded under the boundedness $\|y_0(t)\|$.

As follows from [5, p. 194], due to the uniform boundedness of the solution to the linear system (2), its solution is stable. Note that the condition under which all eigenvalues $|\rho_k|$=1 correspond to Jordan blocks of size 1 is essential. If this condition is not met, the degenerate (difference) system

$$\bar{y}(t) = -A^{-1}B\bar{y}(\mu t)$$

is unstable. It is shown in [10] that in this case the solution to the original system (2) is also unstable.

The asymptotic stability of systems of the form (0.4) for sufficiently small μ was studied earlier in [6]. The instability of some systems, such as (0.3), for example, for positive μ sufficiently close to unity, can be solved using alternating Lyapunov-Krasovsky functionals of type

$$V = \hat{W}(x) + \sum_{j=1}^{m} \nu_j \int\limits_{\mu t}^{t} x_j^2(s)ds,$$

where $W(x)$ is a quadratic form (not necessarily positive definite); scalar values $\nu_j > 0$. Let us give a simple example. Consider the first order equation

$$dx(t)/dt = ax(t) + b(t)x(\mu t) + \bar{r}(t)dx(\mu t)/dt \tag{1.11}$$

where $a =$ const, $a > 0$, $b(t)$, and $\bar{r}(t)$ are scalar functions of time (argument) t. Consider the equation without neutral terms

$$dx_0(t)/dt = ax^0(t) + b(t)x^0(\mu t) \tag{1.12}$$

To obtain sufficient conditions for the instability of the solution to equation (1.12), for the instability of the solution to equation (1.12), we introduce the functional $V^0(x^0,t) = (x^0)^2 + \bar{\alpha}\int\limits_{\mu t}^{t}(x^0)^2(s)ds$. Here the constant $\bar{\alpha}$ is negative. Assuming $\bar{\alpha} = -a$, calculating the derivative $dV^0(x^0,t)/dt = a(x^0)^2 + 2b(t)x^0(t)x_0(\mu t) + \mu a(x^0(\mu t))^2$ by accoding of equation (1.12), and requiring positive definiteness of the resulting quadratic form of the variables $x^0(t), x_0(\mu t)$ [9, p.147], we obtain sufficient conditions for the instability of the solution to shortened equation (1.12)

$$a > 0, |b(t)| < \sqrt{\mu}(a - \varepsilon), \tag{1.13}$$

where ε is a sufficiently small positive number. In fact, there is a region $V > 0$, the boundaries of which are the curve of form $x(t) \approx \mathbf{O}(e^{(a-\varepsilon)t})$ [1] and the line $x = 0$. In this region, the derivative $dV^0(x_0,t)/dt$ is also definitely positive, therefore, by virtue of the analogue of Chetaev's theorem [5, p. 182], this implies instability of the solution to equation (1.12).

Let us now consider a shortened system (i.e. a system without neutral terms, resulting from system (0.4) To study the instability of such systems, one can use alternating functionals of the form

$$V_1(\tau, z(\tau), z_\tau) = \frac{e^{-\tau}}{t_0}W(z) + \sum_{j=1}^{m} \nu_j \int\limits_{\tau-\sigma}^{\tau} z_j^2(s)ds, \quad (1.14)$$

$W(z)$ – is a quadratic form (not necessarily definitely positive), and ν_j are constants. In this case, using the functional

$$dz_0(\tau)/d\tau = e^{\tau}[az_0(\tau) + bz^0(\tau - \sigma)], \quad (1.15)$$

and calculating its derivative by virtue of a shortened equation of the form (0.4),

$$(dz_0(\tau))/d\tau = e^{\tau}[az_0(\tau) + bz^0(\tau - \sigma)],$$

we get the relation

$$dV_1(\tau, z^0(\tau), z_\tau^0)/d\tau = -\frac{e^{-\tau}}{t_0}(z_0(\tau))^2 + [a(z^0(\tau))^2 + 2bz^0(\tau)z^0(\tau - \sigma) + a(z^0(\tau - \sigma))^2].$$

The expression in square brackets on the right side is definitely a positive quadratic form when the inequalities are satisfied:

$$a > 0, |b| < a. \quad (1.16)$$

Let us show that inequalities (1.16) are sufficient conditions for the instability of the shortened equation (1.15).

Consider the functional $V_1(\tau, z^0(\tau), z_\tau^0)$. Obviously, for the initial function $\bar{\phi}_0(\theta) = 0 : t_0 - \sigma \le (\theta) < t_0$, $\phi_0(t_0) = t_0$ $V_1(0, z^0(0), z_\tau^0(\bar{\phi}_0(\theta){=}1$, i.e. $dV_1(\tau, z^0(\tau), z_\tau^0)/d\tau > 0$. Assume that the solution of the shortened system is stable, i.e. $\exists\ 0 < m < M$: $0 < m < \|z^0(\tau)\| < M$ $(\tau > 0)$. Then for sufficiently large τ we have the inequality $V_1(\tau, z^0(\tau), z_\tau^0) \approx -\int\limits_{t-\sigma}^{t} (m - \epsilon)ds) = -(m - \epsilon)\sigma < 0$. But in the region $0 < m < \|z^0(\tau)\| < M$ the derivative $dV_1(\tau, z^0(\tau), z_\tau^0)/d\tau > 0$, since not only the value $\frac{e^{-\tau}}{t_0}(z_0(\tau))^2 > 0$ is small for sufficiently large $t_0 > 0$, but also the integral $\int\limits_0^{\infty} e^{-s}ds = 1$ converges. Consequently, the function $z^0(\tau)$ cannot remain in any bounded region $0 < m < \|z^0(\tau)\| < \bar{M}$ $(\tau > 0)$, while $V_1(0, z^0(0), z_\tau^0(\bar{\phi}_0(\theta)) \to \infty$. We obtain the instability of the solution to equation (1.15). Obtaining sufficient conditions for instability is also possible for a variable value of $b(\tau)$, and the instability region has the form (1.13) for $\mu = 1$, i.e. it is wider than the region obtained using the functional $V^0(x^0, t)$.

Let us now consider the asymptotic behavior of the solution to a first-order equation of the form (0.4)

$$dz(\tau)/d\tau = t_0e^{\tau}[az(\tau) + bz(\tau - \sigma)] + r(\tau)dz(\tau - \sigma)/d\tau.$$

Assume that the function $r(\tau)$ has a bounded continuous derivative. We represent this equation in the form

$$\begin{gathered} d\{z(\tau) + r(\tau)z(\tau - \sigma)\}/d\tau = e^{\tau}[az(\tau) + bz(\tau - \sigma) + \tfrac{e^{-\tau}}{t_0}r'(\tau)z(\tau - \sigma)], \\ r'(\tau) = dr(\tau)/d\tau. \end{gathered} \quad (1.17)$$

In the expression in square brackets on the right side of (1.17), for sufficiently large t_0, it is possible to neglect the term containing the multipliers of the value $e^(- \tau)/t_0$ due to its smallness [11]. Let us now consider the first approximation equation

$$d\{z^1(\tau)+r(\tau)z^1(\tau-\sigma)\}/d\tau = e^{\tau}[az^1(\tau)+bz^1(\tau-\sigma)] = F(\tau, z^1_\tau) \;\; r'(\tau) = dr(\tau)/d\tau. \quad (1.18)$$

Let $Z(\tau, z^1_\tau)) = z^1(\tau) + r(\tau)z^1(\tau - \sigma)$. With the initial function we have chosen, the following conditions are met:

$$1) \;\; 0 < V(\tau, z^1_\tau), Z(\tau, z^1_\tau)) \leq \hat{M}$$

$$2) \;\; dV/d\tau = \lim_{\Delta\tau \to 0} \inf[V(\tau + \Delta\tau) - V(\tau) \geq 0$$

3) if $V(\tau, z^1_\tau, Z(\tau, z^1_\tau)) \geq \alpha$, then $dV/d\tau \geq \beta(\alpha)$. Then, as follows from [5, p. 182], the solution $z^1(\tau)$ to the first approximation equation is unstable. Consequently, the solution to the original equation is also unstable, which can be proven using the methods proposed in [11] and the convergence $\int\limits_0^\infty e^{-s} ds)$. Let us again consider the system (0.3) and assume that among the eigenvalues λ_j of matrix A, there is an eigenvalue λ_0: $\mathrm{Re}\,(\lambda_0) = \bar{\beta} = \max\limits_j \mathrm{Re}\,(\lambda_j)$, $\bar{\beta} > 0$. We assume that the matrices $B(t)$ and $R(t)$ are continuous and have bounded first derivatives.

Lemma 1. *If the eigenvalue λ_0 exists, the solution to system (0.3) is unstable. Let us give a brief proof of this statement.*

We search particular solution in form $\bar{x}(t) = \bar{\gamma}e^{\lambda t}$. Here $\bar{\gamma}$ is m-vector and quantity λ — scalar. If we substitute this expression in the system (0.3), then get an expression

$$\lambda\bar{\gamma}e^{\lambda t}E = A\bar{\gamma}e^{\lambda t} + B(t)\bar{\gamma}e^{\lambda\mu t} + R(t)d/dt\left(\bar{\gamma}e^{\lambda\mu t}\right). \quad (1.19)$$

We divide both parts on the quantity $e^{\lambda t}$, we receive next correlation

$$\lambda\bar{\gamma}E = A\bar{\gamma} + B(t)\bar{\gamma}e^{\lambda(\mu-1)t} + R(t)\lambda\mu\bar{\gamma}e^{\lambda(\mu-1)t}$$

We believe, that $Re(\lambda) > 0$. Then two last members in right part of this correlation aspire to zero at $t \to \infty$. Top equation has the look

$$(A - \lambda E)\bar{\gamma} = 0$$

and by $\bar{\gamma} \not\equiv 0$ we obtain

$$\det A - \lambda E = 0.$$

So far so among own numbers of matrix A there is quantity λ_0 and $Re(\lambda_0) > 0$, then by sufficiently large t we see, that system (0.3) has the particular decision $\bar{x}(t)$ of kind

$$\bar{x}(t) \approx \bar{\gamma}e^{\lambda_0 t} + o(1)(e^{\lambda_0 t}). \quad (1.20)$$

Hence, system (0.3) is unstable at any matrices $B(t)$, $R(t)$. Statement is proven.

2. Study of the Asymptotic Properties of Some Systems of Neutral Type

Let us consider the asymptotic behavior of the solution to the system (0.4), assuming that the corresponding system without neutral terms is stable, but not asymptotically. We also assume that the inequality is true:

$$\|R_j\|_\sigma < L\bar{q}^j,\ \ L,\bar{q} = \text{const}, L > 1, 0 < \bar{q} < 1,\ \ \|R_j\|_\sigma = \max_{\tau\in[j\sigma,(j+1)\sigma]} \|R(\tau)\|. \quad (2.1)$$

Theorem 2. Under the conditions of Theorem 1, inequality (2.1), and the boundeness of $\|z_0(\tau)\|$ and $\|z_0'(\tau)\|$ values, the solution to system (0.4) is uniformly bounded.

Proof. Now it is more convenient to move to a countable system of equations on a finite interval $[0,\sigma]$, namely, by setting $z_n(\tau) = z(\tau + n\sigma)$, we obtain a countable system of differential-difference equations

$$dz_{n+1}(\tau)/d\tau = \mu^{-n}t_0e^\tau[Az_{n+1}(\tau) + Bz_n(\tau)] + \mu^{-(n-1)}t_0e^\tau R_{n+1}(\tau)[Az_n(\tau) + Bz_{n-1}(\tau)]+$$
$$+\mu^{-(n-2)}t_0e^\tau R_{n+1}(\tau)R_n(\tau)[Az_{n-1}(\tau) + Bz_{n-2}(\tau)] + ...+$$
$$+t_0e^\tau R_{n+1}(\tau)R_n(\tau)...R_2(\tau)[Az_1(\tau) + Bz_0(\tau)]+$$
$$+R_{n+1}(\tau)R_n(\tau)...R_1(\tau)dz_0(\tau)/d\tau,\ \ 0 \le \tau \le \sigma, \quad (2.2)$$

defined by the initial vector function $z_0(\tau - \sigma)$ under the boundary conditions

$$z_{n+1}(0) = z_n(\sigma). \quad (2.3)$$

We introduce the following operators in the Banach space:

$$T_{n,\tau}w(s) = U_n(\tau,0)w(\sigma) + \int_0^\tau U_n(\tau,s)B\frac{t_0e^s}{\mu^n}w(s)ds,\ \ U_n(\tau,s) = \exp(\frac{t_0A}{\mu^n}(e^\tau - e^s)), \quad (2.4)$$

that is, the stepoperator [8, p. 213], and the integral operators

$$\begin{aligned} &T^R_{n+1,\tau}w(s) = \int_0^\tau U_n(\tau,s)\tfrac{R_{n+1}(s)A}{\mu^{n-1}}t_0e^s\mu^{n-1}w(s)ds,\\ &I^R_{n,j}(\tau)w(s) = \int_0^\tau U_n(\tau,s)R_{n+1}(s)R_n(s)...R_{j+2}(s)\tfrac{t_0}{\mu^j}[B + \mu R_{j+1}(s)A]w(s)ds,\\ &j = 1,...,n-1,\\ &I^R_{n,0}(\tau)w(s) = \int_0^\tau U_n(\tau,s)R_{n+1}(s)R_n(s)...R_2(s)\tfrac{t_0}{\mu^j}Bw(s)ds,\\ &D^R_n(\tau)w'(s) = \int_0^\tau U_n(\tau,s)R_{n+1}(s)R_n(s)...R_1(s)w'(s)ds,\ \ w'(s) = d/dsw(s). \end{aligned} \quad (2.5)$$

Let's consider some properties of the given operators. Obviously, $z^0_n(\tau)$, which is the solution to the unperturbed equation, i.e., equation without neutral terms, similar to (1.2) can be written in operator form:

$$z^0_{n+1}(\tau) = T_{n,\tau}z^0_n(s),\ \ n = 0,1,2,... \quad (2.6)$$

A very rough estimate $\|T_{n,\tau}\| \le K_0$ is valid. Due to the fact that the product of operators is defined in the Banach space, the solution to the unperturbed equation can be represented step by step in the form

$$\begin{aligned} &z_1^0(\tau) = T_0(\tau)z_0^0(s), \\ &z_2^0(\tau) = T_1(\tau)T_0(s_1)z_0^0(s) = U_1(\tau,0)\left[U_0(\sigma,0)z_0(\sigma) + \int_0^\sigma e^s Bz_0(s)ds\right] + \\ &+\int_0^\tau U_1 e^{s_1}\mu^{-1}\left[U_0(s_1,0)z_0(\sigma) + \int_0^{s_1} U_0(s_1,s)Be^s z_0(s)ds\right], \dots \end{aligned} \tag{2.7}$$

Generally,

$$z_{n+1}^0(\tau) = T_n(\tau)T_{n-1}(s_n)\dots T_2(s_3)T_1(s_2)T_0(s_1)w(s), \;\; 0 \le s \le s_1 \le s_2 \le \dots \le s_n \le \tau, \;\; \tau \in [0,\sigma].$$

In view of the estimate (1.10), the inequality is valid for any natural N:

$$\|\prod_{j=1}^{N} T_{j-1}(s_j)\| < M_2, \;\; M_2 = M_1 + M_1\|A^{-1}\|\left[\frac{\|A\|K_0 + \|B\|}{\mu(1-\mu^\beta)}\right]$$

$$0 \le s_1 \le s_2 \le \dots \le s_N \le \sigma.$$

For the rest of the integral operators, in the space C^0, taking into account the relation (0.1), in view of the norm estimate of the Cauchy matrix $U(\tau,s)$, the following very rough estimates are valid:

$$\|T_{n+1}^R(\tau)\| < \frac{M_0\mu L\|A\|}{\beta_0}\bar{q}^{n+1}, \tag{2.8}$$

$$\begin{aligned} \|I_{n,j}^R(\tau)\| &< \frac{M_0(L\mu)^{n-j}\|B\|}{\beta_0}(q_1)^{(n+j+3)(n-j)} + \frac{M_0(L\mu)^{n+1-j}\|A\|}{\beta_0}(q_1)^{(n-j)(n+j)} < \\ &< \frac{M_0}{\beta_0}[\|B\| + L\mu\|A\|](q_1)^n\left(L\mu(q_1)^n\right)^{n-j}, \;\; q_1 = \sqrt{\bar{q}}, \end{aligned} \tag{2.9}$$

$$\|I_{n,0}^R(\tau)\| < \frac{M_0\|B\|}{\beta_0}\left(L\mu(q_1)^n\right)^n. \tag{2.10}$$

For the operators $D_n^R(\tau)$, due to the boundedness of the derivative $z_0'(\tau)$, a similar estimate is valid:

$$\|D_n^R(\tau)\| \le \frac{(\mu L)^n LM_0}{\beta_0 t_0}(q_1)^{(n+1)(n+2)} < \frac{LM_0}{\beta_0 t_0}\left(\mu L(q_1)^n\right)^n. \tag{2.11}$$

Thus, we have obtained a family of linear operators bounded in the space C^1. It is obvious that for sufficiently large $n \ge N$, the value $\mu L(q_1)^n < 1$, therefore, the series

$$\sum_{j=1}^{\infty}\left(\mu L(q_1)^n\right)^j < \infty. \tag{2.12}$$

Using a formula similar to the formula for variation of constants, taking into account (2.4), (2.5), the solution $z_(n+1)(\tau)$ to the perturbed equation (2.2) can be represented as

$$z_{n+1}(\tau) = T_n(\tau)z_n(s) + F_{n+1}(\tau), \;\; F_{n+1}(\tau) = T_{n+1}^R(\tau)z_n(s) +$$

$$+I^R_{n,n-1}(\tau)z_{n-1}(s) + ... + I^R_{n,1}(\tau)z_1(s) + I^R_{n,0}(\tau)z_0(s) + D^R_n(\tau)z'_0(s), \qquad (2.13)$$

from which, taking into account (1.1) and (2.2), we obtain

$$z_{n+1}(\tau) = y_{n+1}(\tau) + T_n(\tau)\prod_{j=1}^{n-1} T_j(s_{j+1})F_1(s_1)+$$
$$+T_n(\tau)\prod_{j=2}^{n-1} T_j(s_{j+1})F_2(s_2) + ... + F_{n+1}(\tau). \qquad (2.14)$$

Here $y_n(\tau)$ is the solution to the finite difference system without neutral members corresponding to the unperturbed system (1.2),

$$\prod_{j=k}^{n-1} T_j(s_{j+1})F_k(s_k) = T_{n-1}(s_n)T_{n-2}(s_{n-1})...T_k(s_{k+1})F_k(s_k), \ \ 0 \le s \le s_1 \le s_2 \le ... \le s_n \le \tau \le \sigma.$$

From (2.14), taking into account (2.7), we obtain the inequality

$$\|z_{n+1}(\tau)\|_\sigma \le \|y_{n+1}(\tau)\|_\sigma + M_2 \sum_{j=1}^{n} \|F_j(\tau)\|_\sigma + \|F_{n+1}(\tau)\|_\sigma. \qquad (2.15)$$

Obviously,

$$\|F_{k+1}(\tau)\|_\sigma \le \|T^R_{k+1}(\tau)\|\|z_k(\tau)\|_\sigma + \sum_{j=0}^{k-1} \|I^R_{k,j}(\tau)\|\|z_j(\tau)\|_\sigma + \|D^R_k(\tau)\|\|dz_0(\tau)/d\tau\|_\sigma. \qquad (2.16)$$

Let us perform the reduction of similar terms at $\|dz_0/d\tau\|_\sigma$, $\|z_j(\tau)\|_\sigma$ $j = 1, 2, ...n$ in the right part of the inequality (2.15), taking into account (2.16). By designating the coefficient at a^n_j to $\|z_j(\tau)\|_\sigma$, the coefficient at δ^n_0 to $\|dz_0/d\tau\|_\sigma$, we consistently get

$$\delta^n_0 = \|D^0_n(\tau)\| + M_2\left[\sum_{j=0}^{n-1} \|D^R_j(\tau)\|\right],$$

$$a^n_j = \|I_{n,j}(\tau)\| + M_2\left[\sum_{k=1}^{n-j-1} \|I_{n-k,j}(\tau)\| + \|T^R_{j+1}\|\right], \ \ j = 0, 1, ..., n-2,$$

$$a^n_{n-1} = \|I_{n,n-1}(\tau)\| + M_2\|T^R_n(\tau)\|, \ \ a^n_n = \|T^R_{n+1}(\tau)\|. \qquad (2.17)$$

Consider the asymptotic behavior of these coefficients, taking into account (2.8)–(2.11). For the value δ^n_0 from the first of the relations (2.17) we get the estimate

$$\delta^n_0 < \frac{LM_0}{\beta_0 t_0}(L\mu(q_1)^n)^n + \frac{LM_2M_0}{\beta_0 t_0}\sum_{j=0}^{n-1}(L\mu(q_1)^j)^j.$$

This value is uniformly bounded by some positive constant δ due to the convergence of the series $\sum\limits_{j=0}^{\infty}(L\mu(q_1)^j)^j$. Let us now consider the asymptotic behavior of the values $a^n_j, j = 0, 1, ..., n$. We get estimates:

$$a^n_0 < \frac{M_0\|B\|}{\beta_0}(L\mu\ (q_1)^n)^n + \frac{M_2M_0(\|B\| + \mu L\|A\|)}{\beta_0}\sum_{k=1}^{n-1}(L\mu(q_1)^k)^k + \frac{M_0M_2L\mu\|A\|}{\beta_0}q,$$

$$a_j^n < \frac{M_0[\|B\| + L\mu\|A\|]}{\beta_0}[(L\mu(q_1)^n)^n + \frac{M_2M_0(\|B\| + \mu L\|A\|)}{\beta_0}\sum_{k=j+1}^{n-1}(L\mu(q_1)^k)^k +$$

$$+\frac{M_0M_2L\mu\|A\|}{\beta_0}q^{j+1}, \quad j = 1, 2, ..., n-2$$

$$a_{n-1}^n < \frac{M_0[\|B\| + L\mu\|A\|]}{\beta_0}(q_1)^n(L\mu(q_1)^n) + \frac{M_0M_2L\mu\|A\|}{\beta_0}q^n,$$

$$a_n^n < \frac{M_0L\mu\|A\|}{\beta_0}q^{n+1}. \tag{2.18}$$

From the first inequality in (2.18) it follows that the value a_0^n is also uniformly bounded, and this is proved exactly in the same methods as the estimate for $?_0^n$. Therefore, for the zero approximation of the solution to the equation (2.15) we have the estimate

$$\|z_n^R(\tau)\| \leq \hat{M}_0[\|z_0(\tau)\|_\sigma + \|dz_0(\tau)/d\tau\|_\sigma], \ \ \hat{M}_0 = \text{const}, \hat{M}_0 > 1, \ \ n = 1, 2, ... \tag{2.19}$$

Next, consider the second estimate in (2.18). For the value a_j^n, we have the following presentation:

$$a_j^n = (q_1)^{j/2}\left\{\frac{M_0[\|B\| + L\mu\|A\|]}{\beta_0(q_1)^{j/2}}[(L\mu(q_1)^n]^n + \right.$$

$$\left. + \frac{M_0M_2(\|B\| + L\mu\|A\|)}{\beta_0(q_1)^{j/2}}\sum_{k=j+1}^{n-1}(L\mu(q_1)^k)^k + \frac{M_0M_2L\mu\|A\|}{\beta_0(q_1)^{j/2}}q^{j+1}\right\}. \tag{2.20}$$

Consider the asymptotic behavior of the expression standing on the right in curly brackets in equality (2.20). Let the natural number $\bar{N}$ be determined as follows: $L\mu q_1^{\bar{N}} \geq 1$, $L\mu q_1^{\bar{N}+1} < 1$. Enter the number $\bar{L} = \max\limits_{0 \leq i \leq \bar{N}}\{(L\mu(q_1)^i)^i\}$. Then, since $1 \leq j \leq n-2$, we have the estimate

$$(L\mu(q_1)^n)^n(q_1)^{-j/2} < \bar{L}(q_2^{n^2-(n-2)}, \ \ q_2 = \sqrt{q_1} \tag{2.21}$$

Consider the value $(q_1)^{-j/2}\sum\limits_{k=j+1}^{n-1}(L\mu(q_1)^k)^k = S_n$. For it, we obtain the estimate

$$S_n < \bar{L}[(q_2)^{(j+1)^2-j} + (q_2)^{(j+2)^2-j} + ...] < \bar{L}\frac{(q_2)^{j+1}}{1-(q_2)^{2j}}. \tag{2.22}$$

Considering that the right term in curly brackets in (2.20) has the form $\frac{M_0M_2L\mu\|A\|q}{\beta_0}(q_2)^{3j}$, taking into account the estimates (2.21),(2.22) we obtain that the values in curly brackets in the right part (2.20) are uniformly bounded for any j. Hence we get that a_j^n, defined by equality (2.20), does not exceed the value $K(q_2)^j$, $K = \text{const}$, $K > 1$. Similar estimates can be proved for the values a_{n-1}^n, a_0^n.

Finally, we obtain that the limiting coefficients a_j^∞ satisfy a similar inequality, therefore, the series a_j^∞ converges. This implies uniform boundedness of the solution to the system (1.1) with boundedness of the functions $z_0(\tau)$ and $z_0'(\tau)$.

Corollary 1. *If conditions of the execution of Theorem 1 and the validity of the estimate (2.1) are fulfilled, the solution to the system (0.4) is stable.*

Proof. We have already shown that the solution to the differential-difference system (2.2) can be represented as a solution to the perturbed inhomogeneous difference system (2.13). The solution to the corresponding unperturbed system is uniformly bounded (hence, stable), and estimate (2.19) is valid for it. For a perturbed system (2.13), perturbations are expressions that do not exceed the norm of the values $\sum_n a_n^\infty \|z_n(\tau)\|_\sigma$ and the series

$$\sum_{n=1}^{\infty} a_n^\infty \tag{2.23}$$

converges. Let us show that the solution to the perturbed system is stable. From (2.13), taking into account (2.8)-(2.11) and (2.19), we have the following inequality:

$$\|z_{n+1}(\tau)\|_\sigma \le \hat{M}_0[\|z_0(\tau)\|_\sigma + \|dz_0(\tau)/d\tau\|_\sigma] + \sum_{k=1}^{n} a_k^\infty \|z_k(\tau)\|_\sigma.$$

Therefore [2, p.70],

$$\|z_{n+1}(\tau)\|_\sigma \le \hat{M}_0[\|z_0(\tau)\|_\sigma + \|dz_0(\tau)/d\tau\|_\sigma] \prod_{k=1}^{n}(1 + a_k^\infty).$$

But the product

$$\prod_{k=1}^{\infty}(1 + a_k^\infty)$$

is bounded due to the convergence of series (2.23). Consequently, the solution to the perturbed linear system (2.2) is uniformly bounded, and hence the solution to system (0.4) is also stable.

Let us now consider a more complex perturbed system

$$dx(t)/dt = Ax(t) + (B + f(t)E)x(\mu t) + \hat{R}(t)dx(\mu t)/dt, \;\; t \ge t_0 > 0, \tag{2.24}$$

Here E is the identity matrix of the corresponding size, and f(t) is a continuously differentiable monotonically decreasing vector function satisfying the estimate

$$\|f(t)\| \le \bar{K}\left(\frac{t}{t_0}\right)^{-\bar{\alpha}}, \;\; \bar{\alpha}, \bar{K} = \text{const}, \;\; \bar{K} > 1, \;\; \bar{\alpha} > 0. \tag{2.25}$$

When estimate (2.25) is fulfilled, the solution to the perturbed system (2.24) is stable. Indeed, if we go to the corresponding system (2.13), then for the perturbations that appear due to the presence of f(t)) and have an estimate due to (1.5), (1.6):

$$\|\int_0^\tau \hat{U}_j(\tau, s)e^s \hat{t}_0 B\hat{z}_j(s)ds\| \le \frac{M_0 f_j(\tau_0)}{\beta_0}\|z_j(\tau)\|_\sigma, \;\; j = 1, 2, ...n \tag{2.26}$$

we see that these terms on the right side are similar to the terms in (2.13) $I^R_{(j,j-1)}(\tau)z_j(s)$ due to the convergence of the series

$$\sum_j \left(\frac{1}{\mu^j}\right)^{\bar{\alpha}}.$$

Therefore, using the methods used in the proof of Theorem 2, the stability of system (2.24) can be proven.

Note that $\int\limits_{t_0}^{\infty} f(t)dt$ can diverge, while in the variable τ, a similar integral converges. Thus, the transition to a constant delay system makes it possible to obtain more accurate results on the asymptotic behavior of solutions to perturbed systems.

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АСИМПТОТИЧЕСКИЕ СВОЙСТВА НЕКОТОРЫХ СИСТЕМ НЕЙТРАЛЬНОГО ТИПА С ЛИНЕЙНЫМ ЗАПАЗДЫВАНИЕМ

Б. Г. Гребенщиков

Изучаются дифференциальные уравнения с линейным запаздыванием нейтрального типа. Уравнения с линейным запаздыванием встречаются в задачах механики, биологии, экономике. Особенностью таких уравнений является неограниченность запаздывания, что существенно сужает применимость традиционных методов для исследования задач устойчивости систем подобного типа. Одним из подходов при изучении асимптотических свойств является замена аргумента при этом система сводится к системе с постоянным запаздыванием, но при этом в правой части полученной системы появляется экспоненциальный множитель и правая часть полученной системы становится неограниченной при $t \to \infty$. Асимптотические свойства систем без нейтральных членов изучались авторами ранее. С учетом асимптотических особенностей этих систем (без нейтральных членов в правой части) производится анализ асимптотических свойств (ограниченность, устойчивость и асимптотическая устойчивость) некоторых систем уже нейтрального типа. Поскольку свойство устойчивости является более тонким свойством нежели свойство асимптотической устойчивости исследуется система нейтрального типа с возмущениями, которая (невозмущенная) является просто устойчивой.

Ключевые слова: устойчивость; асимптотическая устойчивость; функционалы Ляпунова-Красовского.

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