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ALGORITHMS FOR CALCULATING EIGENVALUES OF SECOND ORDER PARABOLIC DIFFERENTIAL OPERATORS ON QUANTUM STAR GRAPHS WITH TIME-VARYING EDGES

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This paper develops algorithms for calculating the eigenvalues of partial differential operators on star graphs with time-varying edges. The obtained analytical formulas allow finding the eigenvalues of such operators of the required order at a given time. Numerical experiments on calculating the eigenvalues of the problems under study are carried out in the Maple mathematical environment. The calculations performed show the high computational efficiency of the developed method. Using the analytical formulas for calculating the eigenvalues of the operators under consideration obtained in the paper, it is possible to develop algorithms for solving inverse spectral problems for operators on quantum graphs with time-varying edges.

Keywords: graphs; eigenvalues and eigenfunctions; discrete and self-adjoint operators; regularized trace method; Galerkin method.

Introduction

In some cases, the development of mathematical modeling methods aims to develop solutions to spectral problems defined on quantum graphs with time-varying edges. The need for research in this direction arises when it is necessary to study some natural phenomena and processes in various fields of science and technology. Nanotechnology and the development of nanosystems and microsystems engineering are the areas that use quantum graph models [1] – [3].

The paper extends the methodology for the numerical solution of spectral problems for differential operators on quantum graphs with constant edges, described in [4] – [7], to spectral problems on quantum graphs with varying edges.

The method constructed in the paper will allow the previously developed methodology for solving inverse spectral problems defined on quantum graphs with constant edges to be applied to quantum graphs with time-varying edges [8].

1. Spectral Problems on Quantum Star Graphs

To develop algorithms for finding the eigenvalues of partial differential operators on quantum star graphs with variable edge lengths, it is necessary to consider the corresponding initial-boundary value problems for these operators. In our case, we study spectral problems for second-order parabolic partial differential equations.

Let $G = G(V, E)$ be an oriented star graph with j_0 edges and j_0+1 vertices. V denotes the set of vertices and E denotes the set of edges of the graph G. We study the problem of finding eigenvalues on the quantum graph G on special function spaces defined on the interval $[0, T](T < \infty)$.

We denote the graph G by G_0 if the lengths of its edges E_j do not change over time and are equal tol_j, and by G_T if the lengths of its edges change over time according to the laws

$$L_j(t) = l_j L(t),\ l_j \in R_+,\ 0 \le t \le T,\ j = \overline{1, j_0}, \tag{1}$$

where $L_j(t)$ are twice differentiated functions that are greater than zero for any t.

On the star graph G we introduce the space of square summable functions $L^2(G)$ with the norm

$$||u||_{L^2(\mathbf{G})} = \Big(\int\limits_{\mathbf{G}} u^2(\mathbf{z})d\mathbf{z}\Big)^{1/2},\ \mathbf{z} = (z_1, z_2, ..., z_{j_0}),\ j = \overline{1, j_0},$$

and on the set $\Gamma = G \times (0, T)$ we introduce the space of functions $L^2(\mathbf{G})$ with the norm

$$||u||_{L^{2,1}(\mathbf{\Gamma})} = \int\limits_0^T \Big(\int\limits_{\mathbf{\Gamma}} u^2(\mathbf{z}, t)d\mathbf{z}\Big)^{1/2} dt,\ \mathbf{z} = (z_1, z_2, ..., z_{j_0})),\ j = \overline{1, j_0}.$$

The restriction of the function $u(z)$ to the edge E_j of the graph we denote as $u_{E_j}(z_j)$. The integral over the graph G of the function $u(y)$ is defined as the sum of the integrals of the restrictions $u_{E_j}(z_j)$ over each edgeE_j, that is,

$$\int\limits_{\mathbf{G}} \mathbf{u}(\mathbf{z})d\mathbf{y} = \sum_{j=1}^{j_0} \int\limits_{E_j} u_{E_j}(z_j)dz_j.$$

Next, we need two problems defined on the star graph G_0, which we consider below.

First, on the graph G_0, we consider the vector operator $\mathbf{S} = (S_1, S_2, \ldots, S_{j_0})$

$$\mathbf{S\Phi} = -\frac{d^2\mathbf{\Phi}}{d\mathbf{y}^2},\quad \mathbf{\Phi} = (\varphi_1, \varphi_2 \ldots \varphi_{j_0}),\quad \mathbf{y} = (y_1, y_2, \ldots, y_{j_0}). \tag{2}$$

with the domain of definition D_S=$L^2(G_0)$. Let us consider spectral problems for the components of the vector operator S, defined on the edges of the graph G_0:

$$-\frac{d^2\varphi_j}{dy_j^2} = \lambda\varphi_j,\ \varphi_j = \varphi_j(y_j),\ 0 < y_j < l_j,\ j = \overline{1, j_0}, \tag{3}$$

$$\varphi_1(0) = \varphi_2(0) = \cdots = \varphi_{j_0} = 0, \tag{4}$$

$$\varphi_1(l_1) = \varphi_2(l_2) = \cdots = \varphi_{j_0}(l_{j_0}), \tag{5}$$

$$\sum_{j=1}^{j_0} \frac{d\varphi_j}{dy_j}(l_j) = 0. \tag{6}$$

Boundary conditions (4) define the Dirichlet condition at the extreme vertices, and (5) and (6) define the continuity condition and the Kirchhoff condition at the central vertex of the star graph G_0.

We are interested in the eigenvalues $\{\lambda_n\}_{n=1}^{\infty}$ and the corresponding eigenvector functions $\{\mathbf{\Phi}_n\}_{n=1}^{\infty}$ of the vector operator S, which are found when solving (3) and (6).

We can show that the eigenvalues $\{\lambda_n\}_{n=1}^{\infty}$ are the roots of the transcendental equation

$$\sum_{j=1}^{j_0} \operatorname{ctg}(l_j\lambda) = 0, \tag{7}$$

and the components of the eigenvector functions Φ_n, which correspond to the eigenvalues ??λ_n, on the j-th edge of the graph G_0 have the form

$$\varphi_{jn} = \frac{B_n}{\sin(\lambda_n l_j)}\sin(\lambda_n y_j),\ j = \overline{1, j_0},\ B_n = \sqrt{\frac{2}{\sum\limits_{j=1}^{j_0} \dfrac{l_j - \dfrac{\sin(2\lambda_n l_j)}{2\lambda_n}}{\sin^2(\lambda_n l_j)}}},\ n \in N. \tag{8}$$

It is easy to verify that the system of functions $\{\Phi_n\}_{n=1}^{\infty}$ is orthonormal.

$$\int\limits_{\mathbf{G}_0} \mathbf{\Phi}_n \mathbf{\Phi}_m d\mathbf{y} = \sum_{j=1}^{j_0} \int\limits_0^{l_j} \varphi_{jn}(y_j)\varphi_{jm}(y_j)dy_j = \begin{cases} 0,\ n \neq m, \\ 1,\ n = m. \end{cases}$$

Moreover, it is a basis of the space $L^2(G_0)$ [3]. Therefore, for any function $f(y) \in L^2(G_0)$, there is a Fourier series expansion in the system of functions $\{\mathbf{\Phi}_n\}_{n=1}^{\infty}$

$$\mathbf{f}(\mathbf{y}) = \sum_{n=1}^{\infty} a_n \mathbf{\Phi}_n(\mathbf{y}),\ a_n = \int\limits_{\mathbf{G}_0} \mathbf{f}(\mathbf{y})\mathbf{\Phi}_n(\mathbf{y})d\mathbf{y} = \sum_{j=1}^{j_0} \int\limits_0^{l_j} f_j(y_j)\varphi_{jn}(y_j)dy_j.$$

Next, on G_0 we consider a vector operator $M = (M_1, M_2, ..., M_{j_0})$, defined by the rule

$$\mathbf{M\Omega} = \frac{\partial \mathbf{\Omega}}{\partial t} - \frac{\partial^2 \mathbf{\Omega}}{\partial \mathbf{y}^2},\ \mathbf{\Omega} = (\omega_1, \omega_2, \dots \omega_{j_0}),\ \mathbf{y} = (y_1, y_2, \dots y_{j_0}),\ y_j \in [0, l_j],\ j = \overline{1, j_0} \tag{9}$$

with the domain $D_{\mathbf{M}} = L^{2,1}(\mathbf{G}_0)$, where the functions $\omega_j(y_j, t) \in W_2^{2,1}(L^{2,1}(0, l_j) \times [0, T])$. For the projections M_j onto the edges of the graph of the vector operator M, we consider the initial boundary value problems

$$\frac{\partial \omega_j}{\partial t} - \frac{\partial^2 \omega_j}{\partial y_j^2} = 0,\ \omega_j = \omega_j(y_j, t),\ j = \overline{1, j_0}, \tag{10}$$

$$\omega_1(0, t) = \omega_2(0, t) = \dots = \omega_{j_0}(0, t) = 0, ,\ t \in [0, T], \tag{11}$$

$$\omega_1(l_1, t) = \omega_2(l_2, t) = \dots = \omega_{j_0}(l_{j_0}, t), ,\ t \in [0, T], \tag{12}$$

$$\sum_{j=1}^{j_0} \frac{\partial \omega_j}{\partial y_j}\Big|_{y_j=l_j} = 0, \ t \in [0,T], \tag{13}$$

$$\omega_j(y_j,0) = \phi(y_j). \tag{14}$$

The physical meaning of boundary conditions (11) – (13) is described above. Condition (14) is the initial condition. To solve problems (10) – (14), using the method of variable separation, we find

$$\omega_j(y_j,t) = \sum_{n=1}^{\infty} \omega_{jn}(y_j,t), \ \omega_{jn}(y_j,t) = a_n e^{-\lambda_n t}\varphi_{jn}(y_j), \ j=\overline{1,j_0}, \ n \in N. \tag{15}$$

Here $a_n = \sum\limits_{j=1}^{j0} \int\limits_0^{l_j} \phi(y_j)\varphi_{jn}(y_j)dy_j, \ n \in N;$ λ_n are the eigenvalues of the boundary value problem (3) – (6) and are found when solving equation (7).

Let us compose a system of vector functions $\{\tilde{\mathbf{\Omega}}_n\}_{n=1}^{\infty}$ with components $\tilde{\omega}_{jn}(y_j,t) = d_n\omega_{jn}(y_j,t)$, где $d_n = \dfrac{1}{a_n}\sqrt{\dfrac{2\lambda_n}{1-e^{-2\lambda_n T}}}$. It is easy to check that it is orthonormal in $L^{2,1}(\mathbf{G}_0)$

$$\int\limits_0^T \int\limits_{\mathbf{G}_0} \tilde{\mathbf{\Omega}}_n(\mathbf{y},t)\tilde{\mathbf{\Omega}}_m(\mathbf{y},t)d\mathbf{y}dt = \int\limits_0^T \sum_{j=1}^{j_0} \int\limits_0^{l_j} \tilde{\omega}_{jn}(y_j,t)\tilde{\omega}_{jm}(y_j,t)dy_jdt = \begin{cases} 0, \ n \neq m, \\ 1, \ n = m \end{cases} \tag{16}$$

and satisfies the boundary conditions (11)–(13).

2. Calculation of Eigenvalues

We introduce a discrete semibounded differential operator $\mathbf{F} = (F_1, F_2, \dots, F_{j_0})$ in partial derivatives of the second order of parabolic type on a quantum star graph G_T, whose edge length varies in time according to the laws (1)

$$\mathbf{FU} = (\mathbf{T}+\mathbf{P})\mathbf{U}, \ \mathbf{TU} = \frac{\partial \mathbf{U}}{\partial t} - \frac{\partial^2 \mathbf{U}}{\partial \mathbf{x}^2}, \ \mathbf{PU} = \mathbf{p}_1(\mathbf{x},t)\frac{\partial \mathbf{U}}{\partial \mathbf{x}} + \mathbf{p}_0(\mathbf{x},t)\mathbf{U} \tag{17}$$

with the definition domain $D_{\mathbf{F}} = L^{2,1}(\mathbf{G}_T)$. Here $\mathbf{U}(\mathbf{x},t) = (u_1(x_1,t), u_2(x_1,t), \ \dots, u_{j_0}(x_{j_0},t))$; $\mathbf{p}_i(\mathbf{x},t) = (p_{i1}(x_1,t), p_{i2}(x_2,t), \dots, p_{ij_0}(x_{j_0},t))$; $\mathbf{x} = (x_1, x_2, \dots, x_{j_0})$; $x_j \in (0, L_j(t))$; $t \in [0,T]$; $i = \overline{0,1}$; $j = \overline{1,j_0}$.

Let us consider spectral problems for the components of the vector operator F, defined on the edges of the graph G_T

$$\frac{\partial u_j}{\partial t} - \frac{\partial^2 u_j}{\partial x_j^2} + p_{1j}\frac{\partial u_j}{\partial x_j} + p_{0j}u_j = \mu u_j, \tag{18}$$

$$u_j = u_j(x_j,t), \ 0 < x_j < L_j(t), \ j=\overline{1,j_0}, \ t \in [0,T],$$

$$u_1(0,t) = u_2(0,t) = \dots = u_{j_0}(0,t) = 0, \ t \in [0,T], \tag{19}$$

$$u_1(L_1(t),t) = u_2(L_2(t),t) = \dots = u_{j_0}(L_{j_0}(t),t), \ t \in [0,T], \tag{20}$$

$$\sum_{j=1}^{j_0} \frac{\partial u_j}{\partial x_j}(L_j(t), t) = 0, ,\ t \in [0, T] \tag{21}$$

$$u_j(x_j, 0) = \xi(x_j). \tag{22}$$

Let us find a solution to the problems (18) – (22) on graph G_T with time-varying edge lengths by the change of variables

$$\begin{cases} y_j = \dfrac{x_j}{L(t)}, \\ t_1 = t. \end{cases} \tag{23}$$

Let's move on to the problems on graph G_0, whose edge lengths do not change. To do this, we express the partial derivatives included in (18) and (21) with respect to t and x_j through the partial derivatives with respect to t_1 and y_j.

$$\frac{\partial}{\partial x_j} u_j(x_j, t_1) = \frac{\partial}{\partial x_j} u_j(y_j(x_j, t_1), t_1) = \frac{\partial}{\partial y_j} u_j(y_j, t_1)\frac{\partial y_j}{\partial x_j} + \frac{\partial}{\partial t_1} u_j(y_j, t_1)\frac{\partial t_1}{\partial x_j} =$$

$$= \frac{1}{L(t_1)}\frac{\partial}{\partial y_j} u_j(y_j, t_1) + \frac{\partial}{\partial t_1} u_j(y_j, t_1) \cdot 0 = \frac{1}{L(t_1)}\frac{\partial}{\partial y_j} u_j(y_j, t_1).$$

By analogy,

$$\frac{\partial^2}{\partial x_j^2} u_j(x_j, t_1) = \frac{1}{L^2(t_1)}\frac{\partial^2}{\partial y_j^2} u_j(y_j, t_1),$$

$$\frac{\partial}{\partial t} u_j(x_j, t) = -\frac{\dot{L}(t_1)}{L(t_1)} y_j \frac{\partial}{\partial y_j} u_j(y_j, t_1) + \frac{\partial}{\partial t_1} u_j(y_j, t_1).$$

Here $\dot{L}$ denotes the time derivative. Further, since the replacement t=t_1 is formal, we denote t_1 as t. As a result of the change of variables (23) in (18) – (22), we obtain spectral problems on the graph G_0

$$\begin{gathered} \frac{\partial}{\partial t} u_j(y_j, t) - \frac{1}{L^2(t)}\frac{\partial^2}{\partial y_j^2} u_j(y_j, t) + \frac{p_{1j}(y_j, t) - y_j \dot{L}(t)}{L(t)}\frac{\partial}{\partial y_j} u_j(y_j, t) + \\ + p_{0j}(y_j, t) u_j(y_j, t) = \mu u_j(y_j, t),\ 0 < y_j < l_j,\ j = \overline{1, j_0}, \end{gathered} \tag{24}$$

$$u_1(0, t) = u_2(0, t) = \cdots = u_{j_0}(0, t) = 0,\ t \in [0, T], \tag{25}$$

$$u_1(l_1, t) = u_2(l_2, t) = \cdots = u_{j_0}(l_{j_0}, t),\ t \in [0, T], \tag{26}$$

$$\sum_{j=1}^{j_0} \frac{\partial u_j}{\partial y_j}(l_j, t) = 0,\ t \in [0, T], \tag{27}$$

$$u_j(y_j, 0) = \xi(y_j). \tag{28}$$

Note that in problems (24) – (28) the functions $u_j(y_j, t)$ satisfy the boundary conditions (25) – (27), which are written on the graph G_0 edges of constant length l_j, and not with time-dependent edge lengths, as in problems (18) – (22) on the graph G_T. This means that they actually satisfy the boundary conditions (4) – (6).

To find the eigenvalues μ of the vector operator F, we use the method developed in the papers [4] – [8] for approximately calculating the eigenvalues of discrete semibounded differential operators defined on Hilbert spaces.

Let us consider the spectral problem

$$Ww = \mu w, \ Gw|_{\Gamma} = 0. \tag{29}$$

for a discrete semibounded differential operator W, which is defined in the Hilbert space H.

We construct a sequence $\{H_n\}_{n=1}^{\infty}$ of finite-dimensional spaces that is complete in H. If orthonormal bases $\{\phi_k\}_{k=1}^{n}$ of spaces $H_n \subseteq H$ satisfying boundary conditions (29), are known, then the following theorem is true.

Theorem 1. *Approximate eigenvalues $\widetilde{\mu}_n$ of the spectral problem* (29) *are found by the linear formulas*

$$\tilde{\mu}_n = (W\phi_n, \phi_n) + \tilde{\delta}_n, \ n \in N, \tag{30}$$

where $\tilde{\delta}_n = \sum\limits_{k=1}^{n-1}[\tilde{\mu}_k(n-1) - \tilde{\mu}_k(n)]$, $\tilde{\mu}_k(n)$ *is the n-th Galerkin approximation to the corresponding values μ_k of the spectral problem* (29). *In this case,* $\lim\limits_{n\to\infty} |\tilde{\delta}_n| = 0$.

To solve our problem of finding the eigenvalues of boundary value problems defined on the star graph (24) – (28) we use the theorem, taking the system of functions $\{\tilde{\mathbf{\Omega}}_n\}_{n=1}^{\infty}$ as the basic functions. As a result, we obtain

$$\begin{gathered}\tilde{\mu}_n(T) = \int\limits_0^T \int\limits_{\mathbf{G}_0} \mathbf{F}\Big(\tilde{\mathbf{\Omega}}_n(\mathbf{y},t)\Big)\tilde{\mathbf{\Omega}}_n(\mathbf{y},t)d\mathbf{y}dt + \tilde{\delta}_n(T) = \int\limits_0^T \sum_{j=1}^{j_0} \int\limits_0^{l_j} \Big[\frac{\partial}{\partial t}\tilde{\omega}_{jn}(y_j,t) - \\ - \frac{1}{(L(t))^2}\frac{\partial^2}{\partial y_j^2}\tilde{\omega}_{jn}(y_j,t) + \frac{p_{1j}(y_j,t) - y_j\dot{L}(t)}{L(t)}\frac{\partial}{\partial y_j}\tilde{\omega}_{jn}(y_j,t) + p_{0j}(y_j,t)\tilde{\omega}_{jn}(y_j,t)\Big] \times \\ \times \tilde{\omega}_{jn}(y_j,t)\ dy_j dt + \tilde{\delta}_n(T), \ \tilde{\delta}_n(T) = \sum_{k=1}^{n-1}[\tilde{\mu}_k^{n-1}(T) - \tilde{\mu}_k^{n}(T)], \ n \in N.\end{gathered} \tag{31}$$

The obtained formulas (31) allow us to calculate the eigenvalues μ_n вектор – оператора $\mathbf{F}$of the vector operator F defined on the star graph G_TT with time-varying edges at the required time points.

To simplify the form of (31) , we substitute the functions $\tilde{\omega}_{in}(y_j,t)$ from (15) into (31) and calculate some definite integrals included in them. For the first term of formula (31)

$$\begin{gathered}\int\limits_0^T \sum_{j=1}^{j_0} \int\limits_0^{l_j} \tilde{\omega}_{jn}(y_j,t)\frac{\partial}{\partial t}\tilde{\omega}_{jn}(y_j,t)dy_j dt = -\lambda_n d_n^2 a_n^2 \int\limits_0^T e^{-2\lambda_n \tau}d\tau \sum_{j=1}^{j_0} \int\limits_0^{l_j} \varphi_{jn}^2(y_j)dy_j = \\ = -\frac{2\lambda_n^2}{1-e^{-2\lambda_n T}}\frac{1-e^{-2\lambda_n T}}{2\lambda_n} = -\lambda_n.\end{gathered} \tag{32}$$

For the second one,

$$\sum_{j=1}^{j_0} \int\limits_0^{l_j} \tilde{\omega}_{jn}(y_j,t)\frac{\partial^2}{\partial y_j^2}\tilde{\omega}_{jn}(y_j,t)dy_j = d_n^2 a_n^2 e^{-2\lambda_n t} \sum_{j=1}^{j_0} \int\limits_0^{l_j} \phi_{jn}\frac{\partial^2}{\partial y_j^2}\phi_{jn}(y_j,t)dy_j =$$

$$= -\lambda_n^2 d_n^2 a_n^2 e^{-2\lambda_n t} \sum_{j=1}^{j_0} \int\limits_0^{l_j} \phi_{jn}^2(y_j)dy_j = -\lambda_n^2 d_n^2 a_n^2 e^{-2\lambda_n t},$$

$$\int_0^T \frac{1}{(L(t))^2} \sum_{j=1}^{j_0} \int_0^{l_j} \tilde{\omega}_{jn}(y_j,t) \frac{\partial^2}{\partial y_j^2} \tilde{\omega}_{jn}(y_j,t) dy_j dt = -\lambda_n^2 d_n^2 a_n^2 \int_0^t \frac{e^{-2\lambda_n t}}{(L(t))^2} dt. \quad (33)$$

For the part of the third term,

$$\sum_{j=1}^{j_0} \int_0^{l_j} y_j \tilde{\omega}_{jn}(y_j,t) \frac{\partial}{\partial y_j} \tilde{\omega}_{jn}(y_j,t) dy_j = d_n^2 a_n^2 e^{-2\lambda_n t} \sum_{j=1}^{j_0} \int_0^{l_j} y_j \phi_{jn}(y_j) \frac{\partial}{\partial y_j} \phi_{jn}(y_j) dy_j =$$

$$= \frac{\lambda_n d_n^2 a_n^2 B_n^2 e^{-2\lambda_n t}}{2} \sum_{j=1}^{j_0} \frac{1}{\sin^2(\lambda_n l_j)} \int_0^{l_j} y_j \sin(2\lambda_n y_j) dy_j =$$

$$= \frac{d_n^2 a_n^2 B_n^2 e^{-2\lambda_n t}}{8\lambda_n} \sum_{j=1}^{j_0} \frac{\sin(2\lambda_n l_j) - 2\lambda_n l_j \cos(2\lambda_n l_j)}{\sin^2(\lambda_n l_j)},$$

$$\begin{gathered} \int_0^T \sum_{j=1}^{j_0} \int_0^{l_j} \frac{\dot{L}(\tau)}{L(t)} y_j \frac{\partial}{\partial y_j} \tilde{\omega}_{jn}(y_j,t) \tilde{\omega}_{jn}(y_j,\tau) dy_j dt = \\ = \frac{d_n^2 a_n^2 B_n^2}{8\lambda_n} \int_0^T \frac{\dot{L}(t)}{L(t)} e^{-2\lambda_n t} dt \sum_{j=1}^{j_0} \frac{\sin(2\lambda_n l_j) - 2\lambda_n l_j \cos(2\lambda_n l_j)}{\sin^2(\lambda_n l_j)}. \end{gathered} \quad (34)$$

After substituting (32) – (34) в (31) and reduction of similar terms, we find

$$\begin{gathered} \tilde{\mu}_n(T) = -\lambda_n + \lambda_n^2 d_n^2 a_n^2 \int_0^T \frac{e^{-2\lambda_n t}}{(L(t))^2} dt - \\ - \frac{d_n^2 a_n^2 B_n^2}{8\lambda_n} \int_0^T \frac{\dot{L}(t)}{L(t)} e^{-2\lambda_n t} dt \sum_{j=1}^{j_0} \frac{\sin(2\lambda_n l_j) - 2\lambda_n l_j \cos(2\lambda_n l_j)}{\sin^2(\lambda_n l_j)} + \\ + \int_0^T \sum_{j=1}^{j_0} \int_0^{l_j} \Big[\frac{p_{1j}(y_j,t)}{L(t)} \frac{\partial}{\partial y_j} \tilde{\omega}_{jn}(y_j,t) + \\ p_{0j}(y_j,t) \tilde{\omega}_{jn}(y_j,t) \Big] \tilde{\omega}_{jn}(y_j,t)\ dy_j dt + \tilde{\delta}_n(T) \end{gathered}$$

or

$$\begin{gathered} \tilde{\mu}_n(T) = -\lambda_n + d_n^2 a_n^2 \int_0^T R_n(t) \frac{e^{-2\lambda_n t}}{L(t)} dt + \\ + \int_0^T \sum_{j=1}^{j_0} \int_0^{l_j} \Big[\frac{p_{1j}(y_j,t)}{L(t)} \frac{\partial}{\partial y_j} \tilde{\omega}_{jn}(y_j,t) + \\ + p_{0j}(y_j,t) \tilde{\omega}_{jn}(y_j,t) \Big] \tilde{\omega}_{jn}(y_j,t)\ dy_j dt + \tilde{\delta}_n(T), \end{gathered} \quad (35)$$

where $R_n(t) = \dfrac{\lambda_n^2}{L(t)} - \dfrac{B_n^2 \dot{L}(t)}{8\lambda_n} \sum_{j=1}^{j_0} \dfrac{\sin(2\lambda_n l_j) - 2\lambda_n l_j \cos(2\lambda_n l_j)}{\sin^2(\lambda_n l_j)}$, $\tilde{\delta}_n(T) = \sum_{k=1}^{n-1} [\tilde{\mu}_k^{n-1}(T) - \tilde{\mu}_k^n(T)]$.

Using formulas (35), we can find approximate eigenvalues $\tilde{\mu}_n(T)$ of the operator F, with the required sequence numbers at the required moments of time T, specified on a quantum star graph G_T with edge length varying in time according to laws (31). In addition, based on (31), it is easy to write down the system of equations

$$\sum_{k=1}^{M_0} a_k \Big\{ \int_0^T \int_{\mathbf{G}_0} \Big[\mathbf{F}\Big(\tilde{\mathbf{\Omega}}_k(\mathbf{y},t)\Big)\tilde{\mathbf{\Omega}}_m(\mathbf{y},t) - \\ -\mu\tilde{\mathbf{\Omega}}_k(\mathbf{y},t)\tilde{\mathbf{\Omega}}_m(\mathbf{y},t)\Big] d\mathbf{y}dt \Big\} = 0, \ m = \overline{1, M_0} \tag{36}$$

that allows finding the eigenvalues $\widetilde{\mu}_n(T)$ of the operator F using the [10].

3. Computational Experiments

Based on the developed algorithms and the obtained formulas (31), (36) , we conducted numerous computational experiments to find approximate values of the first eigenvalues of the operator F defined on the quantum graph G_T with varying edges.

Let $\hat{\mu}$ denote the approximate eigenvalues of the operator F found by the Galerkin method using (36). As an example, in formula (1), in numerical calculations, we use the harmonic dependence of the function L(t) on time t in the form

$$L(t) = a + b\cos(\omega t), \ a, b \in R_+,$$

where $\omega = 2\pi T_k$ is the oscillation frequency, T_k is the oscillation period. The following simulation parameters were chosen:

$$j_0 = 3, \ l_1 = 1, \ l_2 = e, \ l_3 = \pi, \\ a = 1, \ b = 0,5, \ \omega = 1.$$

Table 1 shows the results of calculating the eigenvalues at $p1_1 = 0$; $p1_2 = 0; 3y_2\sin(\omega t)$; $p1_3 = 0$; $p0_1 = y_1\sin(\omega t)$; $p0_2 = y_2\cos(\omega t)$; $p0_3 = y_3$.

Table 1

Results of calculating the first eigenvalues at time $T = 0,3\pi$

n	$\tilde{\mu}_n$	$\hat{\mu}_n$	$\lvert\tilde{\mu}_n - \hat{\mu}_n\rvert$
1	0.8415	−0.5949	1.4363
2	2.1290	0.3092	1.8205
3	2.4452	1.1565	1.2887
4	2.4528	2.1611	0.2917
5	2.6942	2.4842	0.2100
6	3.1572	2.7523	0.4049
7	5.1334	3.7003	1.4331
8	5.1830	4.4951	0.6879
9	6.0688	5.8683	0.2005
10	7.5333	7.2015	0.3318
11	9.3789	9.2671	0.1117
12	10.4772	10.2329	0.2443
13	12.8859	12.7686	0.1174
14	14.3653	13.8972	0.4681
15	16.9494	16.9494	0.0000

Numerous calculations have shown high computational efficiency of the developed method for finding eigenvalues of partial differential operators given on star-type graphs with time-varying edges.

Conclusion

The paper proposes a method for finding approximate eigenvalues of discrete semibounded partial differential operators of parabolic type on star-type graphs with time-varying edges. The found formulas (31) will allow us in the future, using the theory of solving inverse spectral problems on quantum graphs with constant edges [8], to develop a method for solving inverse spectral problems on quantum graphs with varying edges.

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АЛГОРИТМЫ ВЫЧИСЛЕНИЯ СОБСТВЕННЫХ ЗНАЧЕНИЙ ДИФФЕРЕНЦИАЛЬНЫХ ОПЕРАТОРОВ ВТОРОГО ПОРЯДКА ПАРАБОЛИЧЕСКОГО ТИПА ЗАДАННЫХ НА КВАНТОВЫХ ГРАФАХ – ЗВЕЗДА С ИЗМЕНЯЮЩИМИСЯ РЕБРАМИ

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В статье разработаны алгоритмы вычисления собственных значений дифференциальных операторов в частных производных, заданных на графах — звезда с переменными рёбрами. Получены аналитические формулы, позволяющие находить собственные значения этих операторов необходимого порядка в заданный момент времени. В математической среде Maple проведены численные эксперименты по вычислению собственных чисел исследуемых задач. Проведённые вычисления показали высокую вычислительную эффективность разработанного метода. Используя полученные в статье аналитические формулы вычисления собственных значений рассматриваемых операторов, можно разработать алгоритмы решения обратных спектральных задач для операторов, заданных на квантовых графах с изменяющимися во времени рёбрами.

Ключевые слова: графы; собственные числа и собственные функции; дискретные и самосопряжённые операторы; метод регуляризованных следов; метод Галеркина.

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