MCS 34K06 **DOI: 10.14529/jcem240202**

CONTINUOUS SOLUTIONS OF LINEAR FUNCTIONAL EQUATIONS ON PIECEWISE SMOOTH CURVES IN MATHEMATICAL MODELS OF BOUNDARY VALUE PROBLEMS WITH A SHIFT

V. L. Dilman[1]**,** dilmanvl@susu.ru**,**
T. V. Karpeta[1]**,** karpetatv@susu.ru**,**
[1]South Ural State University, Chelyabinsk, Russian Federation

Linear functional equations on an arbitrary piecewise smooth curve are considered. Such equations are studied in connection with the theory of singular integral equations, which are a mathematical tool in the study of mathematical models of elasticity theory in which the conjugation conditions contain a boundary shift. Such equations also arise in the mathematical modeling of the transfer of charged particles and ionized radiation. The shift function is assumed to act cyclically on a set of simple curves forming a given curve, with only the ends of simple curves being periodic points. The aim of the work is to find the conditions for the existence and cardinality of a set of continuous solutions to such equations in the classes of Helder functions and primitive Lebesgue integrable functions with coefficients and the right side of the equation from the same classes. The solutions obtained have the form of convergent series and can be calculated with any degree of accuracy. The method of operation consists in reducing this equation to an equation of a special type in which all periodic points are fixed, which allows you to use the results for the case of a simple smooth curve.

Keywords: linear functional equations from one variable; classes of primitive Lebesgue integrable functions, piecewise smooth curves.

Inroduction

The theory of boundary value problems for analytical functions with a boundary condition containing a shift function [1–3] can be interpreted in terms of the theory of singular integral equations "with a shift". In such equations, the kernels of integrals are themselves integrals with a Cauchy kernel with two variable limits dependent on each other [4, 5]. Let's denote this dependence α. In general, these equations are given on arbitrary piecewise smooth curves on the complex plane in the sense of work [6]. The study of such equations necessarily leads to the consideration of purely functional linear equations (LFE) of the form:

$$(F_g(\psi))(t) \equiv \psi(\alpha(t)) - g(t)\psi(t) = h(t), \tag{1}$$

defined on piecewise smooth curves of the complex plane.In addition to the issues related to the existence, uniqueness, and number (power of the set) of solutions to equation (1), it is necessary to investigate, as can be seen from [4,5], the operator $(F_g(\psi))^{-1}$ from the point of view of its invariance with respect to classes of Helder and Lebesgue functions, usually used in the study of singular integral equations, and classes of primitive ones from them. These classes include classes of Helder functions: $H_\mu,\ 0 < \mu \le 1$; $H = \bigcup\limits_{0<\mu\le 1} H_\mu$; classes of Lebesgue functions: $L_p, 1 \le p < \infty; L = \bigcup\limits_{1<p<\infty} L_p$, as well as classes of primitive from Lebesgue functions $A_p, p > 1$ and $\tilde{A}_p = \bigcup_{q<p} A_q$. The latter play an important role in the study of the above-mentioned equations with generalized logarithmic kernels, more precisely, equations whose kernels are Cauchy-type integrals with two related variable limits. In general, the operator $(F_g(\psi))^{-1}$ changes the parameters of the specified classes of functions, so it is important to find estimates of these parameters for solutions of (1). In the case of two fixed points of the shift function located at the ends of the curves these questions for (1) on simple smooth open curves were considered in [5, 7–9]. A more general situation where there are a finite number of periodic points on a piecewise smooth curve was studied in [10]. In many publications devoted to equation (1) and its generalizations, the equations were given on a real straight line or its semi-open segment, and the solutions were mainly in classes of continuous functions [11–18] or some of their subclasses. In [17, 18], the function α had a finite order with respect to the superposition. Such LFE find applications in mathematical modeling of radiation protection optimization methods [18, 19].

When equation (1) is studied in the general case, on an arbitrary piecewise smooth curve, it is necessary to consider the situation when the function α determines some permutation at the ends of simple arcs (these ends are not necessarily point fixed relative to α). Then a certain degree of mapping acts on each simple arc as an automorphism, but not identical, as in [17, 18], but having infinite order. In [10], (1) was studied in this situation when the mapping α generates a cyclic group on a set of periodic points. In addition, in [10] the case was considered when on each simple curve the solution be continuous at one of the ends. As noted in [5, 7–9], a simple curve containing one of the ends and not containing the other: $[a; b)$ or $(a; b]$,is the natural domain for setting (1).Then a continuous continuation of the solution at the appropriate end is possible. The existence of a continuous solution at all ends (in the case of a simple open curve on two) requires additional constraints on the solution and, in some cases, leads to the absence of such a solution. This is described in detail in [9] for the case of a simple curve and two fixed points at its ends. The aim of the work is to transfer the results of [9] to the case of a piecewise smooth curve considered in [10].

1. Notation and Assumptions

The designations and assumptions are given in detail in [10]. The main ones are here for ease of reading. Let $\Gamma = \bigcup_{j=1}^{n} \Gamma_j, \;\; \Gamma_j = [a_j; b_j], \;\; j = \overline{1,n}$ is a simple open smooth arc. Denote by $\overset{\circ}{\Gamma}$ a curve Γ without ends $a_j; b_j$ of arcs Γ_j; by Γ^a a curve Γ without ends b_j; by Γ^b a curve Γ without ends a_j. If M is an arbitrary class of functions defined on the curve Γ, $c_1, ..., c_s \in \Gamma$, then let $h \in M({c_1}^0, ..., {c_s}^0)$ if $h \in M$ and $h(c_i) = 0, \;\; i = \overline{1; s}$. We denote the degree of the invertible defined on Γ mapping $\alpha = \alpha(t), \;\; t \in \Gamma$ by the subscript $\alpha_0(t) \equiv t$, $\alpha_1(t) \equiv \alpha(t)$, $\alpha_n(t) \equiv \alpha(\alpha_{n-1}(t))$, $\alpha_{-1}(t)$ is an inverse mapping to α, $\alpha_{-n}(t) \equiv \alpha_{-1}(\alpha_{-n+1}(t))$, $n = \overline{1, \infty}$. Note $\alpha_n(\alpha_{-n}(t)) \equiv \alpha_{-n}(\alpha_n(t)) \equiv \alpha_0(t) \equiv t$. Let's assume that α has the following properties.

1) Narrowing to map the arc $\Gamma_j, \;\; j = \overline{1,n}$ to a certain arc $\Gamma_{\hat{\alpha}(j)}$, continuously, mutually unambiguously and preserves orientation. The last mean the condition: $\alpha(a_j) = a_{\alpha(j)}$. The mapping of numbers $j \to \hat{\alpha}(j)$ is a substitution $\hat{\alpha}$ on the set $\overline{1, n}$.

2) Mapping α has no other periodic points except the ends a_j, b_j, $j \in \overline{1, n}$.

Thus, α acts on $\overset{\circ}{\Gamma}$ mutually unambiguous and mutually continuous. Condition 2) does not limit generality, since it is always possible to divide curves Γ_j into parts by periodic internal points and their orbits. These parts are assumed to be those simple curves that form the entire curve Γ.

3) $\forall t \in \Gamma \;\; \exists \alpha'(t) \neq 0$ and $\alpha' \in H_\theta$ on Γ, $\theta \in (0; 1]$.

Let be $\{N_1, ..., N_m\}$ is the splitting of the set into the orbits of the substitution $\hat{\alpha}$ (in other terminology, into independent cycles of this substitution). Let $\hat{\alpha}_j$, $j = \overline{1, m}$ be a narrowing $\hat{\alpha}$ on N_j, that is, an independent substitution cycle $\hat{\alpha}$ acting on the orbit N_j. Let be p_j is the order of this cycle (obviously $\sum\limits_{j=1}^{m} p_j = n$). Consider $k \in N_j$. Note that the mapping α_{p_j} translates the arc Γ_k to itself, and satisfies properties 1–3 of [5, 7–9] (properties 1–3 of these works are the same properties 1–3 listed above provided that $n = 1$).

We will assume that for each specified mapping α_{p_j}, property 4 of the work [10] is also fulfilled. This assumption is equivalent to a condition:

4) $\forall j \in \overline{1, m} \quad \prod\limits_{k \in N_j} |\alpha'(a_k)| \neq 1, \quad \prod\limits_{k \in N_j} |\alpha'(b_k)| \neq 1.$

As in [5, 7–10], we will assume that the letters a_k and b_k denote, respectively, the attracting and pushing away fixed points of the maps α_{p_j} $(k \in N_j)$. Note that the attracting fixed points are translated into each other by mapping α. The same applies to pushing away fixed points. In [10] it is shown that condition 4 can be written as:

4′) $\forall j \in \overline{1, m} \quad \prod\limits_{k \in N_j} |\alpha'(a_k)| < 1, \quad \prod\limits_{k \in N_j} |\alpha'(b_k)| > 1.$

2. Supporting Statements

We will consider the curve Γ^a and assume that $\hat{\alpha}$ is a cyclic substitution on the set$\left\{\overline{1, n}\right\}$: $\hat{\alpha}(j) = j + 1$, $j = \overline{1, n-1}$, $\hat{\alpha}(n) = 1$, that is, we assume that $m = 1$ there is only one orbit where α is acting on Γ_j, $j = \overline{1, n}$. The key role in the study of equation (1) is played by the equation:

$$\psi(\tilde{\alpha}(t)) - \tilde{g}(t)\,\psi(t) = \tilde{h}(t), \tag{2}$$

$$\tilde{g}(t) = \prod_{j=0}^{n-1} g(\alpha_j(t)),$$

$$\tilde{h}(t) = h(\alpha_{n-1}(t)) + \sum_{j=1}^{n-1} h(\alpha_{n-1-j}(t)) \prod_{k=1}^{j} g(\alpha_{n-k}(t)), \quad \tilde{\alpha}(t) = \alpha_n(t) \tag{3}$$

The relationship of these equations is investigated in [10], but only for solutions $\psi \in C^{[a_j;b_j)}$, $\forall\, j \in \overline{1, n}$. However, the case $\forall\, j \in \overline{1, n}$ $\psi \in C^{[a_j;b_j]}$ gives significantly different results (see Theorems 4–7 below). We will assume that

$$g(a_j) \neq 0, \;\; j = \overline{1, n}. \tag{4}$$

Theorem 1. [10, Lemmas 1 and 2] *Equation (2) is a consequence of (1). Let $g \in C^{[a_j;b_j)}$ for $\forall\, j \in \overline{1, n}$. If*

$$\prod_{j=1}^{n} |g(a_j)| \geq 1, \quad \prod_{j=1}^{n} g(a_j) \neq 1, \tag{5}$$

then equations (1) and (2), (3) are equivalent in class C^{Γ^a}.

Theorem 2. [10, Lemma 3] *Let* $g \in C^{[a_j;b_j)}$ *for* $\forall\, j \in \overline{1,n}$. *If*

$$\prod_{j=1}^{n} g\left(a_j\right) = 1, \tag{6}$$

and $h\left(a_j\right) = 0,\ j = \overline{1,n}$, *then equations (1) and (2) are equivalent in class* $C^{\Gamma^a}\left(a_j^0,\ j \in \overline{1,n}\right)$.

Theorem 3. [10, Theorem 1] *Let* $g \in C^{[a_j;b_j)}$ *for* $\forall\, j \in \overline{1,n}$. *If*

$$\prod_{j=1}^{n} \left|g\left(a_j\right)\right| < 1, \tag{7}$$

then equations (1) and (2) are equivalent in class C^{Γ^a}.

There are dual statements for theorems 1–3. They will be obtained if we replace $[a_j;b_j)$ with $(a_j;b_j]$ in the formulations of these theorems, and replace conditions (5) – (7) with conditions, respectively

$$\prod_{j=1}^{n} \left|g\left(b_j\right)\right| \le 1, \quad \prod_{j=1}^{n} g\left(b_j\right) \ne 1; \quad \prod_{j=1}^{n} g\left(b_j\right) = 1; \quad \prod_{j=1}^{n} \left|g\left(b_j\right)\right| > 1.$$

Theorems 1–3 and the dualities to them allow us to reduce the study of equation (1) to the study of equation (2), which splits into several equations, each of which is given on a simple curve $\Gamma_j = [a_j b_j]$, $j = \overline{1,n}$, that is, instead of studying (1) on an arbitrary piecewise smooth curve, to study (2) on a simple smooth curve.

3. Theorems of Existence and Number of Solutions for Cyclic Substitution

In all the theorems formulated below, it is assumed $h\left(b_j\right) = 0,\ j = \overline{1,p}$. As shown in [9], this condition does not reduce the generality of statements about equation (1) solutions. In applications to singular integral equations desired function φ associated with ψ by the relation

$$\psi\left(t\right) = \int_{t}^{b_j} \varphi\left(\tau\right) d\tau,\ \ t \in \left[a_j;b_j\right],\ \ j = \overline{1,p}.$$

If $b_k = a_{k+1}$, then there will be two values at this point (one of which is zero). This only means that function $\varphi\left(t\right) = -\psi'(t)$ will not be defined at the “junction” point. We will assume the presence of two or more different values of the desired function at the point of “junction” of two or more simple curves, calling such solutions continuous on Γ. If we want to find solutions that are continuous in the usual sense, then we need constraints on (1) at the "nodes", that is, at the "junction" points of the ends of simple curves forming a curve.

Lemma 1. *Let* $\Gamma_k = \left[c_k;d_k\right]$, $\Gamma_l = \left[c_l;d_l\right]$ *– simple curves included in* Γ, *and* $d_k = c_l$. *Then, for the continuity of the solution at this point on the curve, it is necessary and* *sufficient that*

$$\frac{\tilde{h}(d_k)}{1-\tilde{g}(d_k)} = \frac{\tilde{h}(c_l)}{1-\tilde{g}(c_l)}.$$

In order for $\psi(b_j) = 0$, *it is necessary and sufficient that* $\tilde{h}(b_j) = 0$, *where* b_j *is the periodic point of function* α.

Proof. The periodic points of function α are the fixed points of function α_n, so the second statement is obvious. Substituting fixed points d_k and c_l of function α_n into equation (2), equivalent to (1), we obtain

$$\psi(d_k) = \frac{\tilde{h}(d_k)}{1-\tilde{g}(d_k)} = \frac{\tilde{h}(c_l)}{1-\tilde{g}(c_l)} = \psi(c_l).$$

This implies continuity of the solution in node $d_k = c_l$.

□

Let's introduce the notation. Let's fix an arbitrary point $c \in \overset{\circ}{\Gamma}$. Since by convention this point is non-periodic, the same is true for $\alpha_k(c)$, therefore $\forall k \ \ \alpha_{k+n}(c) \neq \ \alpha_k(c)$. Let's say for any integer k $I_k(c) = [\alpha_{k+n}(c); \alpha_k(c)]$. Note $I_k(c) \subseteq \Gamma_j \Leftrightarrow k \equiv j \pmod n$. Denote by $C_{c,g,h}$ the class of functions $f$continuous on $I_0(c) = [\alpha_p(c); c]$ and satisfying the condition:

$$f(\alpha_p(c)) - \tilde{g}(c) f(c) = \tilde{h}(c). \tag{8}$$

Let $\psi_0 \in C_{c,g,h}$, but otherwise it is arbitrary. If K is an arbitrary class of functions on $I_0(c)$, then we assume by definition $K_{c,g,h} = C_{c,g,h} \cap K$. We will use the icon $\exists!$ to indicate the existence of a single solution to equation (1) in the specified class. Recall that the functions $\tilde{g}, \ \tilde{h}, \ \tilde{\alpha}$ are defined in (3). Condition (4) is assumed in all theorems.

Theorem 4. *Let* $h, g \in H_\mu, \ \ g(t) \neq 0, \ t \in \Gamma$. *If* $\prod_{j=1}^{n} |g(a_j)| > 1, \ \prod_{j=1}^{n} |g(b_j)| > 1$, *then* $\exists!$ *solution of (1) on* C^Γ, *defined by formulas:*

$$\psi(a_j) = \frac{\tilde{h}(a_j)}{1-\tilde{g}(a_j)}; \quad \psi(b_j) = 0;$$

$$\psi(t) = -\sum_{k=0}^{\infty} \frac{\tilde{h}(\tilde{\alpha}_k(t))}{\prod\limits_{j=0}^{k} \tilde{g}(\tilde{\alpha}_j(t))}, \quad t \in \Gamma_j, \quad j = \overline{1,n}. \tag{9}$$

If $\forall j \in \overline{1;n} \ \ \mu < \log_{\left|\tilde{\alpha'}(b_j)\right|} |\tilde{g}(b_j)|$, *then* $\psi \in \widetilde{H}_\mu^\Gamma$.

If $g, h \in \widetilde{A}_p^\Gamma, \ \ p > 1, \ \forall j \in \overline{1;n} \ \ \frac{p-1}{p} < \log_{\left|\tilde{\alpha'}(b_j)\right|} |\tilde{g}(b_j)|$, *then* $\psi \in \widetilde{A}_p^\Gamma$.

Remark 1. $\forall j \in \overline{1;n} \ \ \prod_{k=1}^{n} g(a_k) = \tilde{g}(a_j)$, therefore, the inequalities in the condition of the theorem can be written, for example, in the form: $|\tilde{g}(a_1)| > 1, \ |\tilde{g}(b_1)| > 1$.

Theorem 5. *(Dual to Theorem 4). Let* $h, g \in H_\mu, \ \ g(t) \neq 0, \ t \in \Gamma$. *If* $\prod_{j=1}^{n} |g(a_j)| < 1$, $\prod_{j=1}^{n} |g(b_j)| < 1$, *then* $\exists!$ *solution of (1) on* C^Γ, *defined by* $\psi(a_j) = 0; \quad \psi(b_j) = \frac{\tilde{h}(b_j)}{1-\tilde{g}(b_j)}$;

$$\psi(t) = \tilde{h}(\tilde{\alpha}_{-1}(t)) + \sum_{k=2}^{\infty} \tilde{h}(\tilde{\alpha}_{-k}(t)) \prod_{j=1}^{k-1} \tilde{g}(\tilde{\alpha}_{-j}(t)), \quad t \in \Gamma_j, \ \ j = \overline{1,n}. \tag{10}$$

If $\forall j \in \overline{1;n} \quad \mu < \log_{\left|\tilde{\alpha}'(a_j)\right|} |g(a_j)|$, *then* $\psi \in \widetilde{H}_\mu^\Gamma$.

If $g, h \in \widetilde{A}_p^\Gamma, \ p > 1$, *and* $\forall j \in \overline{1;n} \quad \frac{p-1}{p} < \log_{\left|\tilde{\alpha}'(a_j)\right|} |g(a_j)|$, *then* $\psi \in \widetilde{A}_p^\Gamma$.

Proof of theorem 4. From Theorems 2 and 4 of [5] it follows that $\forall j \in \overline{1,n}$, provided $|\tilde{g}(a_j)| > 1$ on $[a_j; b_j)\exists!$ solution ψ^*, and it has the form (9). Let $\in (a_j; b_j)$ be any point. Then, for ψ^*, as for any solution of equation (1) on $I_0(c)$, condition (8) is satisfied. Therefore, it follows from the proof of Theorem 2 [7] that $\exists!$ solution ψ^{**} (1) on $(a_j; b_j]$, whose narrowing on $I_0(c)$ coincides with ψ^*. Then ψ^* and ψ^{**} are the same. Therefore, function

$$\psi = \begin{bmatrix} \psi^*, \ t \in [a_j; b_j), \\ \psi^{**}, \ t \in (a_j; b_j] \end{bmatrix}$$

is the only continuous solution (1) for $t \in \Gamma_j, \ \ j = \overline{1,p}$. The affiliation of this solution to the functional classes indicated in the formulation of the theorem directly follows from the results of [5, 7].

□

Remark 2. The above reasoning actually repeats the proof of Theorem 1 of [9] with replacement $\Gamma = [a;b]$ by $\Gamma_j = [a_j; b_j], \ \ j = \overline{1,n}$. A similar situation holds for the theorems formulated below, so their proofs are not given.

Theorem 6. *Let* $h, g \in H_\mu, \ \ g(t) \neq 0, \ t \in \Gamma$. *If* $\prod_{j=1}^{n} |g(a_j)| < 1, \ \prod_{j=1}^{n} |g(b_j)| > 1$, *then (1) has a continuum of linearly independent solutions in the class* C^Γ, *which have the form:*

$$\forall\, j \in \overline{1,n} \ \ \psi(a_j) = \frac{\tilde{h}(a_j)}{1 - \tilde{g}(a_j)}; \quad \psi(b_j) = 0;$$

$$\psi(t) = \begin{cases} \prod\limits_{j=0}^{n-1} \tilde{g}(\tilde{\alpha}_{j-n}(t))\, \psi_0(\tilde{\alpha}_{-n}(t)) + \sum\limits_{k=0}^{n-1} \tilde{h}(\tilde{\alpha}_{k-n}(t)) \prod\limits_{j=k+1}^{n-1} \tilde{g}(\tilde{\alpha}_{j-n}(t)), t \in I_n(c), \\ \tilde{g}(\tilde{\alpha}_{-1}(t))\ \psi_0(\tilde{\alpha}_{-1}(t)) + \tilde{h}(\tilde{\alpha}_{-1}(t)), t \in I_1(c), \\ \psi_0(t), t \in I_0(c), \\ \prod\limits_{j=1}^{n-1} \tilde{g}^{-1}(\tilde{\alpha}_{j-n}(t))\, \psi_0(\tilde{\alpha}_n(t)) - \sum\limits_{k=1}^{n} \tilde{h}(\tilde{\alpha}_{n-k}(t)) \prod\limits_{j=k}^{n} \tilde{g}^{-1}(\tilde{\alpha}_{n-j}(t)), t \in I_{-n}(c), \end{cases} \tag{11}$$

where $n \in \mathbb{N}$ *and function* $\psi_0 \in C_{c,g,h}$ *is arbitrary.*

If $\forall\, j \in \overline{1,n}\ \mu < \min\left\{\log_{\left|\tilde{\alpha}'(a_j)\right|} |\tilde{g}(a_j)|;\ \log_{\left|\tilde{\alpha}'(b_j)\right|} |\tilde{g}(b_j)|\right\}$ *and* $\psi_0 \in \widetilde{H}_\mu$, *then* $\psi \in \widetilde{H}_\mu^\Gamma$.

If $g, h \in \widetilde{A}_p^\Gamma, \ \ p > 1$ *and* $\forall\, j \in \overline{1,n}\ \frac{p-1}{p} < \min\left\{\log_{\left|\tilde{\alpha}'(a_j)\right|} |\tilde{g}(a_j)|;\ \log_{\left|\tilde{\alpha}'(b_j)\right|} |\tilde{g}(b_j)|\right\}$, $\psi_0 \in A_{p_0}, \ p_0 > 1$, *then* $\psi \in \widetilde{A}_{p^*}^\Gamma$ *for* $p^* = \min\limits_{j=\overline{1;n}} \{p; p_0; p_{1j}; p_{2j}\}$, *where* $p_{1j} = \left(1 - \log_{\left|\tilde{\alpha}'(a_j)\right|} |\tilde{g}(a_j)|\right)^{-1}$, *if* $|\tilde{g}(a_j)| > \left|\tilde{\alpha}'(a_j)\right|$, *and* $p_{2j} = \left(1 - \log_{\left|\tilde{\alpha}'(b_j)\right|} |\tilde{g}(b_j)|\right)^{-1}$, *if* $|\tilde{g}(b_j)| < \left|\tilde{\alpha}'(b_j)\right|$, *otherwise* $p_1 = p_2 = +\infty$.

Remark 3. Condition (8) is necessary in order for function $\psi_0 \in C_{c,g,h}$ to have a continuous continuation in (11) (according to formula (11)). Theorem 6 does not have a dual one – if the signs in the inequalities of the condition of Theorem 6 are reversed, equation (1) turns out to be generally insoluble. The condition for the existence of a solution under such conditions for a simple curve is formulated [9, Theorem 5].

Theorem 7. *Let* $h, g \in H_\mu,\ \ g(t) \neq 0,\ \ t \in \Gamma, \prod\limits_{j=1}^{n} |g(a_j)| = 1,\ \prod\limits_{j=1}^{n} g(a_j) \neq 1,\ \prod\limits_{j=1}^{p} |g(b_j)| > 1.$ *Then* $\exists!$ *solution of (1) in* C^Γ*, defined by the formulas:*

$$\psi(t) = \frac{\tilde{h}(a_1)}{1-\tilde{g}(a_1)} - \frac{1}{1-\tilde{g}(a_1)} \sum_{k=0}^{\infty} \frac{(1-\tilde{g}(a_1))\,\tilde{h}(\tilde{\alpha}_k(t)) - \tilde{h}(a_1)(1-\tilde{g}(\tilde{\alpha}_k(t)))}{\prod\limits_{j=0}^{k} \tilde{g}(\tilde{\alpha}_j(t))},\ \ t \in \Gamma.$$

If $\forall j \in \overline{1,n}\ \mu < \log_{\left|\tilde{\alpha}'(b_j)\right|} |\tilde{g}(b_j)|$*, then* $\psi \in H^\Gamma_{\mu_1},\ \ \mu_1 = \frac{\mu\theta}{1+\theta}$ *(number*θ *is entered in condinion 3 of shift function* α *definition).*

If $g, h \in \tilde{A}^\Gamma_p,\ \ p > 1,$ *and* $\forall j \in \overline{1,n}\ \frac{p-1}{p} < \log_{\left|\tilde{\alpha}'(b_j)\right|} |\tilde{g}(b_j)|$*, then* $\psi \in \widetilde{A}^\Gamma_{p_1},\ p_1 = \frac{p(1+\theta)}{p+\theta}.$

Theorem 8. *(Dual to Theorem 6). Let* $h, g \in H_\mu,\ \ g(t) \neq 0,\ \ t \in \Gamma,\ \prod\limits_{j=1}^{p} |g(a_j)| < 1,$ $\prod\limits_{j=1}^{n} |g(b_j)| = 1\ \prod\limits_{j=1}^{n} g(b_j) \neq 1$*. Then* $\exists!$ *solution of (1) in* C^Γ*, defined by the formulas:*

$$\psi(t) = \frac{\tilde{h}(b_1)}{1-\tilde{g}(b_1)} - \frac{1}{1-\tilde{g}(b_1)} \sum_{k=0}^{\infty} \frac{(1-\tilde{g}(b_1))\,\tilde{h}(\tilde{\alpha}_k(t)) - \tilde{h}(b_1)(1-\tilde{g}(\tilde{\alpha}_k(t)))}{\prod\limits_{j=0}^{k} \tilde{g}(\tilde{\alpha}_j(t))},\ \ t \in \Gamma.$$

If $\forall j \in \overline{1,n}\ \mu < \log_{\left|\tilde{\alpha}'(a_j)\right|} |\tilde{g}(a_j)|$*, then* $\psi \in H^\Gamma_{\mu_1},\ \ \mu_1 = \frac{\mu\theta}{1+\theta}.$

If $g, h \in \tilde{A}^\Gamma_p,\ \ p > 1,$ *and* $\forall j \in \overline{1,n}\frac{p-1}{p} < \log_{\left|\tilde{\alpha}'(a_j)\right|} |\tilde{g}(a_j)|$*, then* $\psi \in \widetilde{A}^\Gamma_{p_1},\ p_1 = \frac{p(1+\theta)}{p+\theta}.$

Theorem 9. *Let* $h, g \in H_\mu,\ \ g(t) \neq 0,\ \ t \in \Gamma.$ *let* $\prod\limits_{j=1}^{n} g(a_j) = 1, \prod\limits_{j=1}^{p} |g(b_j)| > 1.$ *Then equation (1) is solvable in class* C^Γ *if and only if* $\forall j \in \overline{1,n}\ h(a_j) = 0$*. In this case, the general solution is a one-parameter family of functions of the form:*

$$\psi(t) = \frac{C}{\prod\limits_{k=0}^{\infty} \tilde{g}(\tilde{\alpha}_k(t))} - \sum_{k=0}^{\infty} \frac{\tilde{h}(\tilde{\alpha}_k(t))}{\prod\limits_{l=0}^{k} \tilde{g}(\tilde{\alpha}_l(t))},\ \ t \in \Gamma. \tag{12}$$

If $\forall j \in \overline{1,n}\ \mu < \log_{\left|\tilde{\alpha}'(a_j)\right|} |\tilde{g}(a_j)|$*, then* $\psi \in \widetilde{H}^\Gamma_\mu.$

If $g, h \in \tilde{A}^\Gamma_p,\ \ p > 1,$ *and* $\forall j \in \overline{1,n}\ \frac{p-1}{p} < \log_{\left|\tilde{\alpha}'(a_j)\right|} |\tilde{g}(a_j)|$*, then* $\psi \in \widetilde{A}^\Gamma_p.$

Remark 4. It follows from (12) $\psi(b_j) = 0, \forall j \in \overline{1,n}.$

Theorem 10. *Let* $h, g \in H_\mu,\ \ g(t) \neq 0,\ \ t \in \Gamma.$ *Let*$\prod\limits_{j=1}^{p} |g(a_j)| < 1,\ \ \prod\limits_{j=1}^{n} g(b_j) = 1.$ *Then equation (1) is solvable in class* C^Γ *if and only if* $\forall j \in \overline{1,n}\ h(b_j) = 0$*. In this* *case, the general solution is a one-parameter family of functions of the form:* $\psi(t) = C\prod_{j=1}^{\infty} g(\alpha_{-j}(t)) + h(\alpha_{-1}(t)) - \sum_{j=2}^{\infty} h(\alpha_{-j}(t)) \prod_{k=1}^{j-1} g(\alpha_{-k}(t)),\ \ t \in (a;b]$.

If $\forall\, j \in \overline{1,n}\mu < \log_{\left|\tilde{\alpha'}(b_j)\right|} |\tilde{g}(b_j)|$, *then* $\psi \in \widetilde{H}_{\mu}^{\Gamma}$.

If $g, h \in \tilde{A}_p^{\Gamma},\ \ p > 1,$ *and* $\forall\, j \in \overline{1,n}\ \frac{p-1}{p} < \log_{\left|\tilde{\alpha'}(b_j)\right|} |\tilde{g}(b_j)|$, *then* $\psi \in \widetilde{A}_p^{\Gamma}$.

Remark 5. Theorems 4–10 consider all options except the conditions:

$$\prod_{j=1}^{n} |\tilde{g}(a_j)| \geq 1, \quad \prod_{j=1}^{n} |\tilde{g}(b_j)| \leq 1. \tag{14}$$

In this case, there is a single continuous solution on Γ^a and a single continuous solution on Γ^b, which in general do not coincide on Γ. Therefore, under conditions (14), there is generally no continuous solution.

Conclusions

Analytical expressions for continuous solutions of equation (1) on an arbitrary piecewise smooth curve are obtained. These solutions have the form of converging series or are given by iterating over an infinite number of cases. It is shown that the set of continuous solutions (1) depends on the values $\prod_{j=1}^{n} g(a_j)$ and $\prod_{j=1}^{n} g(b_j)$ compared to the unit and can be empty, singleton, be a one-parameter family or contain an infinite set of linearly independent solutions. The solutions still belong to classes H_μ and classes A_p that functions $g(t)$ and $h(t)$ belong to, but, in general, with other parameters.

References

1. Litvinchuk G.S. *Solvability Theory of Boundary Value Problems and Singular Integral Equations with Shift*, Dodrecht, Springer Science, Business Media, 2012. DOI: 10.1007/978-94-011-4363-9

2. Kravchenko V.G., Litvinchuk G.S. *Introduction to the Theory of Singular Integral Operators with Shift*, Kluwer Academic Publishers, Dordrecht, Boston, London, 1994. DOI: 10.1007/978-94-011-1180-5

3. Karlovich Y.I., Kravchenko V.G., Litvinchuk G.S. Noether Theory of Singular Integral Operators with Shift. *Soviet Math. (Iz. VUZ)*, 1983, no. 4, pp. 3–27.

4. Dilman V.L., Chibrikova L.I. On Solutions of an Integral Equation with a Generalized Logarithmic Kernel in Lp, p>1. *Soviet Math. (Iz. VUZ)*, 1986, no. 4, pp. 26–36.

5. Dilman V.L. Linear Functional Equations in Helder Classes of Functions on a Simple Smooth Curve. *Bulletin of the South Ural State University. Series: Mathematics. Mechanics. Physics,* 2020, vol. 12, no. 2, pp. 5–12. DOI: 10.14529/mmph200201 (in Russian)

6. Muskhelishvili N.I. *Singular Integral Equations.* Moscow, Nauka, 1968. (in Russian)

7. Dilman V.L., Komissarova D.A. Conditions for the Existence and Uniqueness of Solutions of Linear Functional Equations in Classes of Primitive Lebesgue

Functions on a Simple Smooth Curve. *Bulletin of the South Ural State University. Series: Mathematics. Mechanics. Physics*, 2021, vol. 13, no. 4, pp. 13–23. DOI: 10.14529/mmph210402 (in Russian)

8. Dilman V.L., Komissarova D.A. Linear Functional Equations in Classes of Primitive Lebesgue Functions on Segments of Curves. *Chelyabinsk Physical and Mathematical Journal*, 2023, vol. 8, iss. 1, pp. 5–17. DOI: 10.47475/2500-0101-2023-18101 (in Russian)
9. Dilman V.L. Properties and Description of Solutions Sets of Linear Functional Equations on a Simple Smooth Curve. *Bulletin of the South Ural State University. Series: Mathematics. Mechanics. Physics*, 2023, vol. 15, no. 4, pp. 5–13. DOI: 10.14529/mmph230401 (in Russian)
10. Dilman V.L. Linear Functional Equations With a Cyclic Set of Periodic Points of the Shift Function. *Bulletin of the South Ural State University. Series: Mathematics. Mechanics. Physics*, 2024, vol. 16, no. 2, pp. 5–11. DOI: 10.14529/mmph240201 (in Russian)
11. Kuczma M. *An Introduction to the Theory of Functional Equations and Inequalities, Cauchy's Equation and Jensen's Inequality*. Katowice, Panstwowe Wydawnictwo Naukowe, 1985.
12. Kravchenko V.G. On a Functional Equation with a Shift in the Space of Continuous Functions. *Mathematical Notes*, 1977, vol. 22, iss. 2, pp. 660–665.
13. Pelyuh G.P., Sharkovskiy A.N. *Method of Invariants in the Functional Equation Theory*. Kiyev, Inst. of Math. NAS, 2013. (in Russian)
14. Brodskii Y.S. *Functional Equations*. Kiyev, Visha shkola, 1983. (in Russian)
15. Ilolov M., Avezov R. on a Class of Linear Functional Equations with Constant Coefficients. *Academy of Sciences of the Republic of Tadjikistan*, 2019, vol. 177, no. 4, pp. 7–12. (in Russian)
16. Antonevich A.B. *Linear Functional Equations: An Operator Approach*. Basel, Boston, Berlin, Birkhauser, 1996.
17. Likhtarnikov L. *An Elementary Introduction to Functional Equations*. Saint Petersburg, Lan', 1997. (in Russian)
18. Cherniavsky V.P. Unambiguity of Solutions when Using a Linear Functional Equation in the Radiation Protection Model. *Global Nuclear Security*, 2019, no. 4(33), pp. 18–26. (in Russian)
19. Kryuk Y.E., Kunets I.E. Mathematical Modeling Methods for Optimizing Radiation Protection. *Bulletin of NTUHPI. Series: Computer Science and Modeling*, 2011, no. 36, pp. 95–100. (in Russian)

Valery L. Dilman, DSc (Physics and Math), Head of Department of Mathematical Analysis and Methods of Teaching Mathematics, South Ural State University (Chelyabinsk, Russian Federation), dilmanvl@susu.ru

Tatyana V. Karpeta, PhD (Physics and Math), Assistant Professor of Department of Applied Mathematics and Programming, South Ural State University, (Chelyabinsk, Russian Federation), karpetatv@susu.ru

Received April 10, 2024

УДК 517.965 DOI: 10.14529/jcem240202

НЕПРЕРЫВНЫЕ РЕШЕНИЯ ЛИНЕЙНЫХ ФУНКЦИОНАЛЬНЫХ УРАВНЕНИЙ НА КУСОЧНО-ГЛАДКИХ КРИВЫХ В МАТЕМАТИЧЕСКИХ МОДЕЛЯХ КРАЕВЫХ ЗАДАЧ СО СДВИГОМ

В. Л. Дильман, Т. В. Карпета

Рассматриваются линейные функциональные уравнения на произвольной кусочно-гладкой кривой. Такие уравнения изучаются в связи с теорией сингулярных интегральных уравнений как математического инструмента при исследовании математических моделей теории упругости, в которых условия сопряжения содержат сдвиг по границе. Такие уравнения возникают также при математическом моделировании процессов переноса заряженных частиц и ионизированных излучений. Функция сдвига, по предположению, действует циклически на множестве простых кривых, образующих заданную кривую. Целью работы является нахождение условий существования и мощности множества непрерывных решений таких уравнений в классах функций Гельдера и первообразных от интегрируемых по Лебегу функций с коэффициентами и правой частью уравнения из тех же классов. Полученные решения имеют вид сходящихся рядов и могут быть вычислены с любой степенью точности. Метод работы заключается в сведении данного уравнения к уравнению специального вида, в котором все периодические точки являются неподвижными, что позволяет воспользоваться результатами для случая простой гладкой кривой.

Ключевые слова: сингулярные интегральные уравнения со сдвигом; линейные функциональные уравнения от одной переменной; классы функций Гельдера; классы первообразных от интегрируемых по Лебегу функций.

Литература

1. Litvinchuk, G.S. Solvability Theory of Boundary Value Problems and Singular Integral Equations with Shift / G.S. Litvinchuk. – Dodrecht: Springer Science, Business Media, 2012. – 378 p.
2. Kravchenko, V.G. Introduction to the Theory of Singular Integral Operators with Shift / V.G. Kravchenko, G.S. Litvinchuk. – Dordrecht, Boston, London: Kluwer Academic Publishers, 1994.
3. Карлович, Ю.И. Теория Нётера сингулярных интегральных операторов со сдвигом / Ю.И. Карлович, В.Г. Кравченко, Г.С. Литвинчук // Изв. вузов. Математика. – 1983. – № 4. – С. 3–27.
4. Дильман, В.Л. О решениях интегрального уравнения с обобщенным логарифмическим ядром в Lp, p>1 / В.Л. Дильман, Л.И. Чибрикова // Изв. вузов. Математика. – 1986. – № 4. – С. 26–36.
5. Дильман, В.Л. Линейные функциональные уравнения в гельдеровых классах функций на простой гладкой кривой / В.Л. Дильман // Вестник ЮУрГУ. Серия: Математика, механика, физика. – 2020. – Т. 12, № 2. – С. 5–12.
6. Мусхелишвили, Н.И. Сингулярные интегральные уравнения / Н.И. Мусхелишвили. – М.: Наука, 1968.
7. Дильман, В.Л. Условия существования и единственности решений линейных функциональных уравнений в классах первообразных от лебеговских функций

на простой гладкой кривой / В.Л. Дильман, Д.А. Комиссарова // Вестник ЮУрГУ. Серия: Математика, механика, физика. – 2021. – Т. 13, № 4. – С. 13–23.

8. Дильман, В.Л. Линейные функциональные уравнения в классах первообразных от лебеговских функций на отрезках кривых / В.Л. Дильман, Д.А. Комиссарова // Челябинский физико-математический журнал. – 2023. – Т. 8, вып. 1. – С. 5–17.
9. Дильман, В.Л. Свойства и описание множеств решений линейных функциональных уравнений на простой гладкой кривой / В.Л. Дильман // Вестник ЮУрГУ. Серия: Математика. Механика. Физика. – 2023. – Т. 15, № 4. – С. 5–13.
10. Дильман, В.Л. Линейные функциональные уравнения с циклическим множеством периодических точек функции сдвига / В.Л. Дильман // Вестник ЮУрГУ. Серия: Математика. Механика. Физика. – 2024. – Т. 16, № 2. – С. 5–11.
11. Kuczma, M. An Introduction to the Theory of Functional Equations and Inequalities, Cauchy's Equation and Jensen's Inequality / M. Kuczma. – Katowice: Panstwowe Wydawnictwo Naukowe, 1985.
12. Кравченко, В.Г. Об одном функциональном уравнении со сдвигом в пространстве непрерывных функций / В.Г. Кравченко // Математические заметки. – 1977. – Т. 22, № 2. – С. 303–311.
13. Пелюх, Г.П. Метод инвариантов в теории функциональных уравнений / Г.П. Пелюх, А.Н. Шарковский. – Киев: Инст. мат. НАН, 2013.
14. Бродский, Я.С. Функциональные уравнения / Я.С. Бродский, А.К. Слипенко. – Киев: Высшая школа, 1983.
15. Илолов, М. Об одном классе линейных функциональных уравнений с постоянными коэффициентами / М. Илолов, Р. Авезов // Изв. Акад. наук Респ. Таджикистан. Отделение физико-математических, химических, геологических и технических наук. – 2019. – Т. 177, № 4. – С. 7–12.
16. Антоневич, А.Б. Линейные функциональные уравнения: операторный подход / А.Б. Антоневич. – Минск: Изд-во Университетское, 1988. – 232 с.
17. Лихтарников, Л.М. Элементарное введение в функциональные уравнения / Л.М. Лихтарников. – СПб: Лань, 1997.
18. Чернявский, В.П. Однозначность решений при использовании линейного функционального уравнения в модели радиационной защиты / В.П. Чернявский // Глобальная ядерная безопасность. – 2019. – № 4(33). – С. 18–26.
19. Крюк, Ю.Е. Математические метды моделирования в оптимизации радиационной защиты / Ю.Е. Крюк, И.Е. Кунец // Вестник НТУ ХПИ. Серия: Информатика и моделирование. – 2011. – № 36. – С. 95–100.

Дильман Валерий Лейзерович, доктор физико-математических наук, заведующий кафедрой математического анализа и методики преподавания математики, Южно-Уральский государственный университет (г. Челябинск, Российская Федерация), dilmanvl@susu.ru

Карпета Татаьяна Васильевна, кандидат физико-математических наук, доцент кафедры прикладной математики и программирования, Южно-Уральский государственный университет (г. Челябинск, Российская Федерация), karpetatv@susu.ru

Поступила в редакцию 10 апреля 2024 г.