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ALGORITHM FOR NUMERICAL SOLUTION OF THE OPTIMAL CONTROL PROBLEM FOR ONE HYDRODYNAMICS MODEL USING THE COBYLA METHOD

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The article discusses an algorithm for approximate solution of the optimal control problem for a nonlinear mathematical model of wave propagation in shallow water. The mathematical model is based on the IMBq equation or the improved modified Boussinesq equation and the Dirichlet boundary conditions and the Showalter-Sidorov initial conditions. The model under study is related to the Sobolev models, since it is based on a degenerate equation. The proposed algorithm combines the phase space method, the Galerkin method, the Ritz method, the decomposition method and the COBYLA method.

Keywords: modified Boussinesq equation; optimal control; numerical study; second order semilinear Sobolev type equation; COBYLA method.

Introduction

Let us consider the inhomogeneous modified Boussinesq equation (IMBq equation)

$$(\lambda - \Delta)x_{tt} - \alpha^2 \Delta x - \Delta(x^3) = u(s,t), \quad (s,t) \in (0,l) \times (0,T) \tag{1}$$

with homogeneous Dirichlet boundary condition

$$x(0,t) = x(l,t) = 0, \quad t \in \times(0,T) \tag{2}$$

and the Showalter – Sidorov initial conditions

$$(\lambda - \Delta)(x(s,0) - x_0(s)) = 0, \quad (\lambda - \Delta)(x_t(s,0) - x_1(s)) = 0, \quad s \in (0,l) \tag{3}$$

or Cauchy initial conditions

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in (0,l), \tag{4}$$

where $\lambda, \alpha \in \mathbb{R}$.

The equation has many applications in various fields of natural science. For example, it models the propagation of waves in shallow water taking into account capillary effects. In this case, the function $u = u(x,t)$ determines the wave height. In [1], the properties of solutions to the Cauchy problem for a non-degenerate IMBq equation in a one-dimensional domain are studied. In [2], the existence of a unique global solution to the Cauchy problem for equation (1) is proved, with $\lambda = 1$, $\alpha = 1$. In [3], conditions are obtained under which the solution collapses.

For the mathematical model (1)–(3) (or (1), (2), (4)) we pose an optimal control problem. To do this, we introduce the control space $\mathfrak{U}$ and select in it a non-empty, closed and convex set $\mathfrak{U}_{ad}$, which we call the set of admissible controls

$$J(x,u) \to \inf, \quad u \in \mathfrak{U}_{ad}, \tag{5}$$

in which the objective functional is given by the formula

$$J(x,u) = \beta \int_0^T \sum_{k=0}^{1} (\|x(t) - z(t)\|_{L^4}^4 + \|x'(t) - z'(t)\|_{L^2}^2) dt + (1-\beta) \int_0^T \|u(t)\|_{\mathfrak{U}}^2 dt, \tag{6}$$

here $z(t)$ is the desired state of the system, $\beta \in (0,1)$ is the weighting coefficient. The choice of the functional is determined by two factors. Firstly, the most general form was chosen, which allows balancing, by choosing the weighting coefficient, between the proximity to the desired state and the volume of labor and energy costs. Secondly, the norms are determined by the theorem on the existence of a solution to the optimal control problem.

The optimal control problem is a classical optimization problem. For the first time, the optimal control problem for first-order semilinear Sobolev-type equations was studied in [4].

In this paper, the COBYLA (Constrained Optimization BY Linear Approximations) method is applied to minimize the functional. This is a powerful optimization algorithm designed to solve constrained and nonlinear optimization problems. It uses a trust region approach that uses linear approximations of the objective function to efficiently navigate the solution space without requiring gradient information, significantly reducing the number of required function calculations compared to traditional methods [5]. The COBYLA method involves constructing successive linear approximations of the objective function and constraints using the simplex method, with these approximations being optimized within a trust region at each step [6].

In addition to the Introduction and bibliography, the paper consists of three sections. The first presents algorithms for the numerical solution of both the optimization problem and the initial boundary value problem that arises during the solution. The second section presents computational experiments on the basis of which the obtained solutions are analyzed, and a comparison of the COBYLA method and the branch and bound method is made.

1. Numerical Study Algorithm

Let us describe the developed algorithm for processing information to find the optimal control function for a mathematical model of wave propagation in shallow water in steps corresponding to the blocks shown in Figure 1.

Step 1. We introduce the parameters of the problem: the parameters of the equation λ, α, the domains of variation of the independent variables, defined by their right ends l and T, the initial profile $x_0(s)$ and the initial velocity $x_1(s)$, the parameters of the functional: the desired state $z(s,t)$, the weighting coefficient β, the domain of admissible controls U_{ad}, the number of terms in the Galerkin sum M and the degree of the Ritz polynomial N; as well as the weighting coefficient for the decomposition method θ, the penalty coefficient r and the value of the admissible error δ.

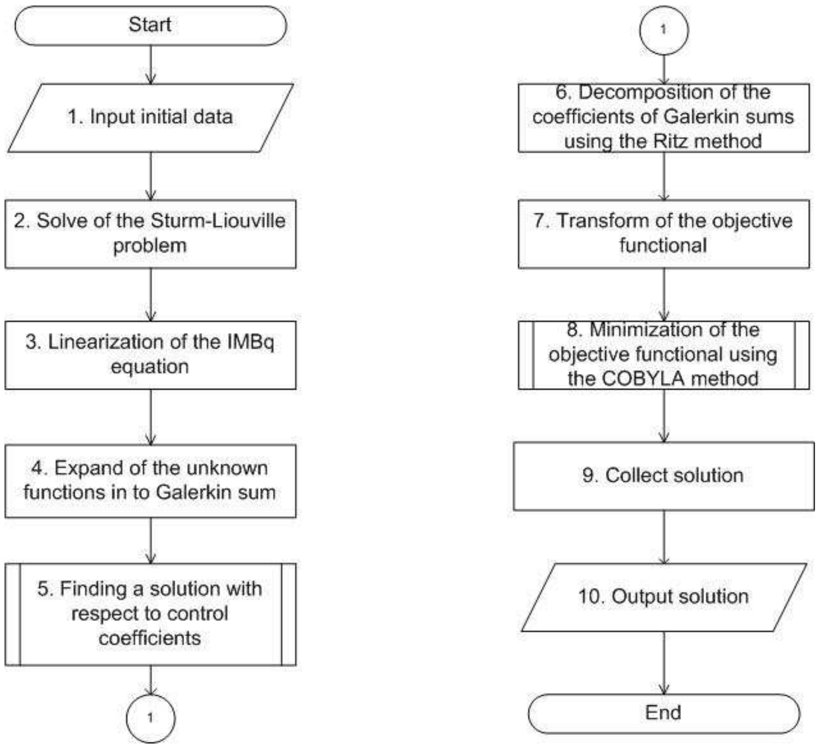


Fig. 1. Algorithm flow chart

Step 2. We find the eigenfunctions φ_i and the eigenvalues λ_i of the homogeneous Dirichlet problem for the Laplace operator.

Step 3. We apply the decomposition method. We linearize the equation for this, we introduce an auxiliary function $y = y(x,t)$, we get

$$(\lambda - \Delta)x_{tt} - \alpha^2 \Delta x - \Delta(y^3) = u(s,t),$$
$$y = x.$$

Now the equation IMBq is linear with respect to the function u.

Step 4. Let us define the desired functions as Galerkin sums

$$x(s,t) = \sum_{i=1}^{M} x_i(t)\varphi_i(s), \quad y(s,t) = \sum_{i=1}^{M} y_i(t)\varphi_i(s), \quad u(s,t) = \sum_{i=1}^{M} u_i(t)\varphi_i(s).$$

Step 5. Study of the solvability of the equation and its solution with respect to the control coefficients by the Laplace method.

Step 5.1. The initial data from the main program are passed to the input.

Step 5.2. Check the equation (1) for degeneracy if $\lambda = -\left(\frac{k\pi}{l}\right)^2$, for some $k \in \mathbb{N}$, then the equation is degenerate, otherwise the equation is non-degenerate.

Step 5.3. If the Cauchy initial conditions are given, then go to step 5.4, otherwise – to step 5.6.

Step 5.4. The membership of the initial functions (initial profile and initial velocity) in the phase space of equation (1) is checked. If they do not belong to the phase space, then go to step 5.5, otherwise – to step 5.6.

Step 5.5. The problem has no solutions. Terminate the subroutine.

Step 5.6. In the loop for i from 1 to M, the equation and initial conditions are multiplied by the eigenfunctions (step 2) in the sense of the scalar product $L^2[0,l]$. At the k-th step, an algebraic equation will appear, and the initial conditions for $u_k(t)$ of the k-th are excluded from further consideration. Then the ordinary differential equations with initial conditions and the algebraic equation are combined into an algebraic-differential system.

Step 5.7. Solve the algebraic-differential system.

Step 5.8. In the loop for i from 1 to M, the equation and initial conditions are multiplied by the eigenfunctions (step 2) in the sense of the scalar product $L^2[0,l]$. Since the equation is non-degenerate, the result is a system of ordinary linear differential equations.

Step 5.9. Solve the system of ordinary differential equations by the method of variation of an arbitrary constant.

Step 5.10. The obtained solution is output and passed to the main program. Termination of the subroutine.

Step 6. The coefficients of the Galerkin sums for the auxiliary and control functions according to the Ritz method are represented as polynomials

$$y_i(t) = \sum_{j=1}^{N} b_{ij}(t)t^j, \quad u_i(t) = \sum_{j=1}^{N} c_{ij}(t)t^j$$

It is necessary to take into account that

$$y_i(0) = x_i(0) = x_{0i}, \quad \frac{\partial y_i}{\partial t}(0) = \frac{\partial x_i}{\partial t}(0) = x_{1i}.$$

Step 7. We transform the penalty functional to take into account the introduced auxiliary function

$$\begin{gathered} J(x,u) = \beta\theta \int_0^T (\|x(s,t) - z(s,t)\|_{L^4}^4 + \|x'(s,t) - z'(s,t)\|_{L^2}^2)dt + \\ +\beta(1-\theta) \int_0^T (\|y(s,t) - z(s,t)\|_{L^4}^4 + \|y'(s,t) - z'(s,t)\|_{L^2}^2)dt + \\ +(1-\beta) \int_0^T \|u(t)\|_{\mathfrak{Y}}^2 dt + r \int_0^T (\|y(s,t) - x(s,t)\|_{L^4}^4 + \|y'(s,t) - x'(s,t)\|_{L^2}^2)dt. \end{gathered} \tag{7}$$

At this step, the parameter θ is chosen from the interval $(0,1)$, and the parameter r is as large as possible so that the solution and the auxiliary are close enough ($r = \frac{1}{\varepsilon}$, $\varepsilon \to 0$). The symbol " $\prime$ " denotes the derivative with respect to "t".

Step 8. Using the COBYLA method, we find the values of the minimum of the functional and the minimum point $c_{ij}, i = \overline{1,M}, j = \overline{0,N}$ and $b_{ij}, i = \overline{1,M}, j = \overline{0,N}$.

The stopping criterion in the optimization procedure using the COBYLA method is the modulus of the difference between two subsequent values of the functional that is less than the predetermined error value δ.

The main difference between the algorithms based on the COBYLA method and the branch and bound method is in this step. Despite the fact that the differences are technical, they are fundamental. To use the branch and bound method, it is necessary to calculate the norms and integrals, and then minimize the resulting function as a function of several variables. This procedure takes up most of the machine time, and with an increase in the number of Galerkin approximations and the degree of the Ritz polynomial, it becomes unmanageable despite the parallelization of this step. In the case of the COBYLA method, preliminary transformations are not required.

Step 9. Substituting the obtained values into the Ritz expansions (step 5) and then into the Galerkin sums (step 4), we obtain an approximate solution to problem (1)–(3), (5) (or (1), (2), (4), (5)).

Step 10. This step of the algorithm is informative in nature and consists of checking the proximity of the solution to problem (1)–(3) (or (1), (2), (4)) $u(x,t)$ obtained above and the solution to the same problem $w(s,t)$ obtained under the assumption that the control function is known. Then compare the obtained value with δ.

Step 11. The functions $u(x,t)$, $y(x,t)$ are output, and their graphs are plotted.

2. Computational Experiments

Let us present the results of information processing according to the developed algorithm. The assignment of the desired functions, the solution of differential equations was carried out in the Maple environment, and the optimization of the functional using the COBYLA method in Python, and using the branch and bound method in Maple.

Information processing was carried out on the basis of computational experiments. Several indicative ones were selected from a series of experiments with different parameters.

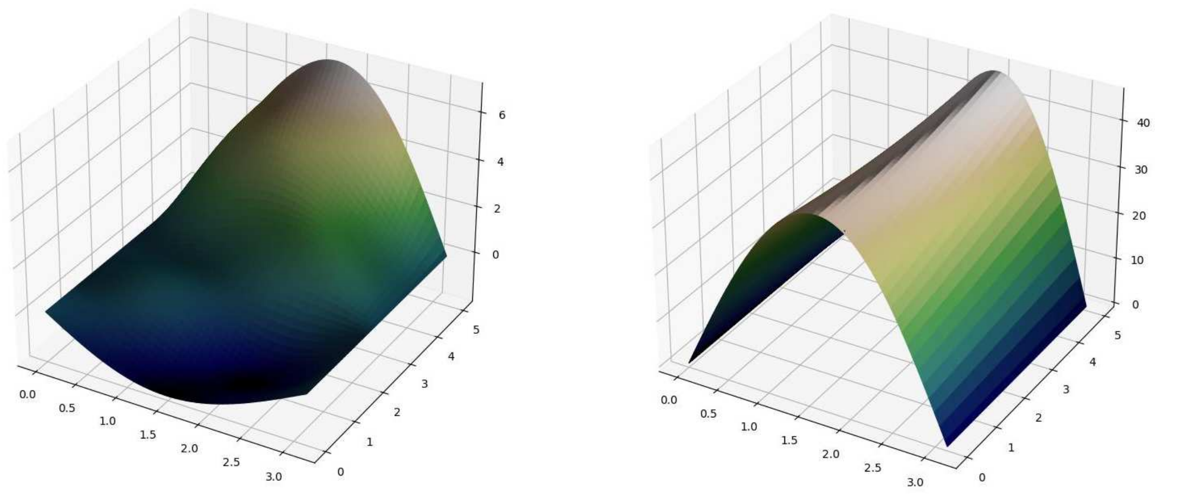
Example 1. Let the following input information be given: $M = 2$, $N = 2$ $\lambda = -1$, $\alpha = 2$, $T = 5$, $l = \pi$, $r = 100$, $\beta = 0.5$, $\theta = 0.5$, $z(s,t) = \sin(s)$, $x_0(s) = s^2(\pi - s)$, $x_1(s) = 0$.

The following results were obtained. The number of iterations performed is 287, $J_{min} = 17823.488$, and the stopping criterion was set as $|J_n - J_{n-1}| \leq 10^{-2}$. The desired coefficients (the minimum point of the objective functional): $b_{12} = 0.001371$, $b_{22} = 0.0215004$, $c_{10} = 80.71561$, $c_{20} = -0.44871$, $c_{21} = 0.028464$ $c_{22} = -0.094583$. Execution time is 1 min 58 sec.

Example 2. Let the following input information be given: $M = 2$, $N = 3$ $\lambda = -1$, $\alpha = 2$, $T = 5$, $l = \pi$, $r = 100$, $\beta = 0.5$, $\theta = 0.5$, $z(s,t) = \sin(s)$, $x_0(s) = s^2(\pi - s)$, $x_1(s) = 0$.

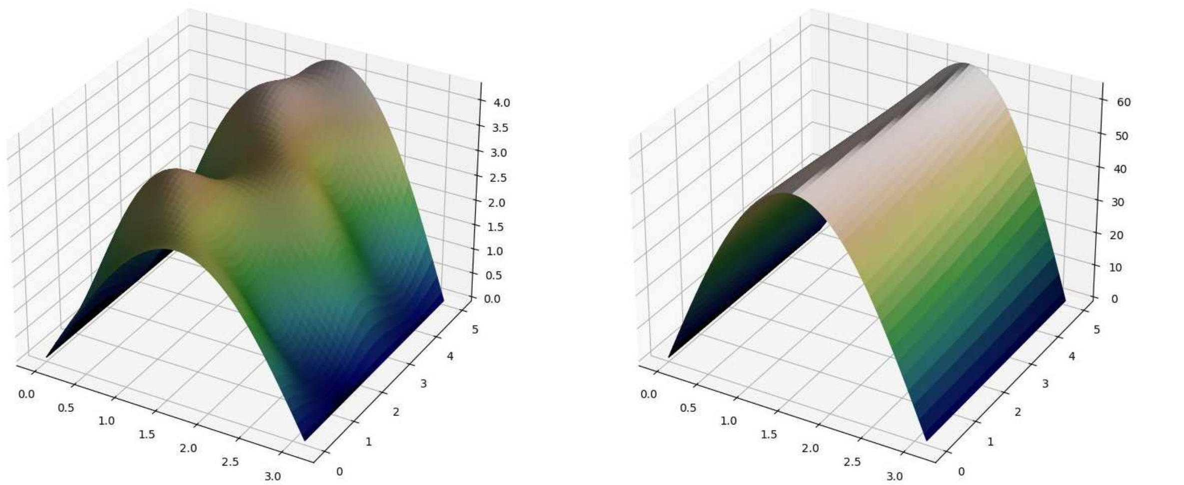
The following results were obtained. The number of iterations performed was 300, $J_{min} = 15715.597$, and the stopping criterion was set as $|J_n - J_{n-1}| \leq 10^{-2}$. The sought coefficients (the minimum point of the functional): $b_{12} = -0.00865$, $b_{13} = 0.000544$, $b_{22} = -0.0066789$, $b_{23} = -0.00146$, $c_{10} = -12.3448$, $c_{20} = -11.13007$, $c_{21} = 0.42685$, $c_{22} = -0.12035$, $c_{23} = 74.985628$. Execution time is 4 min 3 sec.

From the two examples given, it is clear that increasing the degree of the Ritz polynomial leads to a slight decrease in the minimum of the functional, all other things being equal.



a) b)

Fig. 2. The graph: a) functions $x(s,t)$; b) functions $u(s,t)$



a) b)

Fig. 3. The graph: a) functions $x(s,t)$; b) functions$u(s,t)$

Example 3. Let the following input information be given: $M = 2$, $N = 2$ $\lambda = -1$, $\alpha = 2$, $T_1 = 5$, $T_2 = 5$, $l = \pi$, $r = 100$, $\beta = 0.5$, $\theta = 0.5$, $z(s,t) = s(\pi - s)$, $x_0(s) = \sin(s) - 0.5\sin(2s)$, $x_1(s) = \sin(2s)$.

The optimization problem with the specified input data was solved using the COBYLA method and the branch and bound method. Moreover, two cases with the time intervals $[0,1]$ and $[0,5]$ were considered, all other things being equal.

The following results were obtained: for the COBYLA method, on the interval $[0,1]$ $J_{min} = 15.806$ and on the interval $[0,5]$ $J_{min} = 192.72$; for the branch and bound method on the interval $[0,1]$ $J_{min} = 15.608$ and on the interval $[0,5]$ $J_{min} = 88.488$.

It is evident that on the interval $[0,1]$ the difference between the values of the functional is insignificant, however, as the interval increases, an increase in the difference between the values of the functional is observed. This is also true for intermediate time intervals.

Note that the comparison was performed for $M = 2$, $N = 2$ since the branch and bound method for a larger number of parameters does not provide a solution to the optimal control problem in the mathematical model of wave propagation in shallow water with this functional. The result $J_{min} = 192.72$ can be improved by taking a new starting point or increasing the number of iterations.

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АЛГОРИТМ ЧИСЛЕННОГО РЕШЕНИЯ ЗАДАЧИ ОПТИМАЛЬНОГО УПРАВЛЕНИЯ ДЛЯ ОДНОЙ МОДЕЛИ ГИДРОДИНАМИКИ МЕТОДОМ COBYLA

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В статье обсуждается алгоритм приближенного решения задачи оптимального управления для нелинейной математической модели распространения волн на мелкой воде. В основе математической модели лежит IMBq уравнение или улучшенное модифицированное уравнение Буссинеска и краевые условия Дирихле и начальные условия Шоуолтера – Сидорова. Исследуемая модель относится к моделям соболевского, поскольку в ее основе лежит вырожденное уравнение. В предложенном алгоритме комбинируются метод фазового пространства, метод Галеркина, метод Ритца, метод декомпозиции и метод COBYLA.

Ключевые слова: модифицированное уравнение Буссинеска; оптимальное управление; численное исследование; полулинейное уравнение соболевского типа второго порядка.

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