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ON PERIODIC SOLUTIONS OF DIFFERENTIAL EQUATIONS WITH CONTINUOUS RIGHT-HAND SIDES ON LIE GROUPS

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We apply the method of guiding potentials to obtain an existence of periodic solution theorem to a differential equation with continuous periodic right-hand side on a Lie group, i.e., the solution of the Cauchy problem for this equation is not unique. To avoid this difficulty we elaborate the machinery of integral operators with parallel translation such that for a T-periodic ordinary differential equation (i.e., a vector field) on a Lie group with continuous right-hand side the fixed points of those operators are T-periodic solutions. It is shown that under some natural conditions the second iteration of such operator is completely continuous. The method of guiding potentials with those operators allows us to obtain the existence theorem we are looking for. The paper contains a short survey of the theory of integral operators with parallel translation and a modification of the construction of topological index applicable on the manifolds.

Keywords: Lie groups; ordinary differential equations; integral operators with parallel translation; topological index on manifold; periodic solutions.

Introduction

We apply the method of guiding potentials to obtain an existence of periodic solution theorem to a differential equation with continuous periodic right-hand side on a Lie group. For such equations the solution of the Cauchy problem is not unique. In [1] we considered analogous problem on a non-compact manifold in the case where the right-hand side was C^1 smooth so that the solution of the Cauchy problem was unique and it was possible to use the operator of translation along the trajectories. In order to avoid the difficulty with non-uniqueness of the solution of the Cauchy problem, we have elaborated the machinery of integral operators with parallel translation whose second iteration is completely continuous and whose fixed points are the periodical solutions. The method of guiding potentials with those integral operators allows us to obtain the existence theorem we are looking for.

The paper contains a short survey of the theory of integral operators with parallel translation and a modification of the construction of topological index applicable on the manifolds.

1. Integral Operators with Parallel Translation

Everywhere below we deal with all objects given on a finite interval $[0,T], T>0$.

Let M be a complete Riemannian manifold, $m_0 \in M$ and $v\colon [0,T] \to T_{m_0}M$ be a continuous curve in the tangent space $T_{m_0}M$. Everywhere below we deal with the Levi-Civita connection on M.

Theorem 1. *There exists a unique C^1-curve $m\colon [0,T] \to M$ such that $m(0) = m_0$ and the tangent vector $m'(t)$ is parallel along this curve to the vector $v(t) \in T_{m_0}M$ for every $t \in [0,T]$.*

The existence of the curve $m(t)$ from Theorem 1 follows from some classical constructions. Let $m(t)$ be a C^1-smooth curve in M, $t \in [0,T]$, $m(0) = m_0$. Denote by Γ the operator of parallel translation of vector fields along $m(\cdot)$ to $T_{m_0}M$. Recall that the curve $C(m(t)) = \int_0^t \Gamma m'(s)ds$ is known as *Cartan's development* of $m(t)$ at $T_{m_0}M$. Note the well-known fact that Cartan's development is convertible and it is obvious that the curve $m(t)$ from Theorem 1 is expressed via Cartan's development as $C^{-1}\left(\int_0^t v(s)ds\right)$.

We denote the operator that sends $v(t)$ to $m(t)$ in Theorem 1 by the symbol $\mathcal{S}$. It is easy to show that $\mathcal{S}$ is continuous.

Since the parallel translation preserves the norm of the vector, the following statement is valid.

Theorem 2. *Let $\mathcal{U}_K$ be the ball of the radius K centered at the origin of the space of continuous curves $C^0([0,T], T_{m_0}M)$. Then, at every point $t \in [0,T]$, the inequality $\|m'(t)\| \le K$ holds for all curves $m(\cdot)$ from the set $\mathcal{S}\mathcal{U}_K$.*

Lemma 1. (Compactness lemma) *Let $\Xi \subset C^0([0,T], TM)$ be such that $\pi\Xi \subset C^1([0,T], M)$, where $\pi\colon TM \to M$ is the natural projection. If Ξ is relatively compact in $C^0([0,T], TM)$, then so is $\Gamma\Xi$.*

The proof of Lemma 1 can be found, e.g., in [2, Lemma 3.51].

Let Ω_K be the set of curves from $C^1([0,T], M)$ satisfying the inequality $\|m'(t)\| \le K$, where $K > 0$ is a real number, at every point $t \in [0,T]$ and such that the set $\{m(0) \mid m(\cdot) \in \Omega_K\}$ is relatively compact in M.

Theorem 3. *The set of curves $\Gamma(\Omega_K)$ is relatively compact in $C^0([0,T], TM)$.*

Proof. Since Ω_K is compact in $C^0([0,T], M)$ and the field $X(t,m)$ is continuous, the set of curves $\{X(t, m(t)) \mid m(\cdot) \in \Omega_K\}$ is compact in $C^0([0,T], TM)$. Then the aqssertion follows from Lemma 1.

□

Let a continuous vector field $X(t,m)$ be given on M. Consider the set $C^1_{m_0}([0,T], M) \subset C^1([0,T], M)$ consisting of curves with initial value $m(0) = m_0$. Introduce the composition operator

$$\mathcal{S} \circ \Gamma X(t, m(t))\colon C^1_{m_0}([0,T], M) \to C^1_{m_0}([0,T], M).$$

One can easily see that this operator is continuous since the parallel translation continuously depends on the curves, along which it is carried out.

Theorem 4. *The fixed point of $\mathcal{S}\circ\Gamma$ is precisely the solution of equation $m'(t) = X(t, m(t))$ with the initial condition $m(0) = m_0$.*

Indeed, if $m(t)$ is a fixed point, $m'(t)$ is parallel along $m(t)$ to $\Gamma X(t, m(t))$. But by construction $\Gamma X(t, m(t))$ is parallel to $X(t, m(t))$. Hence $m'(t) = X(t, m(t))$.

Now let M be a Lie group being a finite-dimensional manifold.

Remark 1. We denote the elements of the Lie group M as points of manifold M, i.e., by the symbol m, $m(\cdot)$ or $m(t)$ are curves on M. But for simplicity of presentation, the element considered as a diffeomorphism in M, is denoted by the symbol g. In particular g_{m_0,m_1} denotes the unique diffeomorphsm that sends m_0 to m_1. $Tg_{m_0,m_1} : T_{m_0}M \to T_{m_1}M$ is its tangent mapping.

Introduce an arbitrary complete Riemannian metric $\langle \cdot, \cdot \rangle$ on M (not necessarily left or right invariant). The corresponding norms in the tangent spaces are denoted by $\| \cdot \|$.

Consider the Banach manifold $C^1([0, T], M)$ of C^1-smooth curves in M. According to Remark 1, for $m(t) \in C^1([0, T], M)$ denote by $g_{m(0),m(t)}$ the element of Lie group (i.e., the operator) that sends $m(0)$ to $m(t)$, and by $Tg_{m(0),m(t)} : T_{m(0)}M \to T_{m(t)}M$ the tangent map of this operator. For $m(t) \in C^1([0, T], M)$ introduce the operator B_s by the formula

$$B_s(m(\cdot)) = \mathcal{S} \circ Tg_{m(0),\mathcal{S}\circ\Gamma\mathcal{X}(t,m(t))(s))}\Gamma X(t, m(t)) \tag{1}$$

that sends the vectors $\Gamma X(t, m(t))$ at $m(0)$ to the points at time instant s of the curves from $\mathcal{S} \circ \Gamma X(t, m(t))$. One can easily see that B_s is continuous.

Let also the vector field $X(t, m)$ be T-periodic, i.e., for every $m \in M$ the equality $X(t, m) = X(t+T, m)$ holds. In this case we will mainly deal with the operator $B_T(m(\cdot)) = \mathcal{S} \circ Tg_{m(0),\mathcal{S}\circ\Gamma X(t,m(t))(T))}\Gamma X(t, m(t))$.

Theorem 5. *Fixed points of operator B_T and only they are T-periodic solutions of the equation $m'(t) = X(t, m(t))$.*

Proof. Let $m(t)$ be a T-periodic solution of the equation $m'(t) = X(t, m(t))$. Then $g_{m(0),\mathcal{S}\circ\Gamma X(m(\cdot)(T))}m(0) = m(0)$ and so for $X(t, m(t))$

$$Tg_{m(0),m(T)}\mathcal{S} \circ Tg_{m(0),\mathcal{S}\circ\Gamma X(t,m(t)))}\Gamma X(t, m(t)) = \Gamma X(t, m(t)).$$

Then $B_T(m(\cdot)) = \mathcal{S} \circ \Gamma X(t, m(t))$. Recall that $\Gamma X(t, m(t))$ is parallel along $m(\cdot)$ to $X(t, m(t))$. On the other hand, $\frac{d}{dt}\mathcal{S} \circ \Gamma X(t, m(t))$ is parallel along $m(\cdot)$ to $\Gamma X(t, m(t))$ and so

$$\frac{d}{dt}\mathcal{S} \circ \Gamma X(t, m(t)) = X(t, m(t)).$$

Thus $m(t)$ is a fixed point of $B_T(m(\cdot)$.

Now let $m(t)$ be an arbitrary curve in $C^1([0, T], M)$. If $\mathcal{S} \circ \Gamma X(t, m(t))(T) = m(0)$, the above arguments are valid and so $m(t)$ is both a fixed point of B_T and a T-periodic solution. If $\mathcal{S} \circ \Gamma X(t, m(t))(T) \neq m(0)$, $m(t)$ is neither a fixed point of B_T nor a periodic solution.

□

Recall the following notion.

Definition 1. *A map from the topological space Y to the topological space Z is called proper, if the preimage of every relatively compact set in Z is relatively compact in Y. In* *particular, a function $\varphi : M \to \mathbb{R}$ is called proper if the preimage of every bounded subset of $\mathbb{R}$ is relatively compact in M.*

Let $\Xi \subset M$ be a compact set. Denote by $\mathfrak{C} \subset C^1([0,T],M)$ the set of curves $\{m(t)|m(0) \in \Xi, t \in [0,T]\}$. Since all curves from $\mathfrak{C}$ are given on the closed interval $[0,T]$ and M is complete, all the curves from $\mathfrak{C}$ lie in another compact set Ξ_1.

Theorem 6. *Let for any compact set $\Xi \subset M$*

$$\sup_{m\in\Xi, t\in[0,T]} \|X(t,m)\| < \sup_{m\in\Xi} \varphi(m) \tag{2}$$

where $\varphi : M \to \mathbb{R}$ is a certain proper function. Then for all $m(\cdot) \in \mathfrak{C} \subset C^1([0,T],M)$, all curves $\mathcal{S} \circ \Gamma X(t,m(t))$ are well-defined on $[0,T]$ and belong to another compact set $\Xi_2 \subset M$.

Proof. From (2) it follows that the norms of all $X(t,m(t))$ for $m(\cdot) \in \mathfrak{C}$ are uniformly bounded by $\sup_{m\in\Xi_1} \varphi(m)$. Since the parallel translation preserves the norms, all norms of the corresponding curves $\Gamma X(t,m(t))$ are uniformly bounded by the same constant. Thus all the C^1-curves $\mathcal{S} \circ \Gamma X(t,m)$ have bounded lengths. Since the metric is complete, those curves lie in a compact set Ξ_2. □

Theorem 7. *The set of curves $B_s\mathfrak{C} \subset C^1([0,T],M)$ is compact in $C^1([0,T],M)$.*

Proof. Since the set Ξ_2 is compact and the operators $g_{m(0),m(T)}$ and $g_{m(0),\mathcal{S}\circ\Gamma X(m(t)(T))}$ are smooth by the definition of the Lie group, the norms of operators

$$Tg_{m(0),m(T)}\mathcal{S} \circ Tg_{m(0),\mathcal{S}\circ\Gamma X(t,m(t)))}$$

are also uniformly bounded. Then the assertion follows from Theorem 6 and Theorem 3. □

Thus, unlike the classical integral operators in Euclidean spaces, only the second iteration of operator B_s is completely continuous.

2. The Topological Index of Maps on the Manifold

Let M be an n-dimensional noncompact manifold and Ω a domain in M homeomorphic to an open ball in $\mathbb{R}^n$. By $\overline{\Omega}$ we denote the closure of Ω, and by $\partial\Omega$ its boundary. We will suppose everywhere that $\overline{\Omega}$ is homeomorphic to a closed ball and hence it is compact. Notice that it does not follow from the homeomorphism of Ω to an open ball.

Let $F : \overline{\Omega} \to M$ be a continuous map which is fixed point free on the boundary $\partial\Omega$ (i.e., $x \neq F(x)$, $\forall x \in \partial\Omega$).

By the Whitney theorem (see, e.g., [3]) the manifold M can be embedded into the Euclidean space $\mathbb{R}^N$ of sufficiently large dimension $N \geq 2n+1$. Let $W \subset \mathbb{R}^N$ be a tubular neighborhood of M and $r\colon W \to M$ a retraction. Let $U \subset W$ be an open set such that $r(U) = \Omega$. Let us extend the map F to $\overline{U}$ as $\overline{F} : \overline{U} \to M \subset \mathbb{R}^N$ by the formula

$$\overline{F}(x) = F(r(x)).$$

By construction, it is clear that the map $\overline{F}$ is fixed point free on the boundary ∂U. This means that for the corresponding vector field $I - \overline{F}$, where $I\colon \mathbb{R}^N \to \mathbb{R}^N$ is the identity, the topological degree (or rotation) $\gamma(I - \overline{F}, \partial U)$ is well defined (see, e.g., [4, 5]).

Definition 2. *The fixed point index of the map F on $\partial\Omega$ is defined in the following way*

$$ind(F, \partial\Omega) := \gamma(I - \overline{F}, \partial U).$$

First of all, let us mention that the above notion is well defined.

Theorem 8. *The fixed point index $ind(F, \partial\Omega)$ does not depend on the choice of space $\mathbb{R}^N$, embedding, open set U, and retraction r.*

The proof of Theorem 8 (even for an infinite-dimensional case) can be found in [6].

Immediately from the construction it follows that the characteristic, defined above, possesses usual properties, including homotopy invariance. It is also easy to see that its difference from zero implies the existence of at least one fixed point of F in Ω.

For dealing with zeros of tangent and cotangent vector fields inside $\overline{\Omega}$ we have to apply another construction of index.

Let a continuous tangent vector field X having no zero singular points be given on $\partial\Omega$. Since $\overline{\Omega}$ is homeomorphic to a closed ball, there exists a neighbourhood V of $\overline{\Omega}$ that is a chart, i.e., it can be presented as a set in $\mathbb{R}^n$, homeomorphic to an open ball. According to this presentation the tangent vectors of X are becoming vectors in $\mathbb{R}^n$ (are embedded into $\mathbb{R}^n$), and for the field X on $\partial\Omega$ the ordinary topological degree of a vector field is well defined. In order not to confuse it with the index of $\gamma(F, \partial\Omega)$ type, we denote it by $\widehat{\gamma}(X, \partial\Omega)$. Note that the presentation of V as a chart is ambiguously determined, but different versions of such presentation are diffeomorphic to each other and so $\widehat{\gamma}$ does not depend on the choice of such presentation, i.e., it is well defined. Since by the use of a scalar product in $\mathbb{R}^n$ the cotangent vectors (1-forms) can be identified with the tangent ones, the index $\widehat{\gamma}$ is well defined for cotangent vectors also.

Let a continuous map $\Phi : \partial\Omega \to M$ be given.

Definition 3. *We call Φ admissible on $\overline{\Omega}$, if for every $m \in \overline{\Omega}$ the point $\Phi(m)$ belongs to V.*

Definition 4. *The fixed point index of $\widehat{\gamma}$ type for an admissible map Φ is defined by the formula*

$$\widehat{ind}(\Phi, \partial\Omega) := \widehat{\gamma}(I - \Phi, \partial\Omega).$$

Consider the case where the above considered map F is admissible. In this situation, besides index $ind(F, \partial\Omega)$ we can deal with the index $\widehat{ind}(F, \partial\Omega)$.

Theorem 9. *Let the mapping F be admissible on $\overline{\Omega}$. Then*

$$ind(F, \partial\Omega) = \widehat{ind}(F, \partial\Omega).$$

Proof. In fact we have to show that $\gamma(I - \overline{F}, \partial U)$ coincides with $\widehat{\gamma}(I - F, \partial\Omega)$. Since $\overline{F}$ sends $\overline{U}$ to V and on V it coincides with F, this fact follows from the principle of a map restriction (see, e.g., [4, 5]).

□

3. The Main Result

Let $\varphi : M \to \mathbb{R}$ be a proper function. By the symbol $X\varphi$ we denote the derivative of a function φ in the direction of a vector field X. Recall that always

$$X\varphi = d\varphi(X),$$

where $d\varphi(X)$ denotes the value of the 1-form $d\varphi$ on the vector X. Notice also that if a Riemannian metric is given on M (e.g., the Euclidean metric $(\cdot,\cdot)$ after embedding of M into $\mathbb{R}^N$ or a complete Riemannian metric $\langle\cdot,\cdot\rangle$ introduced of the Lie group M above), then

$$X\varphi = d\varphi(X) = (grad\,\varphi, X).$$

Notice that if a non-autonomous vector field $X(t,m)$ for all t does not equal to zero on $\partial\Omega$, the fields at different values of t are homotopic to each other without zeroes on $\partial\Omega$, i.e., they have the same indices $\widehat{ind}$. In particular the fields $X(t,m)$ on $\partial\Omega$ for all t do not equal to zero if for all t on $\partial\Omega$ the relation $X\varphi > 0$ holds for φ.

In the next theorem we deal with the compact set $\overline{\Omega}$ as in Section 2 instead of an arbitrary compact set Ξ as in Theorem 6 but keep notation Ξ_1 and Ξ_2 for the corresponding compact sets in Theorem 6.

Theorem 10. *Let $X(t,m)$ be a continuous T-periodic vector field on $\mathbb{R}\times M$, i.e.,*

$$X(t+T,m) = X(t,m), \quad \forall t\in\mathbb{R},\ m\in M$$

and let a smooth proper function $\varphi : M \to \mathbb{R}_+$ be such that:

(i) for every $X(0,m)$, $m\in\partial\Omega$, the relation

$$d\varphi(X(0,m)) > 0 \tag{3}$$

holds;

(ii) the hypothesis of Theorem 6 is fulfilled;

(iii) for every C^1-smooth curve $m(t)$, $t\in[0,T]$, $m(0)\in\partial\Omega$, the property $m(0)\notin \mathcal{S}\circ\Gamma X(t,m(t))$ and $m(0)\notin B_s(m(\cdot))$ for every s and t is fulfilled;

(iv) the relation

$$\widehat{ind}(grad\,\varphi, \overline{\Omega}) \neq 0 \tag{4}$$

holds.

Then there exists a T-periodic solution of equation $m'(t) = X(t,m(t))$ with an initial condition in Ω.

Proof. From (3) it follows that on $\partial\Omega$ for all t the field $X(t,m)$ does not equal zero, i.e., the index $\widehat{ind}(\mathbf{X},\overline{U})$ is well defined. Since from (3) for every t the inequality

$$(X, grad\,\varphi) > 0$$

holds, the angle between the vectors X and $grad\,\varphi$ is acute. Thus, the linear homotopy between these vectors has no zeroes on $\partial\Omega$ and so

$$\widehat{ind}(X,\overline{\Omega}) \neq 0.$$

From Theorem 7 and the material of Section 2 it follows that on $\overline{\mathfrak{C}}$ (the closure of $\mathfrak{C}$ introduced in Section 3) the index $ind(B_T,\overline{\mathfrak{C}})$ is well-defined. For a curve $m(\cdot)\in\overline{\mathfrak{C}}$ consider $X(t,m(t))$, then consider the curve $\mathcal{S}\circ\Gamma X(t,m(t))$ and finely the curve

$$\mathcal{S}\circ Tg_{m(0),\mathcal{S}\circ\Gamma X(t,m(t))(T))}\Gamma X(t,m(t)) = B_T(m(\cdot)).$$

For all curves of the last type we can introduce the following homotopy

$$\mathcal{S} \circ Tg_{m(0),\mathcal{S}\circ\Gamma X(t,m(t))(t_0+\lambda T))}\Gamma X(t, m(\lambda t))$$

for $\lambda \in [0,1]$. For $\lambda = 1$ this homotopy takes the value of B_T and for $\lambda = 0$ it sends $C^1([0,T], M)$ into its subspace consisting of constant functions that is naturally isomorphic to M. Note that this homotopy for $\lambda = 0$ takes the value $B_T(m(\cdot))(0) = \mathcal{S}\circ\Gamma X(t, m(t))(t_0)$. By (iv) this homotopy has no singular points if $m(0) \in \partial U$. So, $ind(B_T, \overline{\mathfrak{C}}) = ind(\mathcal{S} \circ \Gamma X(t, m(t))(t_0), \overline{\mathfrak{C}})$. By the restriction property of the degree

$$ind(\mathcal{S} \circ \Gamma X(t, m(t))(t_0), \mathcal{C}) = ind(\mathcal{S} \circ \Gamma X(t, m(t))(t_0), \overline{U}).$$

For t_0 sufficiently small the mapping $\mathcal{S} \circ \Gamma X(t, m(t))(t_0)$ is admissible. So,

$$\widehat{ind}(\mathcal{S} \circ \Gamma X(t, m(t))(t_0), \overline{U})$$

is well defined. Notice that $X(0, m(0))$ is the initial tangent vector of the curve $\mathcal{S} \circ \Gamma X(t.m(t))$. Thus one can easily see that

$$\widehat{ind}(\mathcal{S} \circ \Gamma X(t, m(t))(t_0), \overline{U}) = \widehat{ind}(X(0, m(0)), \overline{U}) \neq 0.$$

Then $ind(B_T, \mathfrak{C}) \neq 0$ and by Theorem 5 equation $m'(t) = X(t, m(t))$ has a T-periodic solution.

□

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О ПЕРИОДИЧЕСКИХ РЕШЕНИЯХ ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ С НЕПРЕРЫВНЫМИ ПРАВЫМИ ЧАСТЯМИ НА ГРУППАХ ЛИ

Ю. Е. Гликлих

В работе применяется метод направляющих функций для получения теоремы существования периодических решений дифференциальных уравнений с непрерывными периодическими правыми частями на группах Ли, т.е. решения задачи Коши для подобных уравнений не единственны. Чтобы избежать трудностей был разработан аппарат интегральных операторов с параллельным переносом таких, что для T-периодического обыкновенного дифференциального уравнения на группе Ли с непрерывной правой частью неподвижные точки являются T-периодическими решениями. Показано, что при выполнении некоторых естественных условий вторые итерации указанных операторов вполне непрерывны. Метод направляющих функций с указанными операторами позволяет получить искомую теорему существования периодических решений. Работа содержит краткий обзор теории интегральных операторов с параллельным переносом и модификацию конструкции топологического индекса, применимую на многообразиях.

Ключевые слова: группы Ли; обыкновенные дифференциальные уравнения; интегральные операторы с переллельным переносом; топологический индекс на многообразиях; периодические решения.

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