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THE TRAJECTORY CONTROLLABILITY OF SECOND ORDER SEMI-LINEAR SYSTEMS USING A FUNCTIONAL ANALYTIC APPROACH

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This article examines the trajectory controllability (TC) of second-order evolution systems while taking impulses into account. A cosine family of operators produced by the linear component of the system, the integral version of Gronwall's inequality, and the idea of nonlinear functional analysis were used to describe the TC findings. Applications for finite- and infinite-dimensional systems of trajectory-controlled systems are given.

Keywords: trajectory controllability; Gronwall's inequality; lipschitz nonlinearity; nonlinear systems.

Introduction

In order to move a system from a given starting state to a desired end state or one that is near to it, one must identify suitable controllers. Kalmann was the first to introduce the notion of controllability using the concept of functional analysis. The monographs [1–3] and articles [4–12], and their references discuss various types of controllabilities for linear and nonlinear systems using a functional analytic approach.

In many systems, the state abruptly changes at a specific moment in time or for short period of time. These systems can be referred to as instantaneous or non-instantaneous impulsive systems. The applications and characteristics of these systems are discussed in [16–20] and the references therein. Shah *et al.* discussed the TC of a first-order non-instantaneous impulsive system in the Banach space, [21, 24]. Determining a controller that moves the system from a given beginning state to the desired final state allows an examination of the system's different types of controllability, however, this controller style is not cost-effective. George [13] introduced Trajectory Controllability (TC). Instead of leading the system from a specific starting condition to the intended end state, the challenge was to build a controller that directs a system along a preset course. A specific path and the intended location are necessary when launching a rocket into space. Consequently, TC has been explored by numerous researchers [14, 15]. Sandilya et al. investigated the TC of a semi-linear parabolic system.

Many evolutionary systems representing wave phenomena are modeled into second-order systems. Therefore, in this article, the authors discuss the TC of the second order system

$$\begin{cases} \bar{\mathfrak{q}}''(t) = \mathcal{A}\bar{\mathfrak{q}} + \mathcal{F}(t, \bar{\mathfrak{q}}(t), \bar{\mathfrak{q}}'(t)) + \varpi(t), \\ \bar{\mathfrak{q}}(0) = \bar{\mathfrak{q}}_{10}, \quad \bar{\mathfrak{q}}'(0) = \bar{\mathfrak{q}}_{20}, \end{cases} \tag{1}$$

by considering non-instantaneous impulses over the finite time interval $\Omega = [0, T_0]$, where, at each time t, the state lies in $\mathbb{X}$, $\mathcal{A}$ is the linear on $\mathbb{X}$, $\mathcal{F} : \Omega \times \mathbb{X}^3 \to \mathbb{X}$ is a non-linear function, and $\varpi(t)$ is the trajectory controller of the system.

1. Preliminaries

Definition 1. *[Complete Controllability] [22] The evolution system completely controllable on the interval $\Omega = [0, T_0]$ if for any $\bar{\mathfrak{q}}_0, \bar{\mathfrak{q}}_1 \in \mathbb{X}$ there exists a controller $\varpi(t)$ in the control space $\mathbb{U}$ such that the state of system moves from the initial state $\bar{\mathfrak{q}}_0$ at $t = 0$ to desire final state $\bar{\mathfrak{q}}_1$ at $t = T$.*

Definition 2. *[Total Controllability] [22] The evolution system is totally controllable over the interval $\Omega = [0, T_0]$ if it is completely controllable over all its subintervals $[t_k, t_{k+1}]$.*

Let $\mathcal{C}_\Omega$ be the set of all functions $\acute{\bar{\mathfrak{q}}}(\cdot)$ defined over Ω satisfying the initial state and final state $\bar{\mathfrak{q}}(0) = \bar{\mathfrak{q}}_0$ and $\acute{\bar{\mathfrak{q}}}(T) = \bar{\mathfrak{q}}_1$, respectively. This set $\mathcal{C}_\Omega$ is called the set of all feasible trajectories. A controller with complete and total controllability for a linear system will be optimal but for a semi-linear or non-linear system it may not be optimal. To overcome this situation one has to design a trajectory having optimum energy or cost and define the controller in such a way that the state of the system steers along this trajectory. Finding the controller which steers the system on the prescribed optimal trajectory from an initial state to desired final state is called TC.

Definition 3. *[TC] [22] The evolution system is trajectory controllable (T-Controllable) if, for any trajectory $\acute{\bar{\mathfrak{q}}} \in \mathcal{C}_\mathcal{T}$, there exist L^2 control function $w \in \mathbb{U}$ such that the state of the system $\bar{\mathfrak{q}}(t)$ satisfy $\bar{\mathfrak{q}}(t) = \acute{\bar{\mathfrak{q}}}(t)$ almost everywhere over Ω.*

In TC, one must identify the controller that will steer the system along a predetermined trajectory from an arbitrary beginning state to the desired final state. Consequently, TC is the most powerful kind of controllability.

2. TC without Impulses

This section discusses the TC of the second-order system

$$\begin{cases} \bar{\mathfrak{q}}''(t) = \mathcal{A}\bar{\mathfrak{q}} + \mathcal{F}(t, \bar{\mathfrak{q}}(t), \bar{\mathfrak{q}}'(t)) + \varpi(t), \\ \bar{\mathfrak{q}}(0) = \bar{\mathfrak{q}}_{10}, \quad \bar{\mathfrak{q}}'(0) = \bar{\mathfrak{q}}_{20}, \end{cases} \tag{2}$$

without considering impulses over Ω. Assuming $\mathcal{F}$ is good enough to have unique mild solution

$$\bar{\mathfrak{q}}(t) = \mathcal{C}(t)\bar{\mathfrak{q}}_{10} + \mathcal{S}(t)\bar{\mathfrak{q}}_{20} + \int_0^t \mathcal{S}(t-\tau)[\mathcal{F}(\tau, \bar{\mathfrak{q}}(\tau), \bar{\mathfrak{q}}'(\tau)) + \varpi(\tau)]\,\mathrm{d}\tau, \tag{3}$$

for all $t \in \Omega$ and any measurable function $\varpi(t)$. Where $\mathcal{C}(\cdot)$ is a strongly continuous cosine family of operators generated by the linear part $\mathcal{A}$, and $\mathcal{S}(\cdot)$ is the associated sine family of operators.

Theorem 1. *The system (2) is T-Controllable over Ω if $\mathcal{F}$ is measurable with t, continuous with respect to other arguments, and there exist positive constants L_{F1} and L_{F2} such that*

$$\|\mathcal{F}(t,\bar{\mathfrak{q}}_1,\acute{\bar{\mathfrak{q}}}_1)-\mathcal{F}(t,\bar{\mathfrak{q}}_2,\acute{\bar{\mathfrak{q}}}_2)\|\leq L_{F1}\|\bar{\mathfrak{q}}_1-\bar{\mathfrak{q}}_2\|+L_{F2}\|\acute{\bar{\mathfrak{q}}}_1-\acute{\bar{\mathfrak{q}}}_2\|.$$

Proof. Let $\bar{\mathfrak{p}}(t)$ be any trajectory from $\mathcal{C}_\Omega$ which steers the evolution equation (2) from the initial state $\bar{\mathfrak{q}}_{10}$ to desired final state $\bar{\mathfrak{q}}_{11}$. Define trajectory controller $\varpi(t)$ as

$$\varpi(t)=\bar{\mathfrak{p}}''(t)-\mathcal{A}\bar{\mathfrak{p}}(t)+\mathcal{F}(t,\bar{\mathfrak{p}}(t),\bar{\mathfrak{p}}'(t)), \tag{4}$$

and plugging it in the system (2), the system becomes:

$$\bar{\mathfrak{q}}''(t)=\mathcal{A}\bar{\mathfrak{q}}+\mathcal{F}(t,\bar{\mathfrak{q}}(t),\bar{\mathfrak{q}}'(t))+\bar{\mathfrak{p}}''(t)-\mathcal{A}u(t)+\mathcal{F}(t,\bar{\mathfrak{p}}(t),\bar{\mathfrak{p}}'(t)). \tag{5}$$

Considering $\mathfrak{z}(t)=\bar{\mathfrak{q}}(t)-\bar{\mathfrak{p}}(t)$, the system (5) becomes

$$\mathfrak{z}''(t)=\mathcal{A}\mathfrak{z}(t)+\mathcal{F}(t,\bar{\mathfrak{q}}(t),\bar{\mathfrak{q}}'(t))-\mathcal{F}(t,\bar{\mathfrak{p}}(t),\bar{\mathfrak{p}}'(t)), \tag{6}$$

with conditions $\mathfrak{z}(0)=0$, $\mathfrak{z}'(0)=0$, and the mild solution of the system (6) satisfies

$$\|\mathfrak{z}(t)\|\leq\int_0^t\|\mathcal{S}(t-\tau)\|\;\|\mathcal{F}(\tau,\bar{\mathfrak{q}}(\tau),\bar{\mathfrak{q}}'(\tau))-\mathcal{F}(t,\bar{\mathfrak{p}}(t),\bar{\mathfrak{p}}'(t))\|\;\mathrm{d}\tau.$$

This assumes the properties of a strongly continuous cosine family of the operators generated by linear part $\mathcal{A}$ and the hypotheses of the theorem,

$$\begin{aligned}\|\mathfrak{z}(t)\|&\leq\int_0^t\|\mathcal{S}(t-\tau)\|\,L_{F1}\,\|\bar{\mathfrak{q}}(\tau)-u(\tau)\|+L_{F2}\,\|\mathfrak{x}'(\tau)-u'(\tau)\|\;\mathrm{d}\tau\\&\leq K\int_0^t\left(L_{F1}\,\|\mathfrak{z}(\tau)\|+L_{F2}\,\|\mathfrak{z}'(\tau)\|\right)\mathrm{d}\tau,\qquad K=\|S(\cdot)\|.\end{aligned}$$

Differentiating ($\|\mathfrak{z}(t)\|$ is differentiable a.e) the above inequality

$$\|\mathfrak{z}'(t)\|\leq K\left(L_{F1}\|\mathfrak{z}(t)\|+L_{F2}\|\mathfrak{z}'(t)\|\right),$$

simplifying

$$\|\mathfrak{z}'(t)\|\leq\frac{KL_{F1}}{1-KL_{F2}}\|\mathfrak{z}(t)\|.$$

Applying a differential form of Grönwall's inequality $\|\mathfrak{z}(t)\|=0$ a.e., and thus $\bar{\mathfrak{q}}(t)=\bar{\mathfrak{p}}(t)$ a.e. Hence the system (2) is T-Controllable over Ω. □

Example 1. The equations of motion for an artificial satellite, due to the oblateness of the earth, are modeled into second-order equations

$$\begin{aligned}\mathfrak{x}''(t)&=-\frac{\mu}{r^3}\mathfrak{x}-\frac{3\mu R^2J_2\mathfrak{x}(\mathfrak{x}^2+\mathfrak{y}^2-4\mathfrak{z}^2)}{2r^7},\\\mathfrak{y}''(t)&=-\frac{\mu}{r^3}\mathfrak{y}-\frac{3\mu R^2J_2\mathfrak{y}(\mathfrak{x}^2+\mathfrak{y}^2-4\mathfrak{z}^2)}{2r^7},\\\mathfrak{z}''(t)&=-\frac{\mu}{r^3}\mathfrak{z}-\frac{3\mu R^2J_2\mathfrak{y}(3\mathfrak{x}^2+3\mathfrak{y}^2-2\mathfrak{z}^2)}{2r^7},\end{aligned} \tag{7}$$

where $\mu = GM$, G is the universal gravitational constant, R, M are radius, mass of earth, respectively, J_2 is a zonal coefficient, and $r = \sqrt{\mathfrak{x}^2 + \mathfrak{y}^2 + \mathfrak{z}^2}$. From the various studies, it was found that the motion of an artificial satellite is unstable for the oblate earth if the initial velocity is low and sometimes it can fall from orbit [23].

Therefore to make the motion in the prescribed orbit one has to plug the controller into the satellite so that it follows a specific path. Let $[u_1(t), u_2(t), u_3(t)]$ and $[w_1(t), w_2(t), w_3(t)]$ be the prescribe trajectory and the trajectory controller for the satellite, respectively. Plugging it in (7), the equations of motion becomes:

$$\begin{aligned} \mathfrak{x}''(t) &= -\frac{\mu}{r^3}\mathfrak{x} - \frac{3\mu R^2 J_2 \mathfrak{x}(\mathfrak{x}^2 + \mathfrak{y}^2 - 4\mathfrak{z}^2)}{2r^7} + w_1(t), \\ \mathfrak{y}''(t) &= -\frac{\mu}{r^3}\mathfrak{y} - \frac{3\mu R^2 J_2 \mathfrak{y}(\mathfrak{x}^2 + \mathfrak{y}^2 - 4\mathfrak{z}^2)}{2r^7} + w_2(t), \\ \mathfrak{z}''(t) &= -\frac{\mu}{r^3}\mathfrak{z} - \frac{3\mu R^2 J_2 \mathfrak{y}(3\mathfrak{x}^2 + 3\mathfrak{y}^2 - 2\mathfrak{z}^2)}{2r^7} + w_3(t). \end{aligned} \tag{8}$$

Since the motion of many low earth satellites has a circular orbit having fixed radius $r = a$ from the center of the earth. Therefore, the equation of motion for the circular orbit $r = a$ becomes:

$$\begin{aligned} \mathfrak{x}''(t) &= -\frac{\mu}{a^3}\mathfrak{x} - \frac{3\mu R^2 J_2 \mathfrak{x}(\mathfrak{x}^2 + \mathfrak{y}^2 - 4\mathfrak{z}^2)}{2a^7} + w_1(t), \\ \mathfrak{y}''(t) &= -\frac{\mu}{a^3}\mathfrak{y} - \frac{3\mu R^2 J_2 \mathfrak{y}(\mathfrak{x}^2 + \mathfrak{y}^2 - 4\mathfrak{z}^2)}{2a^7} + w_2(t) \\ \mathfrak{z}''(t) &= -\frac{\mu}{a^3}\mathfrak{z} - \frac{3\mu R^2 J_2 \mathfrak{y}(3\mathfrak{x}^2 + 3\mathfrak{y}^2 - 2\mathfrak{z}^2)}{2a^7} + w_3(t). \end{aligned} \tag{9}$$

These motion equations have the following form

$$\bar{r}''(t) = \mathcal{A}\bar{r}(t) + \mathcal{F}(t, \bar{r}(t)) + \varpi(t), \tag{10}$$

where, $\bar{r} = [\mathfrak{x}(t), \mathfrak{y}(t), \mathfrak{z}(t)]$ the position vector of the satellite,

$$\mathcal{A} = \begin{bmatrix} -\frac{\mu}{a^3} & 0 & 0 \\ 0 & -\frac{\mu}{a^3} & 0 \\ 0 & 0 & -\frac{\mu}{a^3} \end{bmatrix}, \qquad \mathcal{F}(\bar{r}(t)) = \begin{bmatrix} -\frac{3\mu R^2 J_2 \mathfrak{x}(\mathfrak{x}^2 + \mathfrak{y}^2 - 4\mathfrak{z}^2)}{2a^7} \\ -\frac{3\mu R^2 J_2 \mathfrak{y}(\mathfrak{x}^2 + \mathfrak{y}^2 - 4\mathfrak{z}^2)}{2a^7} \\ -\frac{3\mu R^2 J_2 \mathfrak{y}(3\mathfrak{x}^2 + 3\mathfrak{y}^2 - 2\mathfrak{z}^2)}{2a^7} \end{bmatrix}.$$

The function $\mathcal{F}(\bar{r}(t))$ is differentiable with respect to $\bar{r}$ as all of its partial derivatives exist and are continuous over any finite time interval. The linear operator $\mathcal{A}$ generates a strongly continuous cosine family of operators

$$\mathcal{C}(t) = \begin{bmatrix} \cos\sqrt{\frac{\mu}{a^3}}t & 0 & 0 \\ 0 & \cos\sqrt{\frac{\mu}{a^3}}t & 0 \\ 0 & 0 & \cos\sqrt{\frac{\mu}{a^3}}t \end{bmatrix},$$

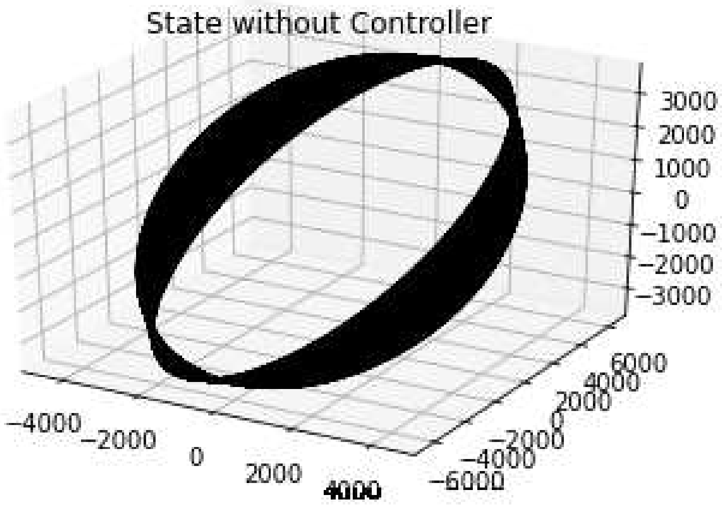


Fig. 1. The motion of the satellite is not stable without a controller.

and the associated sine family

$$\mathcal{S}(t) = \sqrt{\frac{a^3}{\mu}} \begin{bmatrix} \sin\sqrt{\frac{\mu}{a^3}}t & 0 & 0 \\ 0 & \sin\sqrt{\frac{\mu}{a^3}}t & 0 \\ 0 & 0 & \sin\sqrt{\frac{\mu}{a^3}}t \end{bmatrix}.$$

Thus the motion of the satellite (10) is T-Controllable for finite time intervals. Let the initial position of the satellite be

$$\bar{r}_0 = \big[0,\ -5888.9727,\ -3400\big],$$

having initial velocity $\bar{v}_0 = [7, 0, 0]$. The Figure 1 shows that the motion of the satellite is not stable without a controller.

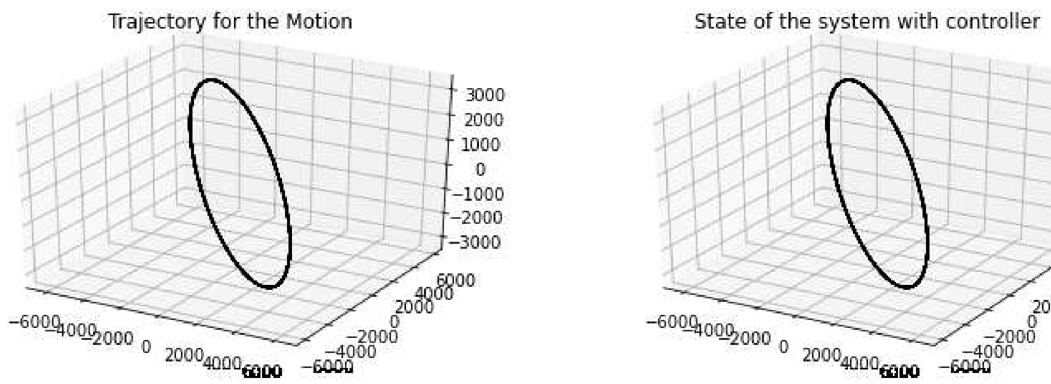
Data:

$$\bar{r_0} = (0, -5888.9727, -3400), \quad \bar{v_0} = (7, 0, 0), \quad R = 6378.1363, \quad a = |\bar{r_0}|,$$

$\mu = G * M$, $J_2 = 1082.63 \times 10^{-6}$, Time Span: 540000 sec. Now considering the trajectory for the motion of the satellite

$$\bar{\mathfrak{p}}(t) = \begin{bmatrix} 7\sqrt{\frac{a^3}{\mu}}\sin\sqrt{\frac{\mu}{a^3}}t & -5888.9727\cos\sqrt{\frac{\mu}{a^3}}t & -3400\cos\sqrt{\frac{\mu}{a^3}}t \end{bmatrix},$$

and define the trajectory controller $\varpi(t) = \bar{u}'' - \mathcal{A}\bar{u} - \mathcal{F}(\bar{u})$ and plugging into the equation of motion (10) the state of the system follows prescribed path. The Figures 2a and 2b show the trajectory and state with the controller.



a) b)

Fig. 2. The trajectory and state with the controller.

3. TC with Impulses

This section discusses the TC of the non-instantaneous impulsive second-order system

$$\begin{cases} \bar{\mathfrak{q}}''(t) = \mathcal{A}\bar{\mathfrak{q}} + \mathcal{F}(t, \bar{\mathfrak{q}}(t), \bar{\mathfrak{q}}'(t)) + \varpi(t), & t \in [0, t_1) \cup [s_1, t_2) \cdots \cup [s_\rho, T_0] \\ \bar{\mathfrak{q}}(t) = \mathcal{G}_k(t, \bar{\mathfrak{q}}(t)) + \varpi_k(t), & t \in [t_1, s_1) \cup [t_2, s_2) \cdots \cup [t_\rho, s_\rho), \\ \bar{\mathfrak{q}}(0) = \bar{\mathfrak{q}}_{10}, \quad \bar{\mathfrak{q}}'(0) = \bar{\mathfrak{q}}_{20}, \end{cases} \tag{11}$$

over Ω. Attributes of the system (11) are good enough to have a unique mild solution

$$\bar{\mathfrak{q}}(t) = \begin{cases} \mathcal{C}(t)\bar{\mathfrak{q}}_{10} + \mathcal{S}(t)\bar{\mathfrak{q}}_{20} \\ \qquad + \displaystyle\int_0^t \mathcal{S}(t-\tau)\ [\mathcal{F}(\tau, \bar{\mathfrak{q}}(\tau), \bar{\mathfrak{q}}'(\tau)) + \varpi(\tau)]\ \mathrm{d}\tau, & t \in [0, t_1), \\ \mathcal{G}_k(t, \bar{\mathfrak{q}}(t)) + \varpi_k(t), & t \in [t_k, s_k), \\ \mathcal{C}(t-s_k)\mathcal{G}_k(s_k, \bar{\mathfrak{q}}(s_k)) + \mathcal{S}(t-s_k)\mathcal{G}_k'(s_k, \bar{\mathfrak{q}}(s_k)) \\ \qquad + \displaystyle\int_{s_k}^t S(t-s)\ [\mathcal{F}(\tau, \bar{\mathfrak{q}}(\tau), \bar{\mathfrak{q}}'(\tau)) + \varpi(\tau)]\ \mathrm{d}\tau, & t \in [s_k, t_{k+1}), \end{cases} \tag{12}$$

for all $t \in \Omega$ and any measurable function $\varpi(t)$, where $\mathcal{C}(\cdot)$, $\mathcal{S}(\cdot)$ are a strongly continuous cosine family of operators generated by the linear part $\mathcal{A}$ and the associated sine family of operators, respectively, and $\mathcal{G}_k'$ denotes the derivative of $\mathcal{G}_k$ with respect to t.

Assumptions:

(A1) The linear part $\mathcal{A}$ of the equation (11) is an infinitesimal generator of a strongly continuous cosine family of operators;

(A2) The nonlinear function $\mathcal{F}$ is measurable with respect to argument t over Ω and there exist constants r_0, L_{F1}, and L_{F2} such that

$$\|\mathcal{F}(t, \bar{\mathfrak{q}}_1, \bar{\mathfrak{q}}_2) - \mathcal{F}(t, \acute{\bar{\mathfrak{q}}}_1, \acute{\bar{\mathfrak{q}}}_2)\| \le L_{F1}\|\bar{\mathfrak{q}}_1 - \acute{\bar{\mathfrak{q}}}_1\| + L_{F2}\|\bar{\mathfrak{q}}_2 - \acute{\bar{\mathfrak{q}}}_2\|, \qquad \forall\, \bar{\mathfrak{q}}_i, \acute{\bar{\mathfrak{q}}}_i \in B_{r_0} \subset \mathbb{X},\ ß = 1, 2;$$

(A3) The nonlinear functions $\mathcal{G}_k$ and its time derivative $\mathcal{G}_K'$ for a known value of $\bar{\mathfrak{q}}(t)$. Moreover there exist $0 < \mathfrak{g}_k < 1$ such that

$$\|\mathcal{G}_k(t, \bar{\mathfrak{q}}) - \mathcal{G}_k(t, \acute{\bar{\mathfrak{q}}})\| \le \mathfrak{g}_k\|\bar{\mathfrak{q}} - \acute{\bar{\mathfrak{q}}}\|, \forall\, \bar{\mathfrak{q}}, \acute{\bar{\mathfrak{q}}} \in B_{r_0}.$$

Theorem 2. *The system* (11) *is T-Controllable over* Ω *if hypotheses* (A1)–(A3) *are satisfied.*

Proof. Let $\bar{\mathfrak{p}}(t)$ be any trajectory from $\mathcal{C}_\Omega$ which steers the evolution equation of (11) from the initial state $\bar{\mathfrak{q}}_{10}$ to desired final state x_{11} satisfying $u(t_k^+) = \bar{\mathfrak{q}}(t_k^+)$. Over the interval $[0, t_1)$ the system becomes:

$$\bar{\mathfrak{q}}''(t) = \mathcal{A}\bar{\mathfrak{q}} + \mathcal{F}(t, \bar{\mathfrak{q}}(t), \bar{\mathfrak{q}}'(t)) + \varpi(t)\bar{\mathfrak{q}}(0) = \bar{\mathfrak{q}}_{10}, \qquad \bar{\mathfrak{q}}'(0) = \bar{\mathfrak{q}}_{20}. \tag{13}$$

Plugging the controller

$$\varpi(t) = \bar{\mathfrak{p}}''(t) - \mathcal{A}\bar{\mathfrak{p}}(t) + \mathcal{F}(t, \mathfrak{p}(\bar{\mathfrak{t}}), \bar{\mathfrak{p}}'(t)), \tag{14}$$

into the system (13), and proceeding in same way as in Theorem 1, the system is controllable over the interval $[0, t_1)$. Over the interval $[t_k, s_k)$, the system becomes:

$$\bar{\mathfrak{q}}(t) = \mathcal{G}_k(t, \bar{\mathfrak{q}}(t)) + \varpi_k(t). \tag{15}$$

Plugging the controller

$$\varpi_k(t) = \bar{\mathfrak{p}}(t) - \mathcal{G}_k(t, \bar{\mathfrak{p}}(t)), \tag{16}$$

into the system (14), the system becomes $\bar{\mathfrak{q}}(t) - \bar{\mathfrak{p}}(t) = \mathcal{G}_k(t, \bar{\mathfrak{q}}(t)) - \mathcal{G}_k(t, \bar{\mathfrak{p}}(t))$. Taking $\mathfrak{z}(t) = \bar{\mathfrak{q}}(t) - \bar{\mathfrak{p}}(t)$ and computing

$$\|\mathfrak{z}(t)\| = \|\mathcal{G}_k(t, \bar{\mathfrak{q}}(t)) - \mathcal{G}_k(t, \bar{\mathfrak{p}}(t))\| \leq \mathfrak{g}_k \|\mathfrak{z}(t)\|.$$

Thus, $(1 - \mathfrak{g}_k)\|z(t)\| \leq 0$. Since $\mathfrak{g}_k < 1$ therefore $\|z(t)\| = 0$. Hence, the system is T-Controllable over $[t_k, s_k)$, $\forall k = 1, 2, \cdots, \rho$.

Over $[s_k, t_{k+1})$ the system becomes:

$$\bar{\mathfrak{q}}''(t) = \mathcal{A}\bar{\mathfrak{q}} + \mathcal{F}(t, \bar{\mathfrak{q}}(t), \bar{\mathfrak{q}}'(t)) + \varpi(t), \tag{17}$$

with initial conditions $\bar{\mathfrak{q}}(s_k) = \mathcal{G}_k(s_k, \bar{\mathfrak{q}}(s_k))$ and $\bar{\mathfrak{q}}'(s_k) = \mathcal{G}_k'(s_k, \bar{\mathfrak{q}}(s_k))$. Since, $\|\mathfrak{z}(t)\| = 0$ for all $t \in [t_k, s_k)$ and the continuity of $\mathcal{G}_k$ leads to $\|\mathfrak{z}(s_k)\| = 0$. Thus, $\bar{\mathfrak{q}}(s_k) = \bar{\mathfrak{p}}(s_k)$. Plugging the controller

$$\varpi(t) = \bar{\mathfrak{p}}''(t) - \mathcal{A}\bar{\mathfrak{p}}(t) + \mathcal{F}(t, \bar{\mathfrak{p}}(t), \bar{\mathfrak{p}}'(t)), \tag{18}$$

into the system (17) and assuming the hypotheses (A1)-(A3) and using the theorem 1, the system is T-Controllable over the interval $[s_k, t_{k+1})$. Hence, the system (11) is T-Controllable over Ω.

□

Example 2. Consider the partial differential equation

$$\begin{cases} \dfrac{\partial^2 Z(t, \bar{\mathfrak{q}})}{\partial t^2} = Z_{\bar{\mathfrak{q}}\bar{\mathfrak{q}}}(t, \bar{\mathfrak{q}}) + e^{-Z(t, \bar{\mathfrak{q}})} + \varpi(t), & t \in \left[0, \frac{1}{3}\right) \cup \left[\frac{2}{3}, 1\right], \\ Z(t, \bar{\mathfrak{q}}) = \dfrac{1}{2} \sin\left(Z(t, \bar{\mathfrak{q}})\right), & t \in \left[\frac{1}{3}, \frac{2}{3}\right), \end{cases} \tag{19}$$

in the Banach space $\mathbb{X} = L^2(\Omega)$, $\Omega = [0, \pi]$, $T_0 = \pi$, and with initial condition

$$Z(0, \bar{\mathfrak{q}}) = Z_0(\bar{\mathfrak{q}}), \qquad Z_t(0, \bar{\mathfrak{q}}) = Z_1(\bar{\mathfrak{q}}),$$

and boundary conditions $Z(t, 0) = Z(t, \pi) = 0$. Define an operator, $\mathcal{A}$, as $\mathcal{A}Z = Z_{\bar{\mathfrak{q}}\bar{\mathfrak{q}}}$ over the domain

$$\mathrm{Dom}(\mathcal{A}) = \Big\{ y \in L^2(\Omega) \, : \, y'' \text{ exist and } z(0) = z(\pi) = 0 \Big\}.$$

Operator $\mathcal{A}$ is represented by

$$\mathcal{A}z = \sum_{n=1}^{\infty} -n^2 \left\langle z, \sqrt{\frac{2}{\pi}} \sin n\bar{\mathfrak{q}} \right\rangle \sqrt{\frac{2}{\pi}} \sin n\bar{\mathfrak{q}}, \qquad z \in \mathrm{Dom}(A).$$

The operator $\mathcal{A}$ is the infinitesimal generator of strongly continuous cosine family $\mathcal{C}(\cdot)$ on $\mathbb{X}$ defined by

$$\mathcal{C}(t)z = \sum_{n=1}^{\infty} \cos nt \left\langle z, \sqrt{\frac{2}{\pi}} \sin n\bar{\mathfrak{q}} \right\rangle \sqrt{\frac{2}{\pi}} \sin n\bar{\mathfrak{q}},$$

and the associated sine family $\mathcal{S}(\cdot)$ on $\mathbb{X}$ defined by

$$\mathcal{S}(t)z = \sum_{n=1}^{\infty} \frac{1}{n} \sin t \left\langle z, \sqrt{\frac{2}{\pi}} \sin n\bar{\mathfrak{q}} \right\rangle \sqrt{\frac{2}{\pi}} \sin n\bar{\mathfrak{q}}.$$

The evolution Eq. (19) can be formulated as the abstract equation in $\mathbb{X} = L^2([0, 1])$ as:

$$\begin{cases} \dfrac{\mathrm{d}^2\upsilon}{\mathrm{d}t^2} = \mathcal{A}\upsilon(t) + \mathcal{F}(t, \upsilon(t)) + \varpi(t), & t \in \left[0, \frac{1}{3}\right) \cup \left[\frac{2}{3}, 1\right], \\ \upsilon(t) = \mathcal{G}_1(t, \upsilon), & t \in \left[\frac{1}{3}, \frac{2}{3}\right), \\ \upsilon(0) = \upsilon_0, \quad \dfrac{\mathrm{d}\upsilon}{\mathrm{d}t}(0) = 0. \end{cases} \tag{20}$$

- The function $\mathcal{F}(t, \upsilon) = e^{-\upsilon}$ is a continuous function and there exist $l_{\mathcal{F}}(r) = 1$ on B_{r_0} satisfying
$$\|\mathcal{F}(t, \upsilon_1) - \mathcal{F}(t, \upsilon_2)\| \leq \|\upsilon_1 - \upsilon_2\|.$$
  Thus, by Theorem the system (20) is T-controllable over $[0, 1]$.
- Assuming that the derivative of $\frac{1}{2} \sin z$, $z'(t)$ exist over the interval $[0, 1]$.

Then, the system (20) is T-Controllable over $[0, 1]$.

Conclusion

This paper discusses the TC of a second-order systems with and without impulses. The TC of the system was obtained using the concept of a cosine family of operators, nonlinear functional analysis, and Grönwall's inequality. Applications to the motion of the artificial satellite and nonlinear one-dimensional wave equations are also added to validate the obtained results.

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ТРАЕКТОРНАЯ УПРАВЛЯЕМОСТЬ ПОЛУЛИНЕЙНЫХ СИСТЕМ ВТОРОГО ПОРЯДКА С ИСПОЛЬЗОВАНИЕМ ФУНКЦИОНАЛЬНО-АНАЛИТИЧЕСКОГО ПОДХОДА

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В данной статье рассматривается траекторная управляемость (ТУ). систем эволюции второго порядка с учетом импульсов. Для описания результатов ТУ использовались косинусное семейство операторов, произведенных линейной составляющей системы, интегральная версия неравенства Гронуолла и идея нелинейного функционального анализа. Приведены приложения для конечномерных и бесконечномерных систем систем с траекторным управлением.

Ключевые слова: управляемость траектории; неравенство Гронуолла; липшицева нелинейность; нелинейные системы.

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