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SOLUTION OF A SCREENED POISSON EQUATION WITH DIRICHLET BOUNDARY CONDITION WITH THE METHOD OF ITERATIVE EXTENSIONS

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In this paper, numerical solution of the screened Poisson equation with the Dirichlet boundary condition in two-dimensional and three-dimensional domains is proposed. The continuation of the boundary value problems is carried out, and then they are approximated by the finite element method. In the developed method of iterative extensions, solutions to extended problems after approximation are iteratively approximated by solutions of the proposed extended problems. This method has optimal asymptotics in terms of the number of operations.

Keywords: fictitious domain method, method of iterative extensions, screened harmonic systems, screened Poisson equation.

Introduction

An iterative method of fictitious domains was proposed for the first time in the work [1] with optimal asymptotics in terms of the number of operations. This method approximated solution of the second-order elliptic boundary value problem only with the Neumann boundary condition. For elliptic boundary value problems in domains with complex geometry, under the Dirichlet boundary condition, numerical methods with logarithmically optimal asymptotics are known, although theoretically there can also be asymptotically optimal asymptotics, i.e. unimprovable asymptotics with respect to the number of operations [2]. When developing numerical methods for solving these problems in domains with complex geometry, they are reduced to problems in rectangular domains for which asymptotically optimal marching methods are known [3]. A methodology of fictitious components for solving second-order elliptic boundary value problems in the presence of a Dirichlet boundary condition was proposed, studied and optimized in the works [4, 5, 6], where they sought to obtain results with optimal asymptotics for elliptic problems with the Neumann boundary condition. Numerical fictitious space iterative method for solving the Dirichlet boundary value problem for a second-order elliptic equation in a domain with complex geometry was proposed in [7], which has optimal asymptotics in terms of computational costs. It can be noted that this method was not further developed, for example, in works [8, 9, 10, 11] for fourth-order elliptic problems.

This work devotes to the development of a numerical method of iterative extensions for an asymptotically optimal, theoretically and practically simple, universal application to the approximate solution of elliptic boundary value problems with the obligatory presence of the Dirichlet boundary condition using the example of screened Poisson equations.

1. Screened Poisson Equation with Dirichlet Boundary Condition

The subject of research and development in this work will be the method of iterative extensions for an approximated numerical solution of screened Poisson equations in bounded domains under mixed boundary conditions and the mandatory presence of the Dirichlet boundary condition:

$$u\colon\ -\Delta\check{u}+\kappa\check{u}=\check{f}|_G,\ G\subset\mathbb{R}^2,\mathbb{R}^3,\check{u}|_{\gamma_1}=0,\frac{\partial\check{u}}{\partial n}|_{\gamma_2}=0. \tag{1}$$

If boundary of domains consists of

$$\partial G=\overline{s},\ s=\gamma_1\cup\gamma_2,\ \gamma_1\cap\gamma_2=\emptyset,$$

and functions $\check{f}\in L_2(G)$, coefficients $\kappa\in(0;+\infty)$, bounded domains G, outer normals n to ∂G are specified.

Let us present boundary value problems as linear functionals in the form of scalar products

$$\check{u}\in\check{H}\colon\langle\check{u},\check{v}\rangle=F(\check{v})\ \forall\check{v}\in\check{H},\ F\in\check{H}', \tag{2}$$

where the functions spaces are Sobolev spaces

$$\check{H}=\check{H}(G)=\left\{\check{v}\in W_2^1(G)\colon\check{v}|_{\gamma_1}=0\right\},$$

if scalar products are considered

$$\langle\check{u},\check{v}\rangle=K(\check{u},\check{v})=\int\limits_G(\check{u}_x\check{v}_x+\check{u}_y\check{v}_y+\kappa\check{u}\check{v})dG,$$

or

$$\langle\check{u},\check{v}\rangle=K(\check{u},\check{v})=\int\limits_G(\check{u}_x\check{v}_x+\check{u}_y\check{v}_y+\check{u}_z\check{v}_z+\kappa\check{u}\check{v})dG,$$

if functions $\check{f}$ are specified, then linear functionals are considered

$$F(\check{v})=(\check{u},\check{v})=\int\limits_G\check{f}\check{v}dG.$$

Let us consider these problems in variational form with the $\omega=1$, and also introduce fictitous homogeneous problems, when $\omega=\mathrm{II}$

$$\check{u}_\omega\in\check{H}_\omega\colon K_\omega(\check{u}_\omega,\check{v}_\omega)=F_\omega(\check{v}_\omega)\ \forall\check{v}_\omega\in\check{H}_\omega,\ \check{F}_\omega\in\check{H}'_\omega,\omega\in\{1,\mathrm{II}\},G_\omega\subset\mathbb{R}^2,\mathbb{R}^3, \tag{3}$$

if functions $\check{f}_1\in L_2(G_1)$, then right sides of the problems are

$$F_\omega(\check{v}_\omega)=\int\limits_{G_\omega}\check{f}_\omega\check{v}_\omega dG_\omega\ \forall\check{v}_\omega\in\check{H}_\omega,\check{f}_{\mathrm{II}}=0,$$

if Sobolev spaces are

$$\check{H}_\omega=\check{H}_\omega(G_\omega)=\left\{\check{v}_\omega\in W_2^1(G_\omega)\colon\check{v}_\omega|_{\gamma_{\omega,1}}=0\right\},$$

where bounded domains G_ω with boundaries

$$\partial G_\omega = \overline{s}_\omega,\ \overline{s} = \gamma_{\omega,1} \cup \gamma_{\omega,2},\ \gamma_{\omega,i} \cap \gamma_{\omega,j} = \emptyset, i \neq j,\ i,j = 1,2,$$

and the scalar products are

$$K_\omega(\check{u}_\omega, \check{v}_\omega) = \int\limits_{G_\omega} (\check{u}_\omega x \check{v}_\omega x + \check{u}_\omega y \check{v}_\omega y + \kappa_\omega \check{u}_\omega \check{v}_\omega) dG_\omega,$$

or

$$K_\omega(\check{u}_\omega, \check{v}_\omega) = \int\limits_{G_\omega} (\check{u}_\omega x \check{v}_\omega x + \check{u}_\omega y \check{v}_\omega y + \check{u}_\omega z \check{v}_\omega z + \kappa_\omega \check{u}_\omega \check{v}_\omega) dG_\omega,$$

and coefficients $\kappa_\omega \in (0; +\infty)$. If $\omega = 1$, then $\kappa_1 \geq 0, \gamma_{1,1} \neq \emptyset$ in these problems, and when $\omega = \text{II}$, then $\kappa_{\text{II}} \geq 0$ in fictitous problems.

Let us consider the continued problem in variational form

$$\check{u} \in \check{V} \colon K_1(\check{u}, I_1 \check{v}) + K_{\text{II}}(\check{u}, \check{v}) = F_1(I_1 \check{v})\ \forall \check{v} \in \check{V}, \tag{4}$$

where the solution of the continued problem (4) belongs to extended space of solutions, which is the Sobolev space of the following form

$$\check{V} = \check{V}(\Pi) = \left\{ \check{v} \in W_2^1(\Pi) \colon \check{v}|_{\gamma_1} = 0 \right\}.$$

We assume that the given and selected domains G_1, G_{II} satisfy the properties $\overline{G}_1 \cup \overline{G}_{\text{II}} = \overline{\Pi},\ \overline{G}_1 \cap \overline{G}_{\text{II}} = \emptyset$, boundaries of domains Π are also the closures of unions of open, disjoint parts

$$\partial \Pi = \overline{s},\ s = \gamma_1 \cup \gamma_2,\ \gamma_i \cap \gamma_j = \emptyset,\ i \neq j,\ i,j = 1,2.$$

We believe that the intersections of the boundaries of the first and second domains will be non-empty sets, being the closures of the intersections of the corresponding parts at the boundaries of the corresponding domains, i.e.

$$\partial G_1 \cap \partial G_{\text{II}} = \overline{S}, S = \gamma_{1,1} \cap \gamma_{\text{II},2} \neq \emptyset.$$

Subspaces of solutions of the continued problems will contain in the extended solution spaces in the following form

$$\check{V}_1 = \check{V}_1(\Pi) = \left\{ \check{v}_1 \in \check{V} \colon \check{v}_1|_{\Pi \setminus G_1} = 0 \right\}.$$

In the formulation of continued problems we use all possible arbitrary projection operators, for example, but not necessarily, orthogonal projection operators from extended solution spaces, necessarily onto all corresponding subspaces of solutions for the corresponding continued problems

$$I_1 \colon \check{V} \mapsto \check{V}_1,\ \check{V}_1 = \text{im} I_1,\ I_1 = I_1^2.$$

2. Method of Iterative Extensions

Let us consider the continued variational problems on finite subspaces, using approximation of Sobolev spaces. In two-dimensional case, if

$$\Pi = (0; b_1) \times (0; b_2),\ \gamma_1 = \{b_1\} \times (0; b_2) \cup (0; b_1) \times \{b_2\},$$

$$\gamma_2 = \{0\} \times (0; b_2) \cup (0; b_1) \times \{0\},\ b_1, b_2 \in (0; +\infty).$$

We consider a grid with nodes in specified rectangular domain Π

$$(x_i; y_j) = \big((i-1,5)h_1; (j-1,5)h_2\big),$$

$$h_1 = b_1/(m-1,5),\ h_2 = b_2/(n-1,5),\ i = 1, 2, \ldots, m,\ j = 1, 2, \ldots, n,\ m-2, n-2 \in \mathbb{N}.$$

Let us consider grid functions on the set of nodes of the grid

$$v_{i,j} = v(x_i; y_j) \in \mathbb{R},\ i = 1, 2, \ldots, m,\ j = 1, 2, \ldots, n,\ m-2, n-2 \in \mathbb{N}.$$

We define the completion of introduced grid functions, using piecewise linear functions

$$\Psi^{i,j}(x; y) = \phi^{1,i}(x)\phi^{2,j}(y),\ i = 2, \ldots, m-1\ , j = 2, \ldots, n-1\ , m-2, n-2 \in \mathbb{N},$$

$$\phi^{1,i}(x) = [2/i]\ \phi(x/h_1 - i + 3,5) + \phi(x/h_1 - i + 2,5),$$
$$\phi^{2,j}(y) = [2/j]\ \phi(y/h_2 - j + 3,5) + \phi(y/h_2 - j + 2,5),$$
$$\phi(t) = \begin{cases} t, & t \in [0; 1], \\ 2 - t, & t \in [1; 2], \\ 0, & t \notin (0; 2). \end{cases}$$

We equate to zero basis functions outside of rectangular domain Π

$$\Psi^{i,j}(x; y) = 0, (x; y) \notin \Pi,\ i = 2, \ldots, m-1\ , j = 2, \ldots, n-1\ , m-2, n-2 \in \mathbb{N}.$$

Sets of linear combinations of the specified basis functions are finite-dimensional continuous subspaces of the extended continuous solution spaces

$$\hat{V} = \left\{ \hat{v} = \sum_{i=2}^{m-1} \sum_{j=2}^{n-1} v_{i,j} \Psi^{i,j}(x; y) \right\} \subset \check{V}.$$

In three-dimensional case, when

$$\Pi = (0; b_1) \times (0; b_2) \times (0; b_3),$$

$$\gamma_1 = \{b_1\} \times (0; b_2) \times (0; b_3) \cup (0; b_1) \times \{b_2\} \times (0; b_3) \cup (0; b_1) \times (0; b_2) \times \{b_3\},$$
$$\gamma_2 = \{0\} \times (0; b_2) \times (0; b_3) \cup (0; b_1) \times \{0\} \times (0; b_3) \cup (0; b_1) \times (0; b_2) \times \{0\},$$
$$b_1, b_2, b_3 \in (0; +\infty).$$

Let us introduce grid with nodes in the domain of the rectangular parallelepiped Π

$$(x_i; y_j; z_p) = \big((i-1,5)h_1; (j-1,5)h_2; (p-1,5)h_3\big),$$

$$h_1 = b_1/(m-1,5),\ h_2 = b_2/(n-1,5),\ h_3 = b_3/(K-1,5),$$
$$i = 1,2,\dots,m,\ j = 1,2,\dots,n,\ p = 1,2,\dots,K,\ m-2, n-2, K-2 \in \mathbb{N}.$$

Let us consider grid functions on the set of nodes of the grid

$$v_{i,j,p} = v(x_i; y_j; z_p) \in \mathbb{R},\ i = 1,2,\dots,m,\ j = 1,2,\dots,n,$$
$$m-2, n-2, K-2 \in \mathbb{N}.$$

We define the completion of introduced grid functions, using piecewise linear functions

$$\Psi^{i,j,p}(x;y;z) = \phi^{1,i}(x)\phi^{2,j}(y)\phi^{3,p}(z),$$

$$i = 1,2,\dots,m,\ j = 1,2,\dots,n,\ p = 1,2,\dots,K,\ m-2, n-2, K-2 \in \mathbb{N},$$
$$\phi^{1,i}(x) = [2/i]\ \phi(x/h_1 - i + 3,5) + \phi(x/h_1 - i + 2,5),$$
$$\phi^{2,j}(y) = [2/j]\ \phi(y/h_2 - j + 3,5) + \phi(y/h_2 - j + 2,5),$$
$$\phi^{3,p}(z) = [2/p]\ \phi(z/h_3 - p + 3,5) + \phi(y/h_2 - j + 2,5).$$

We equate to zero basis functions outside of the domain of the rectangular parallelepiped Π

$$\Psi^{i,j,p}(x;y;z) = 0, (x;y;z) \notin \Pi,$$
$$i = 1,2,\dots,m,\ j = 1,2,\dots,n,\ p = 1,2,\dots,K,\ m-2, n-2, K-2 \in \mathbb{N}.$$

Sets of linear combinations of the specified basis functions are finite-dimensional continuous subspaces of the extended continuous solution spaces

$$\hat{V} = \left\{ \hat{v} = \sum_{i=2}^{m-1} \sum_{j=2}^{n-1} \sum_{p=2}^{K-1} v_{i,j,p} \Psi^{i,j,p}(x;y;z) \right\} \subset \check{V}.$$

Reduction of the introduced continued problems using the given approximation leads to linear systems of algebraic equations, i.e. problems of the following matrix form

$$\overline{u} \in \mathbb{R}^N \colon C\overline{u} = \overline{f},\ \overline{f} \in \mathbb{R}^N. \tag{5}$$

Now let us specify the choice of projection operators I_1, i.e. these operators set to zero the coefficients of the basis functions whose supports do not contain in the closure of the first domains. With such reduction of the introduced continued problems, they are obtained in matrix form, if we assume that the extended matrices, and the right-hand sides of the approximation satisfy the following equalities

$$[C\overline{u}, \overline{v}] = K_1(\hat{u}, I_1\hat{v}) + K_{\Pi}(\hat{u}, \hat{v})\ \forall \hat{u}, \hat{v} \in \hat{V},\ [\overline{f}, \overline{v}] = F_1(I_1\hat{v})\ \forall \hat{v} \in \hat{V},$$

$$[\overline{f}, \overline{v}] = (\overline{f}, \overline{v})h_1h_2 = \overline{f}\overline{v}h_1h_2,\ \overline{v} = (v_1, v_2, \dots, v_N)' \in \mathbb{R}^N,\ N = (m-2)(n-2).$$

When we enumerate the coefficients of the basis functions, we distinguish them in three blocks. In the first block, we include coefficients of the basis functions, whose supports contain in the closure of the first domains. In the third block, we include coefficients of the basis functions, whose supports contain in the closure of the second domains. In the second block, we include coefficients of the basis functions, that are not yet enumerated. With this numbering, the resulting vectors have the following structure.

$$\overline{v} = (\overline{v}_1', \overline{v}_2', \overline{v}_3')', \ \overline{u} = (\overline{u}_1', \overline{0}', \overline{0}')', \ \overline{f} = (\overline{f}_1', \overline{0}', \overline{0}')'.$$

After the numbering we have matrices in the following form

$$C = \begin{bmatrix} K_{11} & K_{12} & 0 \\ 0 & K_{02} & K_{23} \\ 0 & K_{32} & K_{33} \end{bmatrix}.$$

As we consider this numbering, we also additionally define the matrices, using the previously introduced scalar products

$$[K_{\mathrm{I}}\overline{u}, \overline{v}] = K_1(\hat{u}, \hat{v}), \ [K_{\mathrm{II}}\overline{u}, \overline{v}] = K_{\mathrm{II}}(\hat{u}, \hat{v}) \ \forall \hat{u}, \hat{v} \in \hat{V}.$$

These matrices have the following form

$$K_{\mathrm{I}} = \begin{bmatrix} K_{11} & K_{12} & 0 \\ K_{21} & K_{20} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ K_{\mathrm{II}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & K_{02} & K_{23} \\ 0 & K_{32} & K_{33} \end{bmatrix}.$$

Let us introduce discrete subspaces corresponding to the continued subspaces of solutions in vector form

$$\overline{V}_1 = \left\{ \overline{v} = (\overline{v}_1', \overline{v}_2', \overline{v}_3')' \in \mathbb{R}^N : \overline{v}_2 = \overline{0}, \overline{v}_3 = \overline{0} \right\}.$$

Then, additionally, using these matrices, we define the following subspaces in vector form

$$\overline{V}_2 = \left\{ \overline{v} = (\overline{v}_1', \overline{v}_2', \overline{v}_3')' \in \mathbb{R}^N : K_{11}\overline{v}_1 + K_{12}\overline{v}_2 = \overline{0}, K_{32}\overline{v}_2 + K_{33}\overline{v}_3 = \overline{0} \right\}.$$

To find approximate solutions to the problem (5) we use the method of iterative extensions. Let us introduce extended matrices, defined as the sum of the first and second matrices multiplied by additionally parameters

$$B = K_{\mathrm{I}} + \beta K_{\mathrm{II}},$$

$$\begin{bmatrix} B_{11} & B_{12} & 0 \\ B_{21} & B_{22} & B_{23} \\ 0 & B_{32} & B_{33} \end{bmatrix} = \begin{bmatrix} K_{11} & K_{12} & 0 \\ K_{21} & K_{20} & 0 \\ 0 & 0 & 0 \end{bmatrix} + \beta \begin{bmatrix} 0 & 0 & 0 \\ 0 & K_{02} & K_{23} \\ 0 & K_{32} & K_{33} \end{bmatrix}, \beta \in (0; +\infty).$$

Let us present condition on continuations of functions that will be sufficient to construct a further convergent process in the method of iterative extensions

$$\exists \beta_1 \in (0; +\infty), \beta_2 \in [\beta_1; +\infty) \colon \beta_1^2 [B\overline{v}_2, B\overline{v}_2] \le [K_{\mathrm{II}}\overline{v}_2, K_{\mathrm{II}}\overline{v}_2] \le \beta_2^2 [B\overline{v}_2, B\overline{v}_2] \ \forall \overline{v}_2 \in \overline{V}_2,$$

$$\exists \alpha \in (0; +\infty) \colon [K_{\mathrm{I}}\overline{v}_2, K_{\mathrm{I}}\overline{v}_2] \le \alpha^2 [K_{\mathrm{II}}\overline{v}_2, K_{\mathrm{II}}\overline{v}_2] \ \forall \overline{v}_2 \in \overline{V}_2.$$

The method of iterative extensions can be interpreted as a generalization of the method of fictitious components when introducing additional non-unit parameters in the definition of extended matrices, as well as when iterative parameters are selected using error minimization in a stronger norm than energy norm of the emerging problem, i.e. we use minimum residual method instead of method of steepest descent.

$$\overline{u}^k \in \mathbb{R}^N\colon B(\overline{u}^k - \overline{u}^{k-1}) = -\tau_{k-1}(C\overline{u}^{k-1} - \overline{f}),\ k \in \mathbb{N}, \tag{6}$$
$$\forall \overline{u}^0 \in \overline{V}_1,\ \beta > \alpha,\ \tau_0 = 1,\ \tau_{k-1} = [\overline{r}^{k-1}, \overline{\eta}^{k-1}]/[\overline{\eta}^{k-1}, \overline{\eta}^{k-1}],\ k \in \mathbb{N}\backslash\{1\},$$

where in the iterative process it is necessary to recalculate residuals, corrections and equivalent residuals step by step

$$\overline{r}^{k-1} = C\overline{u}^{k-1} - \overline{f},\ \overline{w}^{k-1} = B^{-1}\overline{r}^{k-1},\ \overline{\eta}^{k-1} = C\overline{w}^{k-1},\ k \in \mathbb{N}.$$

Let us introduce norms stronger than energy norms in emerging problems at each step of the iterative process in the method of iterative extensions

$$\|\overline{v}\|_{B^2} = \sqrt{[B^2\overline{v}, \overline{v}]}\ \forall \overline{v} \in \mathbb{R}^N.$$

Theorem 1. *In the developed method of iterative extensions* (6)*, convergence estimations is*

$$\|\overline{u}^k - \overline{u}\|_{B^2} \leq \varepsilon \|\overline{u}^0 - \overline{u}\|_{B^2},\ \varepsilon = 2(\beta_2/\beta_1)(\alpha/\beta)^{k-1},\ k \in \mathbb{N}.$$

Similar results were obtained in particular, similar cases in works [12, 13].

Let us write down the implementation of the method of iterative extensions in the form of an algorithm for an approximate solution of the emerged problems after the approximation applied to them, and then their fictitious continuation. The choice of iteration parameters is based on the method of minimal residuals.

1. Choose arbitrary initial approximations from the subspaces of solutions that approximate the subspaces of solutions of continued problems, and unit initial iteration parameter
$$\forall \overline{u}^0 \in \overline{V}_1, \tau_0 = 1.$$
2. Calculate residual
$$\overline{r}^{k-1} = C\overline{u}^{k-1} - \overline{f},\ k \in \mathbb{N}.$$
3. Find the norms for absolute errors
$$e_{k-1} = [\overline{r}^{k-1}, \overline{r}^{k-1}],\ k \in \mathbb{N}.$$
4. Find corrections
$$\overline{w}^{k-1}\colon B\overline{w}^{k-1} = \overline{r}^{k-1},\ k \in \mathbb{N}.$$
5. Find equivalent residuals
$$\overline{\eta}^{k-1} = C\overline{w}^{k-1},\ k \in \mathbb{N}\backslash\{1\}.$$
6. Find optimal iterative parameters
$$\tau_{k-1} = [\overline{r}^{k-1}, \overline{\eta}^{k-1}]/[\overline{\eta}^{k-1}, \overline{\eta}^{k-1}],\ k \in \mathbb{N}\backslash\{1\}.$$
7. Find new approximation
$$\overline{u}^k = \overline{u}^{k-1} - \tau_{k-1}\overline{w}^{k-1},\ k \in \mathbb{N}.$$
8. Check the stop criterion of iterative processes based on predetermined estimates of permissible relative errors
$$e_{k-1} \leq e^2 e_0,\ k \in \mathbb{N}\backslash\{1\},\ e \in (0; 1).$$

3. Computational Experiments

Example 1. We consider domains

$$G_{\mathrm{I}} = (0; 2.5) \times (0; 2.5) \backslash [1.5; 2.5) \times [1.5; 2.5),$$

$$G_{\mathrm{II}} = (1.5; 2.5) \times (1.5; 2.5), \Pi = (0; 2.5) \times (0; 2.5),$$

with boundaries

$$\gamma_{1,1} = \{2.5\} \times (0; 1.5) \cup (0; 1.5) \times \{2.5\} \cup \{1.5\} \times (1.5; 2.5) \cup (1.5; 2.5) \times \{1.5\},$$

$$\gamma_{1,2} = \{0\} \times (0; 2.5) \cup (0; 2.5) \times \{0\}, \gamma_{\mathrm{II},1} = \{2.5\} \times (1.5; 2.5) \cup (1.5; 2.5) \times \{2.5\},$$

$$\gamma_{\mathrm{II},2} = \{1.5\} \times (1.5; 2.5) \cup (1.5; 2.5) \times \{1.5\}, \gamma_1 = \{2.5\} \times (0; 2.5) \cup (0; 2.5) \times \{2.5\},$$

$$\gamma_2 = \{0\} \times (0; 2.5) \cup (0; 2.5) \times \{0\}.$$

We consider function with $\kappa_1 = 1$

$$\check{f}_1 = ((392 - 384x)(64y^3 - 196y^2 + 225) + (64x^3 - 196x^2 + 225)(392 - 384y))/184^2 +$$

$$+(64x^3 - 196x^2 + 225)(64y^3 - 196y^2 + 225)/184^2,$$

$$\check{u}_1 = (64x^3 - 196x^2 + 225)(64y^3 - 196y^2 + 225)/184^2.$$

Select $e = 0.00001$, $n = 254$, initial zero approximation, then in the method of iterative extensions, the iterative process terminates at the second iteration. Fig. 1 shows the last approximation and the solution.

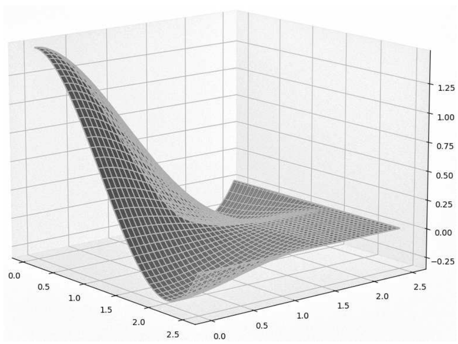

Fig. 1. Solution and last approximation

Value of the maximum error is $\dfrac{\max |u_{i,j} - \check{u}_{i,j}|}{\max |\check{u}_{i,j}|} \leq 0.00002.$

Example 2. We consider domains

$$G_{\mathrm{I}} = (0; 2.5) \times (0; 2.5) \times (0; 2.5) \backslash [1.5; 2.5) \times [1.5; 2.5) \times [1.5; 2.5),$$

$$G_{\mathrm{II}} = (1.5; 2.5) \times (1.5; 2.5) \times (1.5; 2.5), \Pi = (0; 2.5) \times (0; 2.5) \times (0; 2.5).$$

Boundaries consist of closures of parts:

$$\gamma_{1,1} = \{2.5\} \times \Big((0;2.5) \times (0;2.5) \backslash [1.5;2.5) \times [1.5;2.5)\Big) \cup$$

$$\cup \Big((0;2.5) \times \{2.5\} \times (0;2.5)\Big) \backslash \Big([1.5;2.5) \times \{2.5\} \times [1.5;2.5)\Big)$$

$$\cup \Big((0;2.5) \times (0;2.5) \backslash [1.5;2.5) \times [1.5;2.5)\Big) \times \{2.5\} \cup \{1.5\} \times (1.5;2.5) \times (1.5;2.5) \cup$$

$$\cup (1.5;2.5) \times \{1.5\} \times (1.5;2.5) \cup (1.5;2.5) \times (1.5;2.5) \times \{1.5\},$$

$$\gamma_{1,2} = \{0\} \times (0;2.5) \times (0;2.5) \cup (0;2.5) \times \{0\} \times (0;2.5) \cup (0;2.5) \times (0;2.5) \times \{0\},$$

$$\gamma_{II,1} = \{2.5\}\times(1.5;2.5)\times(1.5;2.5)\cup(1.5;2.5)\times\{2.5\}\times(1.5;2.5)\cup(1.5;2.5)\times(1.5;2.5)\times\{2.5\},$$

$$\gamma_{II,2} = \{1.5\}\times(1.5;2.5)\times(1.5;2.5)\cup(1.5;2.5)\times\{1.5\}\times(1.5;2.5)\cup(1.5;2.5)\times(1.5;2.5)\times\{1.5\},$$

$$\gamma_1 = \{2.5\} \times (0;2.5) \times (0;2.5) \cup (0;2.5) \times \{2.5\} \times (0;2.5) \cup (0;2.5) \times (0;2.5) \times \{2.5\},$$

$$\gamma_2 = \{0\} \times (0;2.5) \times (0;2.5) \cup (0;2.5) \times \{0\} \times (0;2.5) \cup (0;2.5) \times (0;2.5) \times \{0\}.$$

We consider function with $\kappa_1 = 1$

$$\check{f}_1 = ((392 - 384x)(64y^3 - 196y^2 + 225)(64z^3 - 196z^2 + 225) +$$

$$+(64x^3 - 196x^2 + 225)(392 - 384y)(64z^3 - 196z^2 + 225) +$$

$$+(64x^3 - 196x^2 + 225)(64y^3 - 196y^2 + 225)(392 - 384z))/225^3,$$

$$\check{u}_1 = (64x^3 - 196x^2 + 225)(64y^3 - 196y^2 + 225)(64z^3 - 196z^2 + 225)/225^3.$$

Select $e = 0.00001$, $n = 79$, initial zero approximation, then in the method of iterative extensions, the iterative process terminates at the second iteration. On Fig. 2 shown the last approximation and the solution on projection of $u_{i,j,p}$ with $p = (4n - 1)/5$.

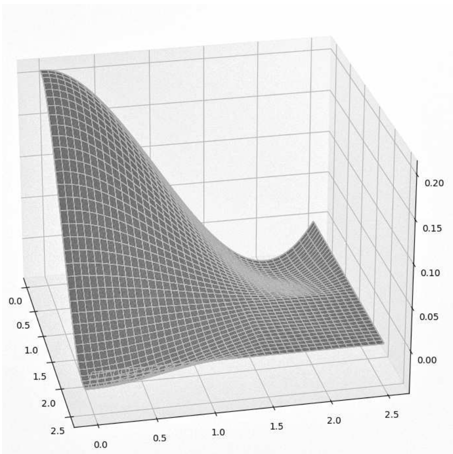

Fig. 2. Solution and last approximation

Value of the maximum error is $\dfrac{\max |u_{i,j,p} - \check{u}_{i,j,p}|}{\max |\check{u}_{i,j,p}|} \leq 0.0002.$

The proposed algorithm of iterative extensions include automation of control for calculating optimal iterative parameters based on iterative processing of information before stopping the iterative processes according to the given criterion, i.e. when performing predefined error estimates. Error minimization in the method of iterative extensions uses stronger norms than energy norms in emerging problems iteratively. The developed method of iterative extensions has unimprovable asymptotics, i.e. optimal asymptotics in terms of the number of required operations, and allows simple, efficient and universal implementation in comparison with the fictitious space method [7].

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РЕШЕНИЕ ЭКРАНИРОВАННЫХ УРАВНЕНИЙ ПУАССОНА С УСЛОВИЕМ ДИРИХЛЕ МЕТОДОМ ИТЕРАЦИОННЫХ РАСШИРЕНИЙ

М. П. Еремчук, А. Л. Ушаков

Предлагается численное решение экранированного уравнения Пуассона с условием Дирихле в двумерных и трехмерных областях. Производится продолжение решаемых краевых задач, а затем их аппроксимация методом конечных элементов. В развиваемом методе итерационных расширений решения продолженных задач после аппроксимации итерационно приближаются решениями предлагаемых расширенных задач. Этот метод для решения исходных задач имеет оптимальную асимптотику по количеству операций.

Ключевые слова: экранированное уравнение Пуассона; метод итерационных расширений.

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