COMPUTATIONAL MATHEMATICS

MSC 35G15, 65N30

INFORMATION PROCESSING IN A NUMERICAL STUDY FOR SOME STOCHASTIC WENTZELL SYSTEMS OF THE HYDRODYNAMIC EQUATIONS IN A BALL AND ON ITS BOUNDARY

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In this paper we study stochastic Wentzell systems: filtration equations describing fluid filtration processes in a fractured porous medium in a three-dimensional ball and on its boundary; free filtration equations describing the evolution of the free surface of the filtered fluid in a three-dimensional ball and on its boundary. In particular, numerical solutions of the Cauchy problem are constructed for the above systems of Wentzell equations and a description of the processing of the results of n experiments at different values of a random variable having a standard normal distribution is given (confidence intervals according to the rule of three sigma are constructed for the obtained cross sections of the stochastic process describing quantitative changes in the geochemical regime of groundwater under non-pressure filtration and quantitative changes in free fluid filtration).

Keywords: stochastic filtration equation; stochastic free filtration equation; Wentzell system of equations; information processing; three sigma rule; Nelson – Glicklich derivative.

Introduction

Let $\Omega = \{(r,\theta,\varphi) : r \in [0,R), \theta \in [0,\pi], \varphi \in [0,2\pi)\}$ be a three-dimensional ball in $\mathbb{R}^3$ with boundary $\Gamma = \{(r,\theta,\varphi) : r = R, \theta \in [0,\pi], \varphi \in [0,2\pi)\}$. For the sake of simplicity, we introduce real separable Hilbert spaces $\mathfrak{U} = \{u \in W_2^2(\Omega) \oplus W_2^2(\Gamma) : \partial_R u = 0\}$, $\mathfrak{F} = L_2(\Omega) \oplus L_2(\Gamma)$ and construct the spaces of *random* $\mathbf{K}$*-values. The random* $\mathbf{K}$*-values* $\eta, \kappa \in \mathbf{U_K L}_2$ have the following form

$$\xi = \sum_{j=1}^{\infty} \lambda_j \xi_j \phi_j, \quad \chi = \sum_{j=1}^{\infty} \lambda_j \chi_j \psi_j, \tag{1}$$

where $\{\phi_k\}$ is the family of eigenfunctions of the modified Laplace – Beltrami operator $\Delta_{r,\theta,\varphi} \in \mathcal{L}(\mathfrak{U};\mathfrak{F})$ orthonormalized in the sense of the scalar product $\langle \cdot, \cdot \rangle$ of $L_2(\Omega)$; $\{\psi_k\}$ is the family of eigenfunctions of the modified Laplace – Beltrami operator $\Delta_{\theta,\varphi} \in \mathcal{L}(\mathfrak{U};\mathfrak{F})$ orthonormalized in the sense of the scalar product $\langle \cdot, \cdot \rangle$ of $L_2(\Omega)$. Let us consider the linear stochastic Wentzell system of the fluid filtration equations (see, e.g., [1]) in a ball and on its boundary

$$(\lambda - \Delta_{r,\theta,\varphi}) \overset{\circ}{\xi} (t) = \alpha \Delta_{r,\theta,\varphi} \xi + \beta \xi, \quad \xi \in C^\infty(\mathbb{R}_+; \mathbf{U_K L}_2), \tag{2}$$

$$(\lambda - \Delta_{\theta,\varphi}) \overset{\circ}{\chi} (t) = \gamma \Delta_{\theta,\varphi} \chi + \partial_R \xi + \delta \chi, \ \chi \in C^{\infty}(\mathbb{R}_+; \mathbf{U_K L}_2), \tag{3}$$

where

$$\Delta_{r,\theta,\varphi} = (r-R)\frac{\partial}{\partial r}\left((R-r)\frac{\partial}{\partial r}\right) + \frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial \varphi^2},$$
$$\Delta_{\theta,\varphi} = \frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial \varphi^2}, \quad \partial_R = \frac{\partial}{\partial r}\bigg|_{r=R}.$$

Here, the symbol $\partial_R = \nu(t, r, \theta, \varphi)$, $(t, r, \theta, \varphi) \in \mathbb{R}_+ \times \Gamma$, denotes the external normal to $\mathbb{R}_+ \times \Omega$; the symbols $\overset{\circ}{\xi} (t)$ and $\overset{\circ}{\chi} (t)$ denote the Nelson – Glicklich derivative for the corresponding stochastic process. The parameters $\alpha, \gamma, \lambda, \beta, \delta \in \mathbb{R}$ characterize the medium. Let us add to this system (2)–(3) a matching condition, which guarantees the uniqueness of the obtained solution (see, e.g., [2]) and equip it with initial conditions

$$\xi(0) = \xi_0, \ \chi(0) = \chi_0. \tag{4}$$

Let us call the solution of the problem (2)–(4) the solution of the Cauchy problem for the stochastic Wentzell system of filtration equations.

In addition, on the $\Omega \cup \Gamma$ compact, we consider the Cauchy problem for a system of two Dzekzer [3], equations modeling the evolution of the free surface of a filtering fluid,

$$(\lambda - \Delta_{r,\theta,\varphi}) \overset{\circ}{\xi} (t) = \alpha_0 \Delta_{r,\theta,\varphi} \xi - \beta_0 \Delta^2_{r,\theta,\varphi} \xi - \gamma_0 \xi, \ \xi \in C^{\infty}(\mathbb{R}_+; \mathbf{U_K L}_2), \tag{5}$$

$$(\lambda - \Delta_{\theta,\varphi}) \overset{\circ}{\chi} (t) = \alpha_1 \Delta_{\theta,\varphi} \chi - \beta_1 \Delta^2_{\theta,\varphi} \chi + \partial_R \eta - \gamma_1 \chi, \ \chi \in C^{\infty}(\mathbb{R}_+; \mathbf{U_K L}_2), \tag{6}$$

$$\xi(0) = \xi_0, \ \chi(0) = \chi_0 \tag{7}$$

subject to the matching condition

$$tr\ u = v \text{ on } \mathbb{R}_+ \times \Gamma. \tag{8}$$

Here, the symbol $\partial_R = \nu(t, r, \theta, \varphi)$, $(t, r, \theta, \varphi) \in \mathbb{R}_+ \times \Gamma$, denotes the external normal to $\mathbb{R}_+ \times \Omega$; the symbols $\overset{\circ}{\xi} (t)$ и $\overset{\circ}{\chi} (t)$ denote the Nelson – Glicklich derivative for the corresponding stochastic process. The parameters $\alpha_0, \alpha_1, \lambda, \beta_0, \beta_1, \gamma_0, \gamma_1 \in \mathbb{R}$ characterize the medium. The solution of the problem (5)–(8) is called the solution of the Cauchy problem for the stochastic Wentzell system of free filtration equations.

The paper, in addition to the introduction and the list of references, consists of four parts. The first part considers the existence and uniqueness of the stochastic Wentzell system of fluid filtration equations in a three-dimensional ball and on its boundary. The second part contains an algorithm for numerical investigation and processing of information obtained from «artificial» experiment for the stochastic Wentzell system of the Barenblatt – Zheltov – Kochina equations. The third part deals with the existence and uniqueness of the stochastic Wentzell system of free fluid filtration equations in a three-dimensional ball and on its boundary. The fourth part contains an algorithm for numerical investigation and processing of information obtained from «artificial» experiment for the stochastic Wentzell system of Dzekzer equations.

1. Analytical Solution for the Stochastic Wentzell System of Fluid Filtration Equations

We give an analytical study for the corresponding system (2)–(4) following the results in [4].

In this case, consider the following series

$$\begin{aligned}\xi = \sum_{k=2}^{\infty} \exp\left(t\frac{\beta-\alpha k^2}{\lambda+k^2}\right)\frac{(R-r)^k}{R^k}\Big(a_k \sin(k\theta)(\sin k\varphi+\cos k\varphi)+\\ + b_k\cos(k\theta)(\sin k\varphi+\cos k\varphi)\Big)+\\ +\sum_{k=1}^{\infty}\exp\left(t\frac{\beta-\alpha k^2}{\lambda+k^2}\right)\Big(c_k\sin(k\theta)(\sin k\varphi+\cos k\varphi)+\\ + d_k\cos(k\theta)(\sin k\varphi+\cos k\varphi)\Big),\end{aligned} \tag{9}$$

where

$$a_k = \int_0^R\int_0^{2\pi}\int_0^{\pi}\xi_0\frac{(R-r)^k}{R^k}\sin(k\theta)(\sin k\varphi+\cos k\varphi)r^2\sin\theta d\theta d\varphi dr,$$

$$b_k = \int_0^R\int_0^{2\pi}\int_0^{\pi}\xi_0\frac{(R-r)^k}{R^k}\cos(k\theta)(\sin k\varphi+\cos k\varphi)r^2\sin\theta d\theta d\varphi dr,$$

$$c_k = \int_0^{2\pi}\int_0^{\pi}\chi_0\sin(k\theta)(\sin k\varphi+\cos k\varphi)d\theta d\varphi,$$

$$d_k = \int_0^{2\pi}\int_0^{\pi}\chi_0\cos(k\theta)(\sin k\varphi+\cos k\varphi)d\theta d\varphi.$$

It is easy to see that the series constructed above is a formal solution of the equation (2). Moreover, if the series in (9) converge uniformly, then we have a solution to the problem (2), (4), where $\partial_R\xi = 0$. Given this, we can construct a solution to the problem (3), (4)

$$v = \sum_{k=1}^{\infty}\exp\left(t\frac{\delta-\gamma k^2}{\lambda+k^2}\right)\big(c_k\sin(k\theta)(\sin k\varphi+\cos k\varphi)+d_k\cos(k\theta)(\sin k\varphi+\cos k\varphi)\big), \tag{10}$$

where in the case $\alpha=\gamma$, $\beta=\delta$ the solutions of the problem (2)–(4) will satisfy the matching condition.

The closure $\mathrm{span}\{(R^k)^{-1}(R-r)^k\sin(k\theta)(\sin k\varphi+\cos k\varphi),\ (R^k)^{-1}(R-r)^k\cos(k\theta)(\sin k\varphi+\cos k\varphi) : k\in\mathbb{N}\backslash\{1\}, r\in(0,R), \theta\in[0,\pi], \varphi\in[0,2\pi)\}$ generated by the inner product

$$\langle\widetilde{\varphi},\widetilde{\psi}\rangle_{A(\Omega)} = \int_0^R\int_0^{2\pi}\int_0^{\pi}\widetilde{\varphi}(r,\theta,\varphi)\widetilde{\psi}(r,\theta,\varphi)r^2\sin\theta d\theta d\varphi dr,$$

denote by the symbol $A(\Omega)$. Next, the closure of span$\{\sin(k\theta)(\sin k\varphi + \cos k\varphi), \cos(k\theta)(\sin k\varphi + \cos k\varphi)\colon k \in \mathbb{N}, \theta \in [0,\pi], \varphi \in [0, 2\pi)\}$ by the norm, generated by the inner product

$$\langle \widetilde{\varphi}, \widetilde{\psi} \rangle_{A(\Gamma)} = \int_0^{2\pi} \int_0^{\pi} \widetilde{\varphi}(\theta,\varphi)\widetilde{\psi}(\theta,\varphi) d\theta d\varphi,$$

denote by the symbol $A(\Gamma)$. Thus, the following theorem holds.

Theorem 1. *For any $\xi_0, \chi_0 \in \mathbf{U_K L}_2(\Omega)$ and the coefficients $\alpha, \beta, \gamma, \delta, \lambda \in \mathbb{R}$, such that $\alpha = \gamma$, $\beta = \delta$, and $\lambda \neq k^2$, where $k \in \mathbb{N}$, there exists a single solution $\xi \in C^\infty(\mathbb{R}_+; \mathbf{U_K L}_2)$ of the stochastic Wentzell system of (2)–(4) filtration equations.*

2. The Numerical Investigation and Information Processing Algorithm for the Stochastic Wentzell System of Filtration Equations

The modified numerical algorithm for the stochastic Wentzell system of filtration equations is based on the numerical solution of the Cauchy problem (2)–(4). In particular, by conducting N computational experiments, an initial condition is set for each experiment, whose random variables have a standard normal distribution, and an approximate numerical solution is constructed in the following form

$$\widetilde{\xi}(t,r,\theta,\varphi) = \xi_N(t,r,\theta,\varphi) = \sum_{k=1}^{N} \xi_k(t)\phi_k(r,\theta,\varphi) + \sum_{k=1}^{N} \chi_k(t)\psi_k(R,\theta,\varphi), \quad (11)$$

where $\{\phi_k : k \in \mathbb{N}\}$ are eigenfunctions of the modified Laplace – Beltrami operator $\Delta_{r,\theta,\varphi}$ and correspond to its eigenvalues orthonormalized by the norm $\langle\cdot,\cdot\rangle_{A(\Omega)}$, numbered in nonincreasing order with multiplicity; $\{\psi_k : k \in \mathbb{N}\}$ are eigenfunctions of the modified Laplace – Beltrami operator $\Delta_{\theta,\varphi}$ and correspond to its eigenvalues, orthonormalized by the norm $\langle\cdot,\cdot\rangle_{A(\Gamma)}$, numbered in nonincreasing order with respect to multiplicity, and $\phi_k(R,\theta,\varphi) \equiv 0, k = 1, \cdots, N$. We substitute the approximate solution (14) into the equation (2) and take the scalar product of the eigenfunctions $\phi_k(r,\theta,\varphi)$ and $\psi_k(R,\theta,\varphi)$ by the following formulas $\langle\cdot,\cdot\rangle_{A(\Omega)}$ и $\langle\cdot,\cdot\rangle_{A(\Gamma)}$. We obtain the following system

$$\begin{cases}
(\lambda - \lambda_1)\overset{\circ}{\xi}_1(t,r,\theta,\varphi) = \alpha\lambda_1\xi_1(t,r,\theta,\varphi) + \beta\xi_1(t,r,\theta,\varphi),\\
(\lambda - \lambda_2)\overset{\circ}{\xi}_2(t,r,\theta,\varphi) = \alpha\lambda_2\xi_2(t,r,\theta,\varphi) + \beta\xi_2(t,r,\theta,\varphi),\\
\cdots\\
(\lambda - \lambda_N)\overset{\circ}{\xi}_N(t,r,\theta,\varphi) = \alpha\lambda_N\xi_N(t,r,\theta,\varphi) + \beta\xi_N(t,r,\theta,\varphi),\\
(\lambda - \mu_1)\overset{\circ}{\chi}_1(t,r,\theta,\varphi) = \gamma\mu_1\chi_1(t,R,\theta,\varphi) + \delta\chi_1(t,R,\theta,\varphi),\\
(\lambda - \mu_2)\overset{\circ}{\chi}_2(t,r,\theta,\varphi) = \gamma\mu_2\chi_2(t,R,\theta,\varphi) + \delta\chi_2(t,R,\theta,\varphi),\\
\cdots\\
(\lambda - \mu_N)\overset{\circ}{\chi}_N(t,r,\theta,\varphi) = \gamma\mu_N\chi_N(t,R,\theta,\varphi) + \delta\chi_N(t,R,\theta,\varphi),\\
\varphi_k(R,\theta,\varphi) \equiv 0, k = 1, \cdots, N.
\end{cases} \quad (12)$$

Depending on the value of the parameter λ, we have algebraic or first order differential equations in the system (12). Let us consider these conditions in more detail.

(i) $\lambda \notin \sigma(\Delta_{r,\theta,\varphi})$. In this case, the mathematical model is nondegenerate, and all equations in the resulting system are first order ordinary differential equations. To make this system solvable with respect to $\xi_k(t)$ and $\chi_k(t)$, we multiply scalarly the initial conditions (4) by the eigenfunctions $\phi_k(r,\theta,\varphi)$ and $\psi_k(R,\theta,\varphi)$ by the norm $\langle\cdot,\cdot\rangle_{A(\Omega)}$ and $\langle\cdot,\cdot\rangle_{A(\Gamma)}$, respectively. We then solve the system (12) with appropriate initial conditions and find the coefficients $\xi_k(t)$ in the approximate solution $\widetilde{\xi}(t,r,\theta,\varphi)$.

(ii) $\lambda \in \sigma(\Delta_{r,\theta,\varphi})$. Let focus on the following equipment $\lambda = \lambda_{m_1} = \cdots = \lambda_{m_r}$, where r is the multiplicity of the root. Then part of the equations will be algebraic and the other part will be first order ordinary differential equations. Let us consider separately the systems consisting of algebraic equations and first order differential equations. Note that the solution of the initial problem exists, according to the theorem, if the initial functions $u_0(r,\theta,\varphi)$ and $v_0(\theta,\varphi)$ in the deterministic case or ξ_0, χ_0 in the stochastic case belong to the phase spaces $\mathfrak{P}_1$ and $\mathfrak{P}_2$, respectively

$$\mathfrak{P}_1 = \Big\{u \in A(\Omega) : \langle\xi,\varphi_k\rangle_{A(\Omega)} = 0, \lambda_k = \lambda\Big\}, \mathfrak{P}_2 = \Big\{v \in A(\Gamma) : \langle\chi,\psi_k\rangle_{A(\Gamma)} = 0, \lambda_k = \lambda\Big\}.$$

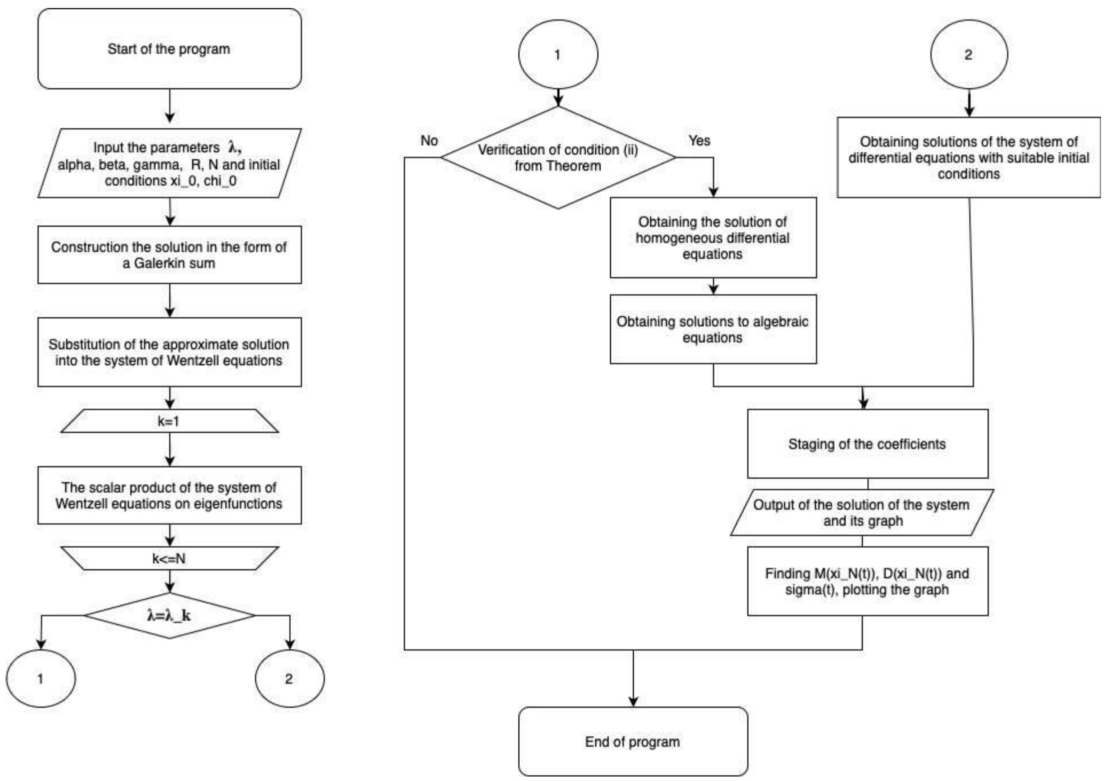

Fig. 1. Information processing algorithm for the stochastic Wentzell system of filtration equations

For further processing of the results, a cycle is run for i, which allows one program to process the results of N experiments, where the necessary characteristics of random processes (mathematical expectation from the section of a stochastic process

$M(\xi(t,r,\theta,\varphi))$, variance from the section of a stochastic process $D(\xi(t,r,\theta,\varphi))$ and standard deviation $\sigma(t))$ are calculated. Here, the mathematical expectation is the non-random function $M(\xi(t,r,\theta,\varphi))$, which at any value of t is equal to the mathematical expectation of section of a stochastic process , i.e. is the average trajectory (realization) obtained as a result of processing N experiments; the variance and mean square deviation will be the non-random functions $D(\xi(t,r,\theta,\varphi))$ and $\sigma(t)$, which at any value of t are equal to the variance and mean square deviation of the corresponding of section of a stochastic process. Thus, using the rule of three sigma, we can state that with probability 0.997 we can estimate $\|\xi(t,r,\theta,\varphi) - M(\xi(t,r,\theta,\varphi))\| < 3\sigma(t)$, which allows us to draw qualitative conclusions about Wentzell systems consisting of the Barenblatt – Zheltov – Kochina equations, taking into account the initial variation of random variables and parameters of the equations characterizing the environment. The figure 1 shows the numerical investigation and information processing algorithm for the stochastic Wentzell system of filtration equations.

3. Analytical Solution for the Stochastic Wentzell System of Free Fluid Filtration Equations

We give an analytical study for the corresponding system (5)–(8) following the results in [5]. In this case, let us consider the following series

$$
\begin{aligned}
u = \sum_{k=2}^{\infty} \exp\left(t\frac{-\beta_0\ k^4-\alpha_0k^2-\gamma_0}{\lambda+k^2}\right)\frac{(R-r)^k}{R^k}\Big(a_k\sin(k\theta)(\sin k\varphi+\cos k\varphi)+ \\
+ b_k\cos(k\theta)(\sin k\varphi+\cos k\varphi)\Big)+ \\
+\sum_{k=1}^{\infty} \exp\left(t\frac{-\beta_0\ k^4-\alpha_0k^2-\gamma_0}{\lambda+k^2}\right)\Big(c_k\sin(k\theta)(\sin k\varphi+\cos k\varphi)+ \\
+ d_k\cos(k\theta)(\sin k\varphi+\cos k\varphi)\Big),
\end{aligned}
\tag{13}
$$

where

$$a_k = \int_0^R\int_0^{2\pi}\int_0^{\pi}\xi_0\frac{(R-r)^k}{R^k}\sin(k\theta)(\sin k\varphi+\cos k\varphi)r^2\sin\theta d\theta d\varphi dr,$$

$$b_k = \int_0^R\int_0^{2\pi}\int_0^{\pi}\xi_0\frac{(R-r)^k}{R^k}\cos(k\theta)(\sin k\varphi+\cos k\varphi)r^2\sin\theta d\theta d\varphi dr,$$

$$c_k = \int_0^{2\pi}\int_0^{\pi}\chi_0\sin(k\theta)(\sin k\varphi+\cos k\varphi)d\theta d\varphi,$$

$$d_k = \int_0^{2\pi}\int_0^{\pi}\chi_0\cos(k\theta)(\sin k\varphi+\cos k\varphi)d\theta d\varphi.$$

It is easy to see that the series constructed above is a formal solution of the equation (5). Moreover, if the series in (13) converge uniformly, then we have a solution to (5), (7), where $\partial_R \xi = 0$. Given this, we can construct a solution to (6), (7)

$$v = \sum_{k=1}^{\infty} \exp\left(t \frac{-\beta_0 k^4 - \alpha_0 k^2 - \gamma_0}{\lambda + k^2} \right) \big(c_k \sin(k\theta)(\sin k\varphi + \cos k\varphi) + d_k \cos(k\theta)(\sin k\varphi + \cos k\varphi) \big),$$

where in the case $\alpha_0 = \alpha_1$, $\beta_0 = \beta_1$, $\gamma_0 = \gamma_1$ the solutions of the problem (5)–(8) will satisfy the matching condition.

The closure span$\{R^{-k}(R-r)^k \sin(k\theta)(\sin k\varphi + \cos k\varphi),\ R^{-k}(R-r)^k \cos(k\theta)(\sin k\varphi + \cos k\varphi) : k \in \mathbb{N}\backslash\{1\}, r \in (0,R), \theta \in [0,\pi], \varphi \in [0, 2\pi)\}$ generated by the inner product

$$\langle \widetilde{\varphi}, \widetilde{\psi} \rangle_{A(\Omega)} = \int_0^R \int_0^{2\pi} \int_0^{\pi} \widetilde{\varphi}(r,\theta,\varphi) \widetilde{\psi}(r,\theta,\varphi) r^2 \sin\theta d\theta d\varphi dr,$$

denote by the symbol $A(\Omega)$. Next, the closure of span$\{\sin(k\theta)(\sin k\varphi + \cos k\varphi), \cos(k\theta)(\sin k\varphi + \cos k\varphi)$: $k \in \mathbb{N}, \theta \in [0,\pi], \varphi \in [0,2\pi)\}$ by the norm, generated by the inner product

$$\langle \widetilde{\varphi}, \widetilde{\psi} \rangle_{A(\Gamma)} = \int_0^{2\pi} \int_0^{\pi} \widetilde{\varphi}(\theta,\varphi) \widetilde{\psi}(\theta,\varphi) d\theta d\varphi,$$

denote by the symbol $A(\Gamma)$.

Thus, the following theorem holds.

Theorem 2. *For any $\xi_0, \chi_0 \in \mathbf{U_K L}_2(\Omega)$ and for the coefficients $\alpha_0, \alpha_1, \lambda, \beta_0, \beta_1, \gamma_0, \gamma_1 \in \mathbb{R}$, such that the condition $\alpha_0 = \alpha_1$, $\beta_0 = \beta_1$, $\gamma_0 = \gamma_1$, and $\lambda \neq k^2$ are hold, where $k \in \mathbb{N}$, there exists a single solution $\xi \in C^\infty(\mathbb{R}_+; \mathbf{U_K L}_2)$ of the stochastic Wentzell system of* (5)–(8) *free filtration equations.*

4. The Numerical Investigation and Information Processing Algorithm for the Stochastic Wentzell System of Free Filtration Equations

The modified numerical algorithm for the stochastic Wentzell system of filtration equations is based on the numerical solution of the Cauchy problem (5)–(8). In particular, by conducting N computational experiments, an initial condition is set for each experiment, the random variables of which have a standard normal distribution, and an approximate numerical solution is constructed in the following form

$$\widetilde{\xi}(t,r,\theta,\varphi) = \xi_N(t,r,\theta,\varphi) = \sum_{k=1}^{N} \xi_k(t) \phi_k(r,\theta,\varphi) + \sum_{k=1}^{N} \chi_k(t) \psi_k(R,\theta,\varphi), \tag{14}$$

where $\{\phi_k : k \in \mathbb{N}\}$ are eigenfunctions of the modified Laplace – Beltrami operator $\Delta_{r,\theta,\varphi}$ and correspond to its eigenvalues orthonormalized by the norm $\langle \cdot, \cdot \rangle_{A(\Omega)}$, numbered in non-increasing order with multiplicity; $\{\psi_k : k \in \mathbb{N}\}$ are eigenfunctions of the modified Laplace – Beltrami operator $\Delta_{\theta,\varphi}$ and correspond to its eigenvalues, orthonormalized by the norm

$\langle\cdot,\cdot\rangle_{A(\Gamma)}$, numbered in nonincreasing order with respect to multiplicity, and $\phi_k(R,\theta,\varphi) \equiv 0, k = 1,\cdots,N$.

Let us substitute the approximate solution (14) into the equation (5) and take the scalar product of the eigenfunctions $\phi_k(r,\theta,\varphi)$ and $\psi_k(R,\theta,\varphi)$ by the following formulas $\langle\cdot,\cdot\rangle_{A(\Omega)}$ and $\langle\cdot,\cdot\rangle_{A(\Gamma)}$. We obtain the following system

$$\begin{cases}(\lambda-\lambda_1)\overset{\circ}{\xi}_1(t,r,\theta,\varphi)=\alpha_0\lambda_1\xi_1(t,r,\theta,\varphi)-\beta_0\lambda_1^2\xi_1(t,r,\theta,\varphi)-\gamma_0\xi_1(t,r,\theta,\varphi),\\(\lambda-\lambda_2)\overset{\circ}{\xi}_2(t,r,\theta,\varphi)=\alpha_0\lambda_1\xi_2(t,r,\theta,\varphi)-\beta_0\lambda_2^2\xi_2(t,r,\theta,\varphi)-\gamma_0\xi_2(t,r,\theta,\varphi),\\\cdots\\(\lambda-\lambda_N)\overset{\circ}{\xi}_N(t,r,\theta,\varphi)=\alpha_0\lambda_1\xi_N(t,r,\theta,\varphi)-\beta_0\lambda_N^2\xi_N(t,r,\theta,\varphi)-\gamma_0\xi_N(t,r,\theta,\varphi),\\(\lambda-\mu_1)\overset{\circ}{\chi}_1(t,r,\theta,\varphi)=\alpha_1\mu_1\chi_1(t,r,\theta,\varphi)-\beta_1\mu_1^2\chi_1(t,r,\theta,\varphi)-\gamma_1\chi_1(t,r,\theta,\varphi),\\(\lambda-\mu_2)\overset{\circ}{\chi}_2(t,r,\theta,\varphi)=\alpha_1\mu_2\chi_2(t,r,\theta,\varphi)-\beta_1\mu_2^2\chi_2(t,r,\theta,\varphi)-\gamma_1\chi_2(t,r,\theta,\varphi),\\\cdots\\(\lambda-\mu_N)\overset{\circ}{\chi}_N(t,r,\theta,\varphi)=\alpha_1\mu_N\chi_N(t,r,\theta,\varphi)-\beta_1\mu_N^2\chi_N(t,r,\theta,\varphi)-\gamma_1\chi_N(t,r,\theta,\varphi),\\\phi_k(R,\theta,\varphi)\equiv 0, k=1,\cdots,N.\end{cases} \tag{15}$$

Depending on the value of the parameter λ, we have algebraic or first order differential equations in the system (12). Let us consider these conditions in more detail.

(i) $\lambda \notin \sigma(\Delta_{r,\theta,\varphi})$. In this case, the mathematical model is nondegenerate, and all equations in the resulting system are first order ordinary differential equations. To make this system solvable with respect to $\xi_k(t)$ and $\chi_k(t)$, we multiply scalarly the initial conditions (4) by the eigenfunctions $\phi_k(r,\theta,\varphi)$ and $\psi_k(R,\theta,\varphi)$ by the norm $\langle\cdot,\cdot\rangle_{A(\Omega)}$ and $\langle\cdot,\cdot\rangle_{A(\Gamma)}$, respectively. We then solve the system (12) with appropriate initial conditions and find the coefficients $\xi_k(t)$ in the approximate solution $\widetilde{\xi}(t,r,\theta,\varphi)$.

(ii) $\lambda \in \sigma(\Delta_{r,\theta,\varphi})$. Let focus on the following equipment $\lambda = \lambda_{m_1} = \cdots = \lambda_{m_r}$, where r is the multiplicity of the root. Then part of the equations will be algebraic and the other part will be first order ordinary differential equations. Let us consider separately the systems consisting of algebraic equations and first order differential equations. Note that the solution of the initial problem exists, according to the theorem, if the initial functions $u_0(r,\theta,\varphi)$ and $v_0(\theta,\varphi)$ in the deterministic case or ξ_0,χ_0 in the stochastic case belong to the phase spaces $\mathfrak{P}_1$ and $\mathfrak{P}_2$, respectively

$$\mathfrak{P}_1 = \Big\{u \in A(\Omega) : \langle\xi,\varphi_k\rangle_{A(\Omega)} = 0, \lambda_k = \lambda\Big\}, \mathfrak{P}_2 = \Big\{v \in A(\Gamma) : \langle\chi,\psi_k\rangle_{A(\Gamma)} = 0, \lambda_k = \lambda\Big\}.$$

For further processing of the results, a cycle is run for i, which allows one program to process the results of N experiments, where the necessary characteristics of random processes (mathematical expectation from the section of a stochastic process $M(\xi(t,r,\theta,\varphi))$, variance from the section of a stochastic process $D(\xi(t,r,\theta,\varphi))$ and standard deviation $\sigma(t)$) are calculated. Here, the mathematical expectation is the non-random function $M(\xi(t,r,\theta,\varphi))$, which at any value of t is equal to the mathematical expectation of section of a stochastic process , i.e. is the average trajectory (realization) obtained as a result of processing N experiments; the variance and mean square deviation will be the non-random functions $D(\xi(t,r,\theta,\varphi))$ and $\sigma(t)$, which at any value of t are equal to the variance and mean square deviation of the corresponding of section of a

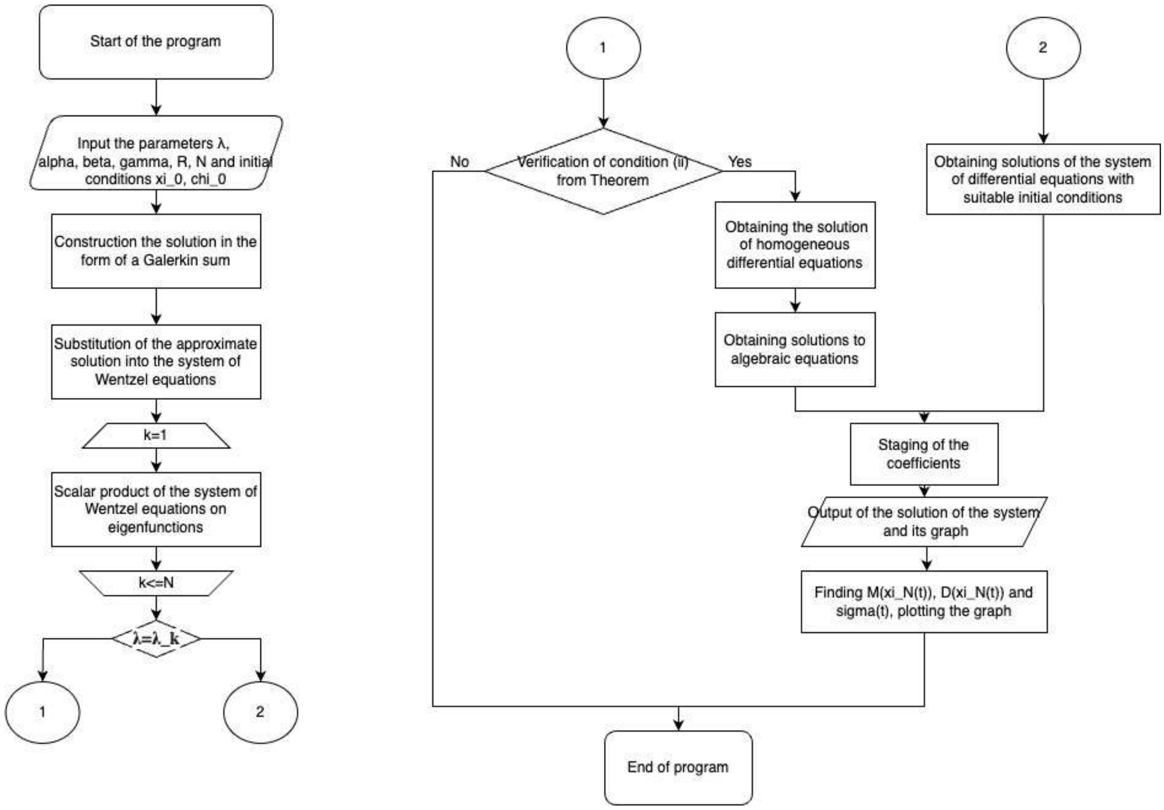

Fig. 2. An information processing algorithm for the stochastic Wentzell system of free filtration equations

stochastic process. Thus, using the rule of three sigma, we can state that with probability 0.997 we can estimate $\|\xi(t,r,\theta,\varphi) - M(\xi(t,r,\theta,\varphi))\| < 3\sigma(t)$, which allows us to draw qualitative conclusions about the Wentzell systems consisting of the Dzekzer equations taking into account the initial variation of random variables and parameters of the equations characterizing the medium. The figure 2 shows the algorithm of numerical investigation and information processing for the stochastic Wentzell system of free filtration equations.

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ОБРАБОТКА ИНФОРМАЦИИ В ЧИСЛЕННОМ ИССЛЕДОВАНИИ ДЛЯ НЕКОТОРЫХ СТОХАСТИЧЕСКИХ СИСТЕМ ВЕНТЦЕЛЯ УРАВНЕНИЙ ГИДРОДИНАМИКИ В ШАРЕ И НА ЕГО ГРАНИЦЕ

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В работе исследуются стохастические системы Вентцеля: уравнения фильтрации, описывающие процессы фильтрации жидкости в трещиновато-пористой среде в трехмерном шаре и на его границе; уравнения свободной фильтрации, описывающие эволюцию свободной поверхности фильтрующейся жидкости в трехмерном шаре и на его границе. В частности, для указанных систем уравнений Вентцеля строятся численные решения задачи Коши и приводится описание обработки результатов n экспериментов при различных значениях случайной величины, имеющей стандартное нормальное распределение (для полученных сечений стохастического процесса, описывающих количественное изменение геохимического режима грунтовых вод при безнапорной фильтрации и количественные изменения свободной фильтрации жидкости строятся доверительные интервалы по правилу трех сигм).

Ключевые слова: стохастическое уравнение фильтрации; стохастическое уравнение свободной фильтрации; система уравнений Вентцеля; обработка информации; правило трех сигм; производная Нельсона – Гликлиха.

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