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THE LINEARIZED OSKOLKOV SYSTEM IN THE AVALOS–TRIGGIANI PROBLEM

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The Avalos-Triggiani problem for a linearized Oskolkov system and a system of wave equations is investigated. The mathematical model contains a linearized Oskolkov system and a wave vector equation corresponding to some structure immersed in the Kelvin–Voight incompressible viscoelastic fluid. The theorem of the existence of the unique solution to the Avalos–Triggiani problem for the indicated systems is proved using the method proposed by the authors of this problem.

Keywords: Avalos–Triggiani problem; incompressible viscoelastic fluid; linearized Oskolkov system.

Formulation of the Problem

Let Ω be a bounded domain in $\mathbb{R}^n, n = 2,3$, with sufficiently smooth boundary $\partial\Omega$. Let $u = \mathrm{col}(u_1, u_2, ..., u_n)$ be a $n-$ dimensional velocity vector $n = 2,3$, the scalar function p be a pressure, and the vector $w = \mathrm{col}(w_1.w_2, ..., w_n)$ be a vector of displacement of a body, which occupies the domain Ω_s, and is immersed in a fluid occupying the domain Ω_f. Therefore, $\Omega = \Omega_s \cup \Omega_f, \overline{\Omega}_s \cap \overline{\Omega}_f = \partial\Omega_s \equiv \Gamma_s$ is the common boundary of Ω_s, and Ω_f. Let us denote the outer boundary of Ω_f by Γ_f (Fig. 1). Our goal is to investigate the Avalos–Triggiani problem (AT problem) [1, 2] for the case when the fluid in Ω_f is an incompressible viscoelastic Kelvin–Voigt fluid discribed by the linearized Oskolkov system. Note that here the vector-function $\tilde{u} = \mathrm{col}(\tilde{u}_1, \tilde{u}_2, ..., \tilde{u}_n)$ corresponds to the stationary solution of the original system [3] . The mathematical model in question is defined by the system

$$(1 - \lambda\nabla^2)u_t - \eta\nabla^2 u - (\tilde{u} \cdot \nabla)u - (u \cdot \nabla)\tilde{u} + \nabla p = 0, \qquad \forall(t,x) \in (0,T] \times \Omega_f \equiv \Omega_{Tf}, \quad (1)$$

$$\nabla \cdot u = 0, \qquad \forall(t,x) \in \Omega_{Tf}, \quad (2)$$

$$w_{tt} - \nabla^2 w + w = 0, \qquad \forall(t,x) \in (0,T] \times \Omega_s \equiv \Omega_{Ts} \quad (3)$$

with the boundary value conditions

$$u|_{\Gamma_f} \equiv 0, \qquad \forall(t,x) \in (0,T] \times \Gamma_f \equiv \Gamma_{Tf}, \quad (4)$$

$$u \equiv w_t, \qquad \forall(t,x) \in (0,T] \times \Gamma_s \equiv \Gamma_{Ts}, \quad (5)$$

$$\frac{\partial u}{\partial \nu} - \frac{\partial w}{\partial \nu} = p\nu, \qquad \forall(t,x) \in \Gamma_{Ts} \quad (6)$$

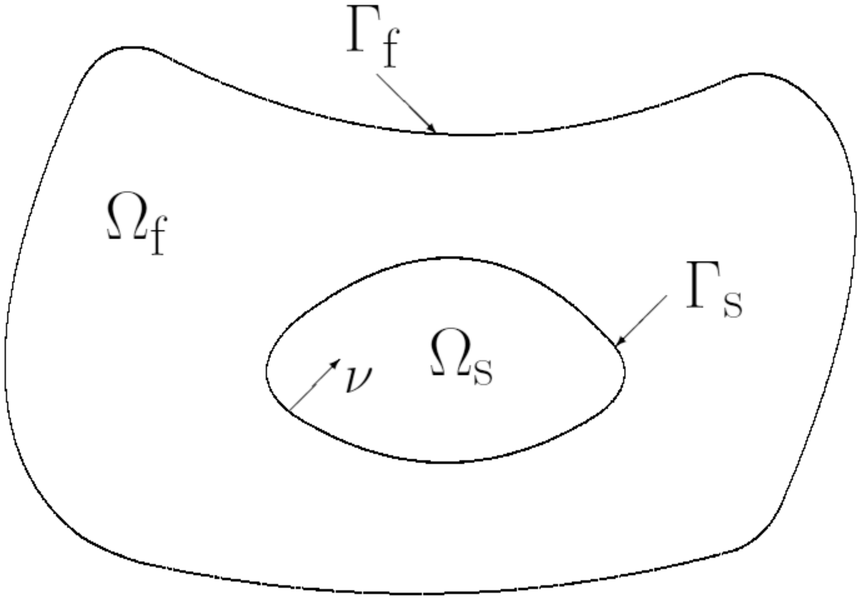

Fig. 1. The physical model

and the initial value condition

$$(w(0,\cdot), w_t(0,\cdot), u(0,\cdot)) = (w_0, w_1, u_0) \in \mathbf{H}, \tag{7}$$

where $\mathbf{H} = (H^1(\Omega_s))^n \times (L^2(\Omega_s))^n \times \mathcal{H}_f$ and $\mathcal{H}_f = \{f \in (L^2(\Omega_f))^n : \nabla \cdot f = 0 \text{ in } \Omega_f$ and $[f \cdot \nu]|_{\Gamma_f} = 0\}$.

In system (1), the parameters λ and η characterize the elastic and viscous properties of the fluid, respectively, ν is an unit normal vector. In the case of $\lambda = 0$, problem (1)–(7) without clauses containing $\tilde{u}$ was investigated in [1, 2]. The AT problem for the linear Oskolkov system and a system of wave equations for $\lambda \neq 0$ was considered in [4, 5], and the AT problem for the linear Oskolkov system of non-zero order and a system of wave equations is investigated in [6, 7] . The AT problem for the linearized Oskolkov system and a system of wave equations is considered at the first time and generalized the results [4].

1. Reduction to the Abstract Cauchy Problem

Following [1], [2], we assume that $p(t)$ satisfies the following elliptic problem:

$$\begin{aligned} &\Delta p = 0 \quad \text{in} \quad \Omega_{Tf}, \\ &p = \frac{\partial u}{\partial \nu} \cdot \nu - \frac{\partial w}{\partial \nu} \cdot \nu \quad \text{on} \quad \Gamma_{Ts}, \\ &\frac{\partial p}{\partial \nu} = \Delta u \cdot \nu \quad \text{on} \quad \Gamma_{Tf}. \end{aligned} \tag{8}$$

Then the pressure p can be represented as follows:

$$p(t) = D_s\{(\frac{\partial u(t)}{\partial \nu} \cdot \nu - \frac{\partial w(t)}{\partial \nu} \cdot \nu)_{\Gamma_{Ts}}\} + N_f((\Delta u(t) \cdot \nu)_{\Gamma_{Tf}}) \quad \text{in} \quad \Omega_{Tf},$$

where the Dirichlet map D_s is defined by the relations

$$h = D_s(g) \Leftrightarrow \begin{cases} \Delta h = 0 \quad \text{in} \quad \Omega_f, \\ h = g \quad \text{on} \quad \Gamma_s, \\ \dfrac{\partial h}{\partial \nu} = 0 \quad \text{on} \quad \Gamma_f, \end{cases}$$

and the Neumann map N_f is defined by the relations

$$h = N_f(g) \Leftrightarrow \begin{cases} \Delta h = 0 \quad \text{in} \quad \Omega_f, \\ h = 0 \quad \text{on} \quad \Gamma_s, \\ \dfrac{\partial h}{\partial \nu} = g \quad \text{on} \quad \Gamma_f. \end{cases}$$

Then original system (1)–(4), which describes the interaction of the fluid and the body immersed in the fluid, takes the form

$$(1 - \lambda\nabla^2)u_t - \eta\nabla^2 u - (\tilde{u}\cdot\nabla)u - (u\cdot\nabla)\tilde{u} - G_1 w - G_2 u = 0, \qquad \forall (t,x) \in \Omega_{Tf}, \tag{9}$$

$$\nabla \cdot u = 0, \tag{10}$$

$$w_{tt} - \nabla^2 w + w = 0, \qquad \forall (t,x) \in \Omega_{Ts} \tag{11}$$

with the boundary value conditions

$$u|_{\Gamma_f} \equiv 0, \qquad \forall (t,x) \in \Gamma_{Tf}, \tag{12}$$

$$u \equiv w_t, \qquad \forall (t,x) \in \Gamma_{Ts}, \tag{13}$$

where

$$G_1 w \equiv \nabla \left\{ D_s \left\{ \left(\frac{\partial w(t)}{\partial \nu} \cdot \nu\right) \right\}_{\Gamma_{Ts}} \right\} \quad \text{in} \quad \Omega_{Tf},$$

$$G_2 u \equiv -\nabla \left\{ D_s \left\{ \left(\frac{\partial u(t)}{\partial \nu} \cdot \nu \right)_{\Gamma_{Ts}} \right\} + N_f\Big((\Delta u(t) \cdot \nu)_{\Gamma_{Tf}} \Big) \right\} \quad \text{in} \quad \Omega_{Tf}.$$

Let us rewrite problem (9)–(13), in which pressure is excluded, in the form of an abstract Cauchy problem

$$L\dot{v} = Mv, \quad v(0) = v_0, \tag{14}$$

where the operators L and M are defined by the matrices respectively

$$\begin{pmatrix} I & O & O \\ O & I & O \\ O & O & A_\lambda \end{pmatrix}, \quad \begin{pmatrix} O & I & O \\ \Delta - I & O & O \\ G_1 & O & B + G_2 \end{pmatrix}$$

and $v = \text{col}(w, w_t, u)$, $A_\lambda = 1 - \lambda\nabla^2, B : u \to \eta\nabla^2 u - (\tilde{u}\cdot\nabla)u - (u\cdot\nabla)\tilde{u}$ [8], I is a unit operator whose domain is clear out of context.

2. Solvability of the Abstract Cauchy Problem

We study problem (14) based on the results obtained in [9]–[12].

Lemma 1. *Let $\lambda \in \mathbb{R}$, $\eta \in \mathbb{R}_+$, the operators L and M be linear continuous operators from $\boldsymbol{G}$ to $\boldsymbol{H}$ ($L, M \in \mathcal{L}(\boldsymbol{G}, \boldsymbol{H})$), then there exists $L^{-1} \in \mathcal{L}(\boldsymbol{H})$. Here is the space $\boldsymbol{G} = (H^2(\Omega_s))^n \times (H^2(\Omega_s))^n \times \mathcal{G}_f$, where $\mathcal{G}_f$ is closure according to the norm of the space $(H^2(\Omega_s))^n$ spaces of infinitely differentiable solenoid functions such that (12)–(13) are fulfilled.*

Theorem 1. *For any $\lambda \in \mathbb{R}, \eta \in \mathbb{R}_+$ and $v_0 \in \boldsymbol{G}$, there is a unique solution to problem (14) $v \in C^\infty(\mathbb{R}, \boldsymbol{G})$*

Remark 1. Received results can be generalized to the AT problem with the linearized Oskolkov system of the more high order [3].

Remark 2. We intend to develop our research in the direction indicated in [13] – [15].

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ЛИНЕАРИЗОВАННАЯ СИСТЕМА ОСКОЛКОВА В ЗАДАЧЕ АВАЛОС – ТРИДЖИАНИ

Т. Г. Сукачева, А. О. Кондюков

Исследуется задача Авалос–Триджиани для линеаризованной системы Осколкова и системы волновых уравнений. Математическая модель содержит линеаризованную систему Осколкова и волновое уравнение, соответствующее некоторой структуре, погруженной в несжимаемую вязкоупругую жидкость Кельвина–Фойгта. С помощью метода, предложенного авторами этой задачи, доказывается теорема существования единственного решения задачи Авалос–Триджиани для указанных систем. Результаты данной статьи обобщают результаты, полученные ранее.

Ключевые слова: задача Авалос – Триджиани; несжимаемая вязкоупругая жидкость; линеаризованная система Осколкова.

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