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INVESTIGATION OF DIFFERENCE METHODS FOR SOLVING STEFAN'S PROBLEM OF FREEZING WET SOIL

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In this paper, we study a special class of mathematical models that describes physical processes with phase transitions. One of the characteristic features of such models is the previously unknown and changing position of the interphase boundary over time. The difference methods for solving the two-phase Stefan problem in the one-dimensional case with a boundary condition of the second kind are investigated. Two numerical methods for solving the problem of freezing wet soil are compared — the method of catching a front into a spatial grid node and the coordinate transformation method. A study has been conducted in the field of finding the functional dependence of the level of displacement of the interphase boundary on time. Numerical modeling of all considered methods for solving the problem of thermal conductivity with phase transitions is carried out. The results obtained showed the advantage of the coordinate transformation method in comparison with the method of catching the front into a spatial grid node. This advantage lies in the possibility of choosing an arbitrary time step.

Keywords: Stefan's problem; interphase boundary; the problem of soil freezing; the method of catching the front; difference scheme.

Introduction

Currently, many heat transfer problems are associated with a change in the state of matter. Research in this direction is applied in practice in many physical processes: the problem of freezing and thawing of wet soil, the formation of ice on the surface of water, the freezing of pipelines, welding, melting, solidification of metal, etc. The process of changing the physical state of a substance occurs when the body temperature changes. For example, when cooled below the melting point, a transition from the liquid phase to the solid phase occurs.

Solving Stefan's problems is of great practical importance in construction, oil and gas production, metallurgy and other fields. An example of a physical process illustrating Stefan's problem in the three-dimensional case is the problem of modeling the process of laser welding of metal plates. During the welding process, a transition from the solid phase of the metal to the liquid phase occurs under the influence of high temperatures. At the same time, an interphase boundary is formed. Determining the change in its position over time is an urgent scientific problem. The difference schemes for modeling the laser welding process are considered in [1].

1. Mathematical Modeling of the Task of Soil Freezing

When modeling the process of soil freezing, we introduce various thermophysical characteristics [2]. They will be the same for both thawed and frozen soil zones.

- λ_1, λ_2 — thermal conductivity coefficients ($W/(m*K)$),

- ρ_1, ρ_2 — soil density (kg/m^3),
- c_1, c_2 — specific heat capacities ($J/(kg*K)$),
- Q — the heat of the phase transition (J/kg),
- $\xi = f(t)$ — the depth of the frozen layer.

This problem is considered in the soil layer with depth L and humidity ω. At the initial moment of time, the soil is in a thawed state at a certain constant temperature T_0. The temperature T_c is set on the ground surface at the initial moment of time. The freezing point is denoted by T_φ.

We will assume that heat transfer in the soil occurs only due to thermal conductivity. Consider a one-dimensional two-phase Stefan problem with a boundary condition of the second kind on the lower boundary [3].

$$\begin{cases} \dfrac{\partial T_1}{\partial t} = \alpha_1 \dfrac{\partial^2 T_1}{\partial x^2}, \quad 0 < x < \xi(t),\ t > 0; \\[2ex] \dfrac{\partial T_2}{\partial t} = \alpha_2 \dfrac{\partial^2 T_2}{\partial x^2}, \quad \xi(t) < x < L,\ t > 0; \end{cases} \tag{1}$$

$$t = 0 : T(x) = T_0,\ 0 \le x \le L;$$
$$x = 0 : T(t) = T_c,\ t > 0; \quad x = L : \ \frac{\partial T}{\partial x} = 0,\ t > 0;$$

$$x = \xi(t) : \begin{cases} T_1 = T_2 = T_\varphi; \\ \lambda_1 \dfrac{\partial T_1}{\partial x} - \lambda_2 \dfrac{\partial T_2}{\partial x} = Q \dfrac{(\rho_1 + \rho_2)}{2} \omega \dfrac{\partial \xi}{\partial t}; \end{cases} \tag{2}$$

It should be noted that in the figure 1 the frozen part of the soil ($0 \le x \le \xi(t)$) is designated as phase 1, and the thawed part of the soil ($\xi(t) \le x \le L$) is designated as phase 2.

Next, we describe the difference methods for solving the problem of thermal conductivity with first-order phase transitions (1)–(2).

2. The Method of Catching the Front in a Spatial Grid Node

Let's apply the method of catching the front into the node of the spatial grid to find an approximate solution to the problem (1)–(2). Let's introduce a uniform grid of nodes according to the spatial variable

$$x_i = (i-1)h,\ h = \frac{L}{N-1},\ i = \overline{1, N},\ x_1 = 0,\ x_N = L. \tag{3}$$

Let's define an uneven grid of nodes by time coordinate

$$t_{n+1} = t_n + \tau_{n+1}, n = \overline{0, M-1},\ t_0 = 0,\ t_M = t_{\text{end}},\ \tau_{n+1} > 0. \tag{4}$$

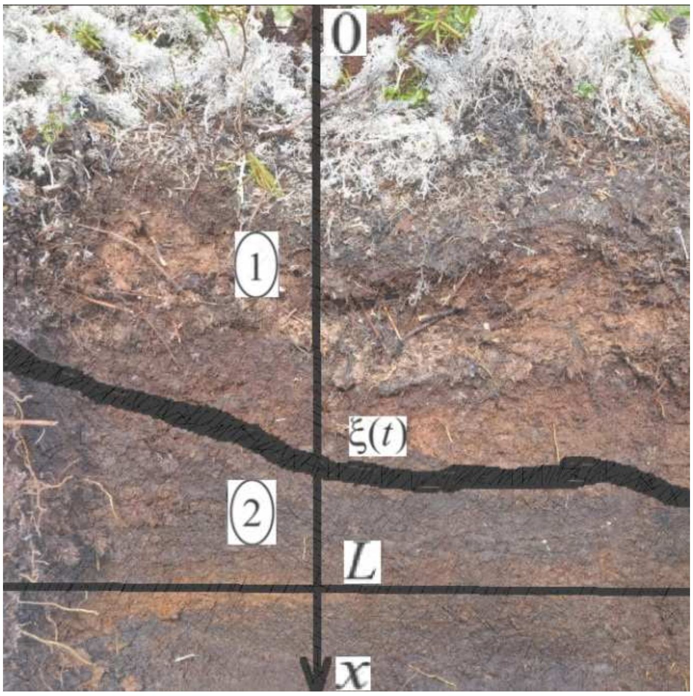

Fig. 1. Diagram of the heat transfer process in the soil

The step along the time coordinate τ_{n+1} is chosen in such a way that during this time the boundary of the phase transition moves exactly one step along the spatial grid. Let's define the following relation

$$\frac{\partial \xi}{\partial t} \approx \frac{h}{\tau_{n+1}}. \tag{5}$$

Let's divide the initial task into two parts (in thawed and frozen soil zones). To implement each difference scheme, we will use the difference run method [4].

The problem of thermal conductivity (1)–(2) in the frozen part of the soil will be approximated by a difference scheme:

$$\begin{gathered} \frac{T_{1,j}^{n+1} - T_{1,j}^{n}}{\tau_{n+1}} = \frac{T_{1,j+1}^{n+1} - 2T_{1,j}^{n+1} + T_{1,j-1}^{n+1}}{h^2}, i = \overline{2, i^* - 1}; \\ T_1|_{i=1} = T_c; \\ T_1|_{i=i^*} = T_3. \end{gathered} \tag{6}$$

To determine the approximate solution of the initial problem in the thawed part of the soil, we apply the following difference scheme

$$\begin{gathered} \frac{T_{2,j}^{n+1} - T_{2,j}^{n}}{\tau_{n+1}} = \frac{T_{2,j+1}^{n+1} - 2T_{2,j}^{n+1} + T_{2,j-1}^{n+1}}{h^2}, i = \overline{i^* + 1, N - 1}; \\ T_2|_{i=i^*} = T_3; \\ \frac{\partial T_2}{\partial x}|_{i=N} = 0; \end{gathered} \tag{7}$$

where $i = i^*$ is the boundary of the phase transition.

It is possible to determine the values of the desired function at the boundary of the phase transition using the equality:

$$T_N^{n+1} = \frac{2\alpha\tau T_{N-1}^{n+1} + h^2 T_N^n}{2\alpha\tau + h^2}. \tag{8}$$

We have obtained a difference scheme for solving the problem (1)–(2) by catching the front into a spatial grid node:

$$\frac{T_{1i}^{n+1} - T_{1i}^n}{\tau_{n+1}} = \alpha_1 \frac{T_{1i+1}^{n+1} - 2T_{1i}^{n+1} + T_{1i-1}^{n+1}}{h^2}, i = 2, ..., i^* - 1;$$
$$\frac{T_{2i}^{n+1} - T_{2i}^n}{\tau_{n+1}} = \alpha_1 \frac{T_{2i+1}^{n+1} - 2T_{2i}^{n+1} + T_{2i-1}^{n+1}}{h^2}, i = i^* - 1, ..., N - 1;$$
$$T_1|_{i=1} = T_c;$$
$$T_1|_{i=i^*} = T_{fr};$$
$$T_2|_{i=i^*} = T_{fr}.$$
$$\frac{T_{2N}^{n+1} - T_{2N-1}^{n+1}}{h} = 0.$$
$$\lambda_1 \frac{T_{1,i^*} - T_{1,i^*-1}}{h} - \lambda_2 \frac{T_{2,i^*+1} - T_{2,i^*}}{h} = Q\rho w \frac{h}{\tau_{n+1}}. \tag{9}$$

3. Coordinate Conversion Method

To solve the one-dimensional problem (1)–(2) using the coordinate transformation method, we will replace the variable x for frozen and thawed soil zones [5]. We will replace it so that the newly entered variable ξ is in a fixed area. Let's introduce the following substitutions:

In a frozen area:

$$\xi = \frac{x}{s(t)};\ 0 \le x \le s(t);\ x = 0 \to \xi = 0;\ x = s(t) \to \xi = 1. \tag{10}$$

In the melt area:

$$\xi = \frac{x - L}{s(t) - L}; s(t) \le x \le L; x = L \to \xi = 0; x = s(t) \to \xi = 1. \tag{11}$$

In phase 1, according to the introduced substitution $(T_1(x,t) = T_1(\xi(s(t)), t))$ partial derivatives in the problem (1)–(2) will have the form:

$$\frac{\partial T_1}{\partial t} = \frac{\partial T_1}{\partial t} - \frac{\xi}{s(t)} \frac{\partial T_1}{\partial \xi} \frac{\partial s}{\partial t}; \quad \frac{\partial T_1}{\partial x} = \frac{1}{s(t)} \frac{\partial T_1}{\partial \xi}; \quad \frac{\partial^2 T_1}{\partial x^2} = \frac{1}{s^2(t)} \frac{\partial^2 T_1}{\partial \xi^2}. \tag{12}$$

In phase 2, the desired function will take the following form $(T_2(x,t) = T_2(\xi(s(t)), t))$.

Partial derivatives in the thawed zone of the soil under the condition (11) will have the form:

$$\frac{\partial T_2}{\partial x} = \frac{1}{s(t) - L} \frac{\partial T_2}{\partial \xi};$$
$$\frac{\partial T_2}{\partial t} = -\frac{\xi}{s(t) - L} \frac{\partial T_2}{\partial \xi} \frac{\partial s}{\partial t} + \frac{\partial T_2}{\partial t};$$
$$\frac{\partial^2 T_2}{\partial x^2} = \frac{1}{(s(t) - L)^2} \frac{\partial^2 T_2}{\partial \xi^2}. \tag{13}$$

Let's define the initial and boundary conditions taking into account the above-described substitutions:

$$t=0,\ 0\le x\le L\to t=0,\ 0\le\xi\le 1;$$
$$T_1(\xi,0)=T_2(\xi,0)=T_0;$$
$$x=0\to\xi=0;\ T_1(0,t)=T_c.$$

$$x=L\to\xi=0;\ \frac{\partial T_2(0,t)}{\partial x}=\frac{1}{s(t)-L}\frac{\partial T_2(0,t)}{\partial\xi}=0. \tag{14}$$

$$x=s(t)\to\xi=1;\begin{cases} T_1(1,t)=T_2(1,t)=T_{fr};\\ \lambda_1\frac{1}{s(t)}\frac{\partial T_1}{\partial\xi}-\lambda_2\frac{1}{s(t)-L}\frac{\partial T_2}{\partial\xi}=Q\rho w\frac{\partial s}{\partial t}.\end{cases} \tag{15}$$

Substituting the equality (12)–(15) into the initial thermal conductivity problem (1)–(2), we obtain a system of differential equations:

$$\frac{\partial T_1}{\partial t}=\alpha_1\frac{1}{s^2(t)}\frac{\partial^2 T_1}{\partial\xi^2}+\frac{\xi}{s(t)}\frac{\partial T_1}{\partial\xi}\frac{ds}{dt},\quad 0<\xi<1,\ t>0;$$
$$\frac{\partial T_2}{\partial t}=\alpha_2\frac{1}{(s(t)-L)^2}\frac{\partial^2 T_2}{\partial\xi^2}+\frac{\xi}{s(t)-L}\frac{\partial T_2}{\partial\xi}\frac{ds}{dt},\quad 0<\xi<1,\ t>0.$$

$$t=0:\ T_1(\xi)=T_2(\xi)=T_0,\ 0\le\xi\le 1; \tag{16}$$

$$\xi=0:\ T_1(0,t)=T_c,\ t>0;\quad \frac{1}{s(t)-L}\frac{\partial T_2(0,t)}{\partial\xi}=0;$$
$$\xi=1:\ T_1(1,t)=T_2(1,t)=T_{fr};$$
$$\lambda_1\frac{1}{s(t)}\frac{\partial T_1(1,t)}{\partial\xi}-\lambda_2\frac{1}{s(t)-L}\frac{\partial T_2(1,t)}{\partial\xi}=Q\rho w\frac{\partial s}{\partial t}$$

where $\alpha_i=\dfrac{k_i}{c_i}, i=1,2.$

We introduce uniform grids of nodes in space and time:

$$\xi_j=jh,\ j=\overline{0,N};\ \xi_0=0,\dots,\xi_N=1;\ h=\frac{1}{N};$$
$$t_n=\tau n,\ n=0,\dots,M;\ t_0=0,\dots,t_M=t_{end};\ \tau=\frac{t_{end}}{M}. \tag{17}$$

We obtain a difference scheme for solving the thermal conductivity equation with phase transitions using the coordinate transformation method:

$$\alpha_1 \frac{T_{1,j+1}^{n+1} - 2T_{1,j}^{n+1} + T_{1,j-1}^{n+1}}{h^2} + s^{n+1}\xi_j \frac{s^{n+1} - s^n}{\tau} \frac{T_{1,j+1}^{n+1} - T_{1,j-1}^{n+1}}{2h} - (s^{n+1})^2 \frac{T_{1,j}^{n+1} - T_{1,j}^{n}}{\tau} = 0;$$

$$\alpha_2 \frac{T_{2,j+1}^{n+1} - 2T_{2,j}^{n+1} + T_{2,j-1}^{n+1}}{h^2} + (s^{n+1} - L)\xi_j \frac{s^{n+1} - s^n}{\tau} \frac{T_{2,j+1}^{n+1} - T_{2,j-1}^{n+1}}{2h} -$$

$$-(s^{n+1} - L)^2 \frac{T_{2,j}^{n+1} - T_{2,j}^{n}}{\tau} = 0;$$

$$T_{1,j}^0 = T_0; T_{2,j}^0 = T_0;\ j = \overline{0, N};$$

$$T_{1,0}^n = T_c;\ n = \overline{0, M};$$

$$T_{1,N}^n = T_{fr};\ n = \overline{0, M};$$

$$\frac{T_{2,1}^n - T_{2,0}^n}{h} = 0;\ n = \overline{0, M}.$$

$$T_{2,N}^n = T_{fr};\ n = \overline{0, M}. \tag{18}$$

$$\lambda_1 \frac{1}{s^n} \frac{T_{1,N}^n - T_{1,N-1}^n}{h} - \lambda_2 \frac{1}{s^n - L} \frac{T_{2,1}^n - T_{2,0}^n}{h} = Q\rho w \frac{s^{n+1} - s^n}{\tau};\ n = \overline{0, M}.$$

To implement the difference scheme, we apply the difference run method. Numerical modeling of the solution of the problem of soil freezing was carried out by catching the front into a spatial grid node and using the coordinate transformation method. The dependence of the change in the boundary of the phase transition on time was obtained. This dependence is nonlinear. This is due to the presence of a condition of the second kind at the lower boundary. The speed of the border movement slows down over time.

Conclusion

The paper considers two numerical methods for solving a one-dimensional two-phase Stefan problem with a boundary condition of the second kind (the method of catching a front into a spatial grid node and the coordinate transformation method). The obtained results of numerical implementation showed the advantage of the coordinate transformation method in comparison with the method of catching the front. This advantage lies in the possibility of choosing an arbitrary time step.

References

1. Kudryashova N.Yu., Pervukhina Yu.V. Investigation of difference methods for solving the problem of thermal conductivity with phase transitions. *Differential equations and their applications in mathematical modeling.* – Saransk: SVMO, 2023. – pp. 96–100. (in Russian)
2. Samarsky A.A., Vabishevich P.K. *Computational Heat Transfer.* Moscow, Editorial URSS, 2003, 784p. (in Russian)

3. Borodin S.L. Numerical Methods for Solving the Stefan Problem. *Tyumen State University Herald. Physical and Mathematical Modeling. Oil, Gas, Energy*, 2015, vol. 1, no. 3, pp. 164–175. (in Russian)
4. Samarsky A.A. *Theory of Difference Schemes.* Moscow, Nauka, 1977. (in Russian)
5. Borisov V.S. Numerical Solution of the Problem of Freezing and Thawing Processes in Perennial Soils. *Vestnik of NEFU*, 2015, vol. 12, no. 2, pp. 36–42. (in Russian)

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ИССЛЕДОВАНИЕ РАЗНОСТНЫХ МЕТОДОВ РЕШЕНИЯ ЗАДАЧИ СТЕФАНА О ПРОМЕРЗАНИИ ВЛАЖНОГО ГРУНТА

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В данной работе исследован особый класс математических моделей, который описывает физические процессы с фазовыми переходами. Одной из характерных особенностей таких моделей является ранее неизвестное и изменяющееся с течением времени положение межфазной границы. Исследуются разностные методы решения двухфазовой задачи Стефана в одномерном случае с граничным условием второго рода. Проведено сравнение двух численных методов решения задачи о промерзании влажного грунта – метода ловли фронта в узел пространственной сетки и метода преобразования координат. Проведено исследование в области нахождения функциональной зависимости уровня смещения межфазной границы от времени. Проведено численное моделирование всех рассмотренных методов решения задачи теплопроводности с фазовыми переходами. Полученные результаты показали преимущество метода преобразования координат по сравнению с методом ловли фронта в узел пространственной сетки. Такое преимущество заключается в возможности выбора произвольного шага по времени.

Ключевые слова: задача Стефана; межфазная граница; задача о промерзании грунта; метод ловли фронта; разностная схема.

Литература

1. Кудряшова, Н.Ю. Исследование разностных методов решения задачи теплопроводности с фазовыми переходами / Н.Ю. Кудряшова, Ю.В. Первухина // Дифференциальные уравнения и их приложения в математическом моделировании: сб. материалов. – Саранск: СВМО, 2023. – С. 96–100.
2. Самарский, А.А. Вычислительная теплопередача / А.А. Самарский, П.К. Вабищевич. – Москва: Едиториал УРСС, 2003.

3. Бородин, С.Л. Численные методы решения задачи Стефана. Вестник Тюменского государственного университета. Физико-математическое моделирование. Нефть, газ, энергетика. – 2015. – Т. 1, №3. – С. 164–175.
4. Самарский, А.А. Теория разностных схем / А.А. Самарский, П.К. Вабищевич. – Москва: Наука, 1977.
5. Борисов, В.С. Численное решение задачи о процессах промерзания и протаивания в многолетних грунтах. Вестник СВФУ. – 2015. – Т. 12, №2. – С. 36–42.

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