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NUMERICAL SIMULATION OF BLOOD FLOW IN A BLOOD VESSEL

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The paper proposes a finite-difference method for solving a boundary value problem for a hyperbolic equation describing the movement of blood in a blood vessel. The stability conditions of the method are given, and numerical results are presented. The method allows to track the amplitude and frequency of heartbeats in various modes, and a numerical model can be used in the study of atrial fibrillation.

Keywords: nonlinear hyperbolic PDE; hydrodynamics of blood circulation; finite-difference method.

Introduction

This paper is devoted to numerical treatment of the mathematical model of blood flow in a blood vessel. To describe the pulse wave in this model, classical methods of hydrodynamics are used in relation to the human cardiovascular system. From the standpoint of classical hydrodynamics, the motion of any real medium is described by the Navier-Stokes equation, which together with the equations of continuity, state and heat balance form a closed system [1]. The problem is reduced to a partial differential equation of the following form:

$$b\frac{\partial^2 V(x,t)}{\partial x^2} = \frac{\partial^2 V(x,t)}{\partial t^2} + aV(x,t)\frac{\partial V(x,t)}{\partial t}, \quad x \in [0,l],\ t \in [0,T] \tag{1}$$

with specified initial and boundary conditions

$$V(0,t) = f_1(t), \quad V(l,t) = f_2(t), \quad V(x,0) = g_1(t), \quad V_t(x,0) = g_2(x). \tag{2}$$

The equation (1) is a nonlinear hyperbolic equation. Two types of partial solutions are known for it and the self-similar solution [2], which are not suitable for practical use. In this regard, the problem of constructing effective methods for the approximate solution of such equations is urgent. A finite-difference scheme for such a problem is proposed in [3]. This paper is a development of the work [3] in relation to the analysis of the amplitude and frequency of heartbeats in various modes, which is important in the study of atrial fibrillation.

1. Finite-difference Approximation

In order to construct an approximate solution, we apply a finite-difference approximation of the operator $FV \equiv bV_{xx} - V_{tt} - aVV_t$. Let us introduce a grid of uniform nodes for each variable

$$x_i = \frac{il}{N},\ i = \overline{0,N},\ t_k = \frac{kT}{M},\ k = \overline{0,M},\ (x_i, t_k) \in [0,l] \times [0,T].$$

Denote $V(x_i, t_k) = V_i^k$, $h = \frac{l}{N}$, $\tau = \frac{T}{M}$.

On the cross template we obtain the following finite-difference scheme

$$\frac{\partial^2 V(x_i, t_k)}{\partial x^2} = \frac{V_{i+1}^k - 2V_i^k + V_{i-1}^k}{h^2} + O(h^2),$$

$$\frac{\partial^2 V(x_i, t_k)}{\partial t^2} = \frac{V_i^{k+1} - 2V_i^k + V_i^{k-1}}{\tau^2} + O(\tau^2),$$

$$\frac{\partial V(x_i, t_k)}{\partial t} = \frac{V_i^{k+1} - V_i^{k-1}}{2\tau} + O(\tau^2) \text{ or } \frac{\partial V(x_i, t_k)}{\partial t} = \frac{V_i^{k+1} - V_i^k}{\tau} + O(\tau).$$

The values of the unknown function at the grid points on the time layer t_0 are determined from the initial data (2):

$$V_i^0 = g_1(x_i),\ i = \overline{0, N}.$$

For $t = t_1$ we have

$$V_0^1 = f_1(t_1),\ V_i^1 \approx \tau g_2(x_i) + V_i^0,\ i = \overline{1, N-1},\ V_N^1 = f_2(t_1).$$

Let us limit ourselves to infinitesimal accuracy of the second order. Then on the layer $k,\ k = 2, 3, \ldots, M-1$, we have

$$V_0^k = f_1(t_k),\ \frac{b}{h^2}\left[V_{i+1}^k - 2V_i^k + V_{i-1}^k\right] = \frac{1}{\tau^2}\left[V_i^{k+1} - 2V_i^k + V_i^{k-1}\right] +$$

$$+\frac{a}{2\tau} V_i^k \left[V_i^{k+1} - V_i^{k-1}\right],\ i = \overline{0, N-1},\ V_N^k = f_2(t_k).$$

Therefore

$$V_0^k = f_1(t_k),\ V_N^k = f_2(t_k),$$

$$V_i^{k+1} = \frac{\frac{b}{h^2}\left[V_{i+1}^k - 2V_i^k + V_{i-1}^k\right] - \frac{1}{\tau^2}(V_i^{k-1} - 2V_i^k) + \frac{a}{2\tau} V_i^k V_i^{k-1}}{\frac{1}{\tau^2} + \frac{a}{2\tau} V_i^k},\ i = \overline{0, N-1}.$$

Finally, we obtain

$$V_0^k = f_1(t_k),\ V_N^k = f_2(t_k),$$

$$V_i^{k+1} = \frac{2b\tau^2\left(V_{i+1}^k - 2V_i^k + V_{i-1}^k\right) + 2h^2(2V_i^k - V_i^{k-1}) + a\tau h^2 V_i^k V_i^{k-1}}{h^2\left(2 + a\tau V_i^k\right)}, \tag{3}$$

$$i = \overline{0, N-1},\ k = \overline{2, M-1}.$$

The scheme (3) is an explicit difference scheme with accuracy order $O(h^2 + \tau)$. The accuracy order of the time variable can be increased to $O(h^2 + \tau^2)$. To do this, we could use a more accurate approximation on the first time layer based on the initial data (see, for example, [4]). However, the above scheme (3) has an important advantage. There is no need to solve nonlinear algebraic equations on each layer, so there are no problems with branching solutions.

1.1. Stability

The explicit difference scheme (3) is conditionally stable. Stability is achieved by fulfilling the Courant condition (see, for example, [5]), linking the grid steps τ and h in a temporal and spatial variable, respectively, with the coefficients a and b of the equation (1). In our case, a sufficient stability condition has the form $b\tau < h$ It should be noted that in the problem of blood movement of interest to us, the parameter b takes sufficiently large values and to fulfill this condition it is necessary to vary the step h. In order to avoid this limitation, we will replace the variables in the equation (1):

$$\tau = at, \ z = \frac{ax}{\sqrt{b}}.$$

Then the main equation takes the form

$$\frac{\partial^2 V(z,\tau)}{\partial z^2} = \frac{\partial^2 V(z,\tau)}{\partial \tau^2} + V(z,\tau)\frac{\partial V(z,\tau)}{\partial \tau}. \tag{4}$$

The finite-difference scheme for the equation (4) is constructed in a similar way.

1.2. Numerical Example

To illustrate the convergence of the proposed numerical scheme, consider the following model boundary value problem

$$V_{xx} = V_{tt} + VV_t, \tag{5}$$

$$\begin{cases} V(0,t) = \frac{6}{6-t}, \\ V(l,t) = \frac{6}{6+2l-t}, \\ V(x,0) = \frac{3}{x+3}, \\ V_t(x,0) = \frac{3}{2(x+3)^2} \end{cases}, \quad x \in [0,l], \ t \in [0,T], \ l = 1, \ T = 1. \tag{6}$$

Here $V(x,t) = \frac{6}{6+2x-t}$ is the exact solution of the problem (5)–(6).

Table 1 shows the results of solving the problem (5)–(6). Here **h** is the grid step by spatial variable, τ is the grid step by time variable, $\varepsilon = \max\limits_{i,k} |V(x_i,t_k) - V_i^k|$.

Table 1

The error of the scheme at different grid steps

h	0.5	0.2	0.1	0.05	0.01	0.005
τ	0.25	0.04	0.01	0.0025	0.0001	0.000025
ε	0.001	0.0002	5.2e-5	1.3e-5	9.2e-7	2.4e-7

The results of Table 1 confirm the theoretical error estimate $\varepsilon = O(h^2 + \tau)$.

2. The Problem of the Movement of Blood in a Blood Vessel

In this section, we will conduct a comparative analysis of solutions to the problem of blood flow in a vessel in a rectangle $(x,t) \in [0,0.2] \times [0,2]$ obtained at different parameter values.

The model parameters that determine the coefficients of the equation (1), as well as the initial and boundary conditions (2) (diastole time, rigidity of the vessel walls after bifurcation, vessel radius, modulus of elasticity of the vessel wall, etc.) vary.

First, let us specify the parameters that kept constant values during calculations:

$$l = 0.2,\ T = 2,\ t_s = 0.3,\ n = 2,\ P_0 = 2 \cdot 10^4,\ \lambda = 0.04,\ \nu = 0.4,\ \delta = 0.001,$$

$$\beta_0 = 10^{-4},\ \rho_0 = 10^3,\ E' = \frac{E}{1-\nu^2},\ \omega = \frac{2\pi}{t_s + t_d},\ \beta_1 = 9.2,\ x_0 = 0.07,\ v_0 = 6,$$

$$\tau = \frac{l}{v_0},\ S_2 = \pi r_0^2,\ E_{pr} = \frac{E}{1 + \frac{2r_0}{\delta E' \beta_0}}.$$

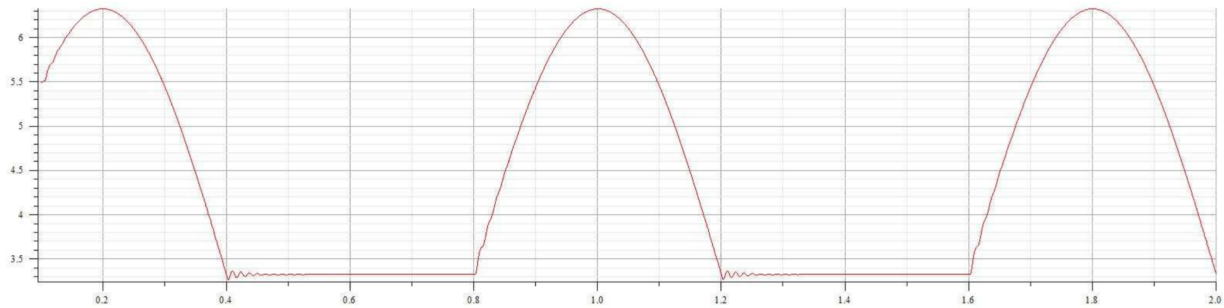

Fig. 1. Pulse wave 1

Figure 1 shows the dependence of the pulse wave velocity on time for the case when the diastole time is a fixed non-random value $t_d = 0.5$. The variable parameters of the boundary value problem here take the following values:

$$E = 10^6,\ \gamma = 3,\ \beta = 0.1,\ r_0 = 0.01.$$

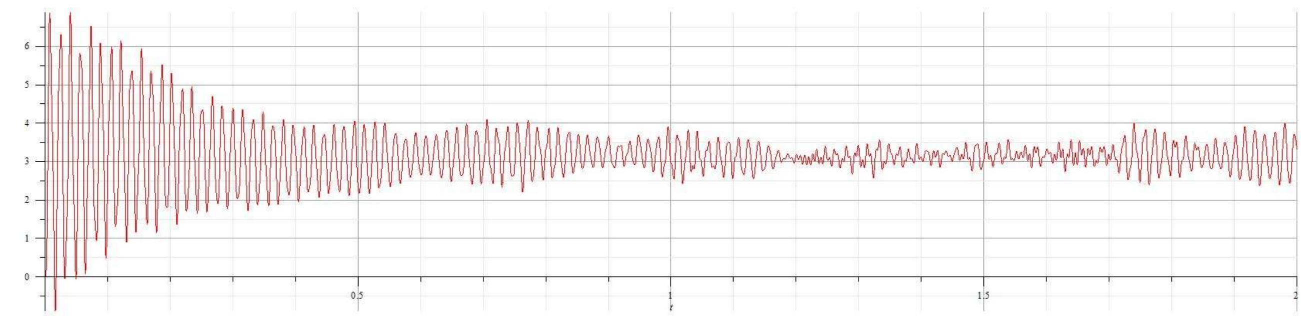

Fig. 2. Pulse wave 2

Figure 2 shows the case when the time of the diastole t_d changes randomly within $[0.1, 0.5]$. It can be noted that the random nature of the change t_d manifests itself in the form of beats, while, as can be seen, the periodicity of the process is lost.

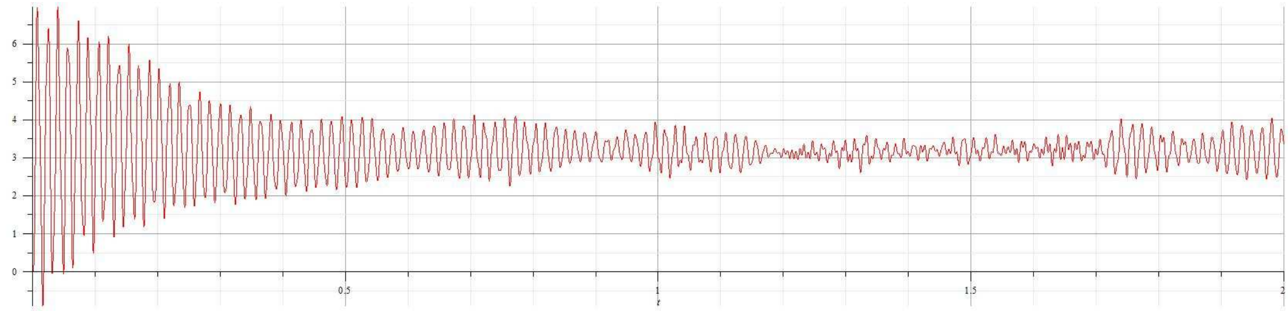

Fig. 3. Pulse wave 3

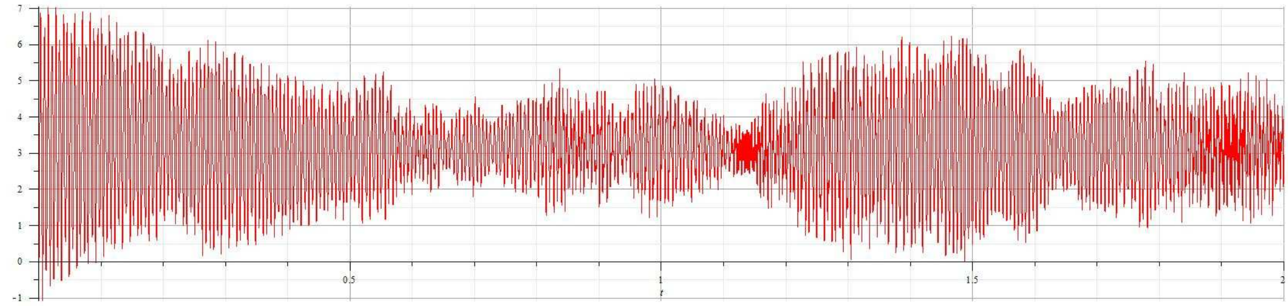

Fig. 4. Pulse wave 4

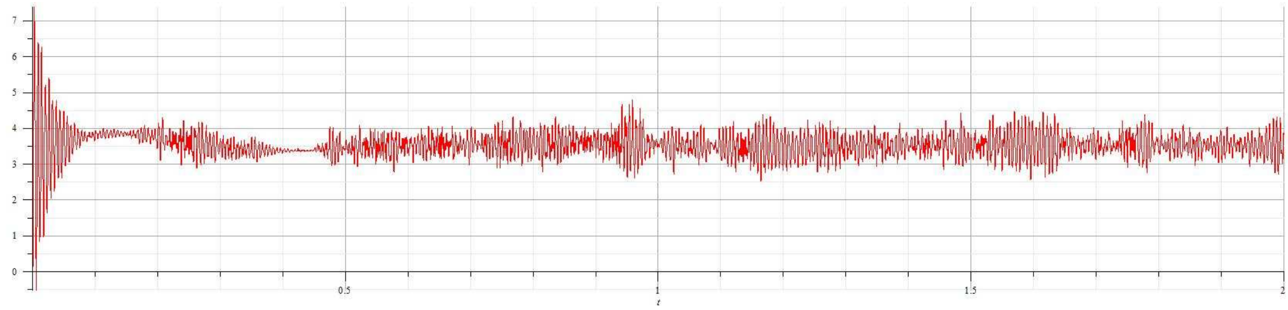

Fig. 5. Pulse wave 5

When changing the parameter γ responsible for the rigidity of the vessel walls after bifurcation, the amplitude of the beats changes slightly, as can be seen from the comparison of Fig. 3 ($\gamma = 1.1$) and Fig. 2 ($\gamma = 3$).

With increasing rigidity of the vessel, the amplitude and frequency of beats change markedly. This is evidenced by the comparison of Fig. 4 ($E = 5 \cdot 10^6$) and Fig. 2 ($E = 10^6$).

With a decrease in the radius of the vessel r_0, as can be seen from the comparison of Fig. 5 ($r_0 = 0.001$) and Fig. 4 ($r_0 = 0.01$), the amplitude of the beats changes significantly and their frequency increases.

Thus, the developed method makes it possible to track the amplitude and frequency of heartbeats in various modes, and the numerical model can be applied in the study of atrial fibrillation.

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ЧИСЛЕННОЕ МОДЕЛИРОВАНИЕ ДВИЖЕНИЯ КРОВИ В КРОВЕНОСНОМ СОСУДЕ

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В работе предложен конечно-разностный метод решения краевой задачи для гиперболического уравнения, описывающего движение крови в кровеносном сосуде. Даны условия устойчивости метода, приведены численные результаты. Метод позволяет отследить амплитуду и частоту биений сердца в различных режимах, а численная модель может быть применена при исследовании мерцательной аритмии.

Ключевые слова: нелинейное гиперболическое уравнение; гидродинамика кровообращения; конечно-разностный метод.

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