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ON A STOCHASTIC ALGEBRAIC-DIFFERENTIAL EQUATION WITH MEAN DERIVATIVES SATISFYING THE RANK-DEGREE CONDITION

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We investigate a stochastic second order algebraic-differential equation with mean derivatives whose matrix pencil is regular and satisfies the rank-degree condition. This equation is modelling dynamically distorted signals in an electronic devise so that the quantum effects are taken into account that turn the deterministic incoming signal into a stochastic process. We prove the existence of solution of this equation.

Keywords: mean derivative; second order stochastic differential equations; stochastic algebraic-differential equations.

Introduction

The notion of mean derivatives was introduced by E. Nelson (see [1, 2, 3]) for the needs of his Stochastic Mechanics, a version of quantum mechanics. The equation of motion in the Stochastic Mechanics was the so called Newton-Nelson equation, a second order stochastic equation with mean derivatives where a special second order mean derivative was in use.

In [4, 5] a new method of measurement of dynamically distorted signals in an electronic device was elaborated on the basis of algebraic-differential equations, called the Leontieff type equations. Later in the works by G.A. Sviridyuk and his school, Yu.E. Gliklikh, E.Yu. Mashkov et al. the noise was taken into account, where the noise was represented in terms of Nelson's symmetric mean derivative (current velocity).

Here we investigate the Leontiev type equation, whose matrix pencil is regular and satisfies the rank-degree condition, and the special second order mean derivative from [1] is in use instead of the symmetric first order mean derivative. This allows us to take into account quantum effects in the device that turn the deterministic incoming signal into a stochastic process. We prove the existence of solution of this equation.

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1. Some Facts from the Theory of Matrices

Everywhere below we deal with the n dimensional linear space $\mathbb{R}^n$, vectors from $\mathbb{R}^n$ and $n \times n$ matrices.

Consider two $n \times n$ constant matrices A and B where A is degenerate while B is non-degenerate. The expression $\lambda A + B$, where λ is real parameter, is called the matrix pencil. The polynomial $\theta(\lambda) = \det(\lambda A + B)$ is called the characteristic polynomial of the pencil $\lambda A + B$. The pencil is called regular, if its characteristic polynomial is not identically equal to zero.

If the matrix pencil $\lambda A + B$ is regular, there exist to non-degenerate linear operators P (acts from the left side) and Q (acts from the right side) that reduce the matrices A and B to the canonical quasi-diagonal form (see [6]). In the canonical quasi-diagonal form, under appropriate numeration of basis vectors, in the matrix PAQ first along diagonal there is the $d \times d$ unit matrix and then along diagonal there are the Jordan boxes with zeros on diagonal. In PBQ in the lines corresponding to Jordan boxes, there is the unit matrix, and in the lines corresponding to the unit matrix there is a certain non-degenerate matrix J. Thus

$$P(\lambda A(t) + B(t))Q = \lambda \begin{pmatrix} I_d & 0 \\ 0 & N(t) \end{pmatrix} + \begin{pmatrix} J & 0 \\ 0 & I_{n-d} \end{pmatrix}, \tag{1}$$

The non-degenerate pencil satisfies the rank-degree condition if

$$rank(A(t)) = \deg(\det(\lambda A(t) + B(t))). \tag{2}$$

If the pencil satisfies the rank-degree condition, formula (1) takes the form

$$P(t)(\lambda A(t) + B(t))Q(t) = \lambda \begin{pmatrix} I_d & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} J & 0 \\ 0 & I_{n-d} \end{pmatrix}. \tag{3}$$

where J is non-degenerate since B is non-degenerate.

2. Preliminaries on the Mean Derivatives

Everywhere below we deal with processes, equations, etc., given on a certain finite time interval $[0, T]$.

Consider a stochastic process $\xi(t)$ in $\mathbb{R}^n$, $t \in [0, T]$, given on a certain probability space $(\Omega, \mathcal{F}, \mathsf{P})$ and such that $\xi(t)$ is L_1-random variable for all t.

Every stochastic process $\xi(t)$ in $\mathbb{R}^n$, $t \in [0, l]$, determines three families of σ-subalgebras of σ-algebra $\mathcal{F}$:

(i) the "past" $\mathcal{P}_t^\xi$ generated by pre-images of Borel sets in $\mathbb{R}^n$ by all mappings $\xi(s) : \Omega \to \mathbb{R}^n$ for $0 \le s \le t$;

(ii) the "future" $\mathcal{F}_t^\xi$ generated by pre-images of Borel sets in $\mathbb{R}^n$ by all mappings $\xi(s) : \Omega \to \mathbb{R}^n$ for $t \le s \le l$;

(iii) the "present" ("now") $\mathcal{N}_t^\xi$ generated by pre-images of Borel sets in $\mathbb{R}^n$ by the mapping $\xi(t)$.

All families are supposed to be complete, i.e., containing all sets of probability 0.

For convenience we denote the conditional expectation of $\xi(t)$ with respect to $\mathcal{N}_t^\xi$ by $E_t^\xi(\cdot)$. Ordinary ("unconditional") expectation is denoted by E.

Strictly speaking, almost surely (a.s.) the sample paths of $\xi(t)$ are not differentiable for almost all t. Thus its "classical" derivatives exist only in the sense of generalized functions. To avoid using the generalized functions, following Nelson (see, e.g., [1, 2, 3]) we give

Definition 1. *(i) Forward mean derivative $D\xi(t)$ of $\xi(t)$ at time $t \in [0, T)$ is an L_1-random variable of the form*

$$D\xi(t) = \lim_{\Delta t \to +0} E_t^\xi \left(\frac{\xi(t + \Delta t) - \xi(t)}{\Delta t} \right) \tag{4}$$

where the limit is supposed to exists in $L_1(\Omega, \mathcal{F}, \mathsf{P})$ and $\Delta t \to +0$ means that Δt tends to 0 and $\Delta t > 0$.

(ii) Backward mean derivative $D_*\xi(t)$ of $\xi(t)$ at $t \in (0, T]$ is an L_1-random variable

$$D_*\xi(t) = \lim_{\Delta t \to +0} E_t^\xi \left(\frac{\xi(t) - \xi(t - \Delta t)}{\Delta t} \right) \tag{5}$$

where the conditions and the notation are the same as in (i).

Please note that the current velocities are natural analogues of physical velocities od deterministic processes.

From the properties of conditional expectation (see [7]) it follows that $D\xi(t)$ and $D_*\xi(t)$ can be represented as compositions of $\xi(t)$ and Borel measurable vector fields (regressions)

$$Y^0(t,x) = \lim_{\Delta t \to +0} E\left(\frac{\xi(t+\Delta t)-\xi(t)}{\Delta t}\Big|\xi(t)=x\right),$$
$$Y^0_*(t,x) = \lim_{\Delta t \to +0} E\left(\frac{\xi(t)-\xi(t-\Delta t)}{\Delta t}\Big|\xi(t)=x\right) \qquad (6)$$

on $\mathbb{R}^n$. This means that $D\xi(t) = Y^0(t,\xi(t))$ and $D_*\xi(t) = Y^0_*(t,\xi(t))$.

Definition 2. *For an L^1-stochastic process $\xi(t)$, $t \in [0,T]$, its quadratic mean derivative $D_2\xi(t)$ is defined by the formula*

$$D_2\xi(t) = \lim_{\triangle t \to +0} E^\xi_t\left(\frac{(\xi(t+\triangle t)-\xi(t))(\xi(t+\triangle t)-\xi(t))^*}{\triangle t}\right), \qquad (7)$$

where $(\xi(t+\triangle t)-\xi(t))$ is a column vector and $(\xi(t+\triangle t)-\xi(t))^*$ is its conjugate, i.e., the row vector, the limit is supposed to exist in $L^1(\Omega,\mathcal{F},\mathsf{P})$.

One can easily derive that for an Ito process $\xi(t) = \int\limits_0^t a(s)ds + \int\limits_0^t A(s)dw(s)$ its quadratic mean derivative takes the form $D_2\xi(t) = AA^*$.

Let $Z(t,x)$ be a C^2-smooth vector field, and $\xi(t)$ be a stochastic process.

Definition 3. *The forward $DZ(t,\xi(t))$ and the backward $D_*Z(t,\xi(t))$ mean derivatives of Z along $\xi(\cdot)$ at t are the L^1-limits of the form*

$$DZ(t,\xi(t)) = \lim_{\Delta t \to +0} E^\xi_t\left(\frac{Z(t+\Delta t,\xi(t+\Delta t)) - Z(t,\xi(t))}{\Delta t}\right) \qquad (8)$$

$$D_*Z(t,\xi(t)) = \lim_{\Delta t \to +0} E^\xi_t\left(\frac{Z(t,\xi(t)) - Z(t-\Delta t,\xi(t-\Delta t))}{\Delta t}\right) \qquad (9)$$

Remark 1. We understand the second mean derivative, say, $DD_*\xi$ as the mean derivative D of the regression (i.e., vector field) of $D_*\xi$. Below we will deal with DD_* and D_*D.

3. The Main Result

We deal with the matrix pencil satisfying the rank-degree condition and transformed to canonical form (3).

Consider the second order stochastic algebraic-differential equation

$$\begin{cases} \frac{1}{2}A(DD_* + D_*D)\eta(t) = B\eta(t) + f(t), \\ D_2\eta(t) = A. \end{cases} \qquad (10)$$

Following [4, 5] we interpret (10) as equation describing dynamically distorted signasls in an electronic device, where the matrices A and B describe the device, the deterministic function $f(t)$ is the ingoing signal, the stochastc process $\eta(t)$ is the outgoing signal. The second order derivative $\frac{1}{2}(DD_* + D_*D)$ was introduced by E. Nelson in the equation (that now is called the Newton–Nelson equation) of motion of a quantum particle in his Stochastic Mechanics (a version of quantum mechanics, see, e.g. [1, 2, 3]). The use of this derivative allows us to take into account the quantum effects that make the outgoing signal a stochastic process.

It follows from the Definition 1 (ii) that (10) is ill-defined at $t = 0$. To avoid thgis difficulty, consider $t_0 \in (0, T)$ small enough and introduce the function $t_0(t)$ by the formula

$$t_0(t) = \begin{cases} \dfrac{1}{t_0}, & \text{if} \quad 0 \le t \le t_0, \\ \dfrac{1}{t}, & \text{if} \quad t_0 \le t. \end{cases}$$

It follows from (3) that (10) is divided into two independent systems

$$\begin{cases} \dfrac{1}{2}(DD_* + D_*D)\eta^{(1)}(t) = J\eta^{(1)}(t) + f^{(1)}(t), \\ D_2\eta^{(1)}(t) = I_d \end{cases} \tag{11}$$

in $\mathbb{R}^d$ and

$$\begin{cases} \eta^{(2)}(t) + f^{(2)}(t) = 0, \\ D_2\eta^{(2)}(t) = 0 \end{cases} \tag{12}$$

in $\mathbb{R}^{n-d}$.

Theorem 1. *There exists a process $\eta(t)$ starting at $t = 0$, that for $t \in [t_0, T]$ satisfies equation (10).*

Proof. The second line in (12) means that the solution of (12) is deterministic. Then it follows from the first line of (12), that $\eta^{(2)}(t) = -f^{(2)}(t)$ with the obvious initial condition $\eta^{(2)}(0) = -f^{(2)}(0)$.

From the second line of (11) it follows that the solution of (11) will take the form $\eta^{(1)}(t) = \int_0^t a(s)ds + w(t)$, where $w(t)$ is a Wiener process and a certain $a(t)$ should be found.

Consider the following stochastic differential equation

$$a(t) = a_0 + \int_0^t f(s)ds + \int_0^t Ja(s)ds + \int_0^t Jdw(s) +$$

$$\frac{1}{2}\int_0^t t_0(t)dw(t) - \frac{1}{2}t_0(t)\left(\int_0^t a(s)ds + w(t)\right). \tag{13}$$

It is a version of [8, equation (15.16)], where we replace the continuous vector $\bar{a}_0$ by f and the linear operator $\bar{a}_1$ by J. Note that here $J' = 0$ since J is constant. Equation (13) has a unique strong solution for $t \in [0, T]$ since its coefficients are either Lipschitz continuous and have linear growth with respect to a or do not depend on a. It follows from [8, Theorem 15.9] that for $\eta^{(1)}(t)(t) = \int_0^t a(s)ds + w(t)$ the following relation holds: $\frac{1}{2}(DD_* + D_*D)\eta^{(1)}(t) = J\eta^{(1)}(t) + f(t)$. □

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ОБ ОДНОМ СТОХАСТИЧЕСКОМ АЛГЕБРО-ДИФФЕРЕНЦИАЛЬНОМ УРАВНЕНИИ С ПРОИЗВОДНЫМИ В СРЕДНЕМ, УДОВЛЕТВОРЯЮЩЕМ УСЛОВИЮ РАНГ-СТЕПЕНЬ

М. Ю. Рязанцев

Мы исследуем одно стохастическое алгебро-дифференциальное уравнение второго порядка с производными в среднем, у которого его матричный пучок регулярен и удовлетворяет условию ранг-степень. Это уравнение моделирует динамическое искажение сигнала в электронном приборе с учетом квантовых эффектов, которые превращают детерминированный входящий сигнал в стохастический процесс. Мы доказываем существование решения этого уравнения.

Ключевые слова: производная в среднем; стохастические дифференциальные уравнения второго порядка; стохастические алгебро-дифференциальные уравнения.

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