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LINEARIZED OSKOLKOV SYSTEM OF NON-ZERO ORDER IN THE AVALOS–TRIGGIANI PROBLEM

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The Avalos-Triggiani problem for a linearized Oskolkov system of non-zero order and a system of wave equations is investigated. The mathematical model contains a linearized Oskolkov system and a wave vector equation corresponding to some structure immersed in the Kelvin–Voight incompressible viscoelastic fluid of non-zero order. Using the method proposed by the authors of this problem the theorem of the existence of the unique solution to the indicated Avalos–Triggiani problem is proved. The results of this article summarize the results obtained earlier.

Keywords: Avalos–Triggiani problem; incompressible viscoelastic fluid; linearized Oskolkov system.

Introduction

The system of equations

$$
\begin{gathered}
(1-\lambda\nabla^2)v_t = \eta\nabla^2 v - (v\cdot\nabla)v + \sum_{l=1}^{K}\beta_l\nabla^2\mathbf{w}_l - \nabla p + f,\\
\nabla\cdot v = 0,\\
\frac{\partial \mathbf{w}_l}{\partial t} = v + \alpha_l\mathbf{w}_l,\\
\alpha_l\in\mathbb{R}_-,\quad l=\overline{1,\ K},
\end{gathered}
$$

models the dynamics of an incompressible viscoelastic Kelvin — Voight fluid of order K [1]. Function $v = (v_1,\ v_2,\ \ldots,\ v_n)$, $v_i = v_i(x,t)$, has the physical meaning of flow velocity, the function $p = p(x,t)$ corresponds to the pressure of the liquid. Here $x\in\Omega$, $\Omega\subset\mathbb{R}^n$, $n = 2,3,$ is a bounded domain with a boundary $\partial\Omega$ of the class $\mathcal{C}^\infty$. The parameters $\eta\in\mathbb{R}_+$ and $\lambda\in\mathbb{R}$ characterize the viscous and elastic properties of the liquid, respectively. The parameters $\beta_l\in\mathbb{R}_+,\ l=\overline{1,\ K}$, determine the pressure retardation (delay) time.

Earlier, we considered the Cauchy — Dirichlet problem

$$
\begin{aligned}
&v(x,0)=v_0(x), && p(x,0)=p_0(x),\\
&\mathbf{w}_l(x,0)=\mathbf{w}_{l_0}(x), && \forall x\in\Omega,\\
&v(x,t)=0, && \mathbf{w}_l(x,t)=0,\quad l=\overline{1,\ K},\quad \forall(x,t)\in\partial\Omega\times\mathbb{R}
\end{aligned}
$$

for this system under the assumption that the free term $f = (f_1,\ f_2,\ \ldots,\ f_n)$, characterizing the external effect on the liquid, does not depend on time [2]. The approach proposed in [3, 4], and developed in [5], allows us to consider the Avalos–Triggiani problem (in the future the AT problem) for the corresponding linearized Oskolkov system of non-zero order.

We will be interested in the unique solvability of the AT problem for the linearized Oskolkov system of non-zero order. It is convenient to study this problem within the framework of the theory of Sobolev-type equations. So we consider the AT problem for the linearized Oskolkov system of non-zero order as a concrete interpretation of the corresponding abstract problem.

1. Physical and Mathematical Model

Let Ω be a bounded domain in $\mathbb{R}^n, n = 2, 3$, with sufficiently smooth boundary $\partial\Omega$. Let $u = \mathrm{col}(u_1, u_2, ..., u_n)$ be a $n-$dimensional velocity vector $n = 2, 3$, the scalar function p be a pressure, and the vector $w = \mathrm{col}(w_1, w_2, ..., w_n)$ be a vector of displacement of a body, which occupies the domain Ω_s, and is immersed in a fluid occupying the domain Ω_f. Therefore, $\Omega = \Omega_s \cup \Omega_f, \overline{\Omega}_s \cap \overline{\Omega}_f = \partial\Omega_s \equiv \Gamma_s$ is the common boundary of Ω_s, and Ω_f and denote $(0, T] \times \Omega_f$ as Ω_{Tf}. Let us denote the outer boundary of Ω_f by Γ_f (Fig. 1). Our goal is to investigate the AT problem [3, 4] for the case when the fluid in Ω_f is an incompressible viscoelastic Kelvin–Voight fluid discribed by the linearized Oskolkov system of non-zero order. Note that here the vector-function $\tilde{u} = \mathrm{col}(\tilde{u}_1, \tilde{u}_2, ..., \tilde{u}_n)$ corresponds to the stationary solution of the original system [1] . The mathematical model in question is defined by the system

$$(1 - \lambda\nabla^2)u_t - \eta\nabla^2 u - (\tilde{u} \cdot \nabla)u - (u \cdot \nabla)\tilde{u} - \sum_{l=1}^{K} \beta_l \nabla^2 \mathbf{w}_l + \nabla p = 0, \ \forall (t, x) \in \Omega_{Tf}, \quad (1)$$

$$\frac{\partial \mathbf{w}_l}{\partial t} = u + \alpha_l \mathbf{w}_l, \qquad \alpha_l \in \mathbb{R}_-, \qquad \beta_l \in \mathbb{R}_+, \qquad l = \overline{1,\ K}, \qquad \forall (t, x) \in \Omega_{Tf}, \quad (2)$$

$$\nabla \cdot u = 0, \qquad \forall (t, x) \in \Omega_{Tf}, \quad (3)$$

$$w_{tt} - \nabla^2 w + w = 0, \qquad \forall (t, x) \in (0, T] \times \Omega_s \equiv \Omega_{Ts} \quad (4)$$

with the boundary value conditions

$$u|_{\Gamma_f} \equiv 0, \qquad \forall (t, x) \in (0, T] \times \Gamma_f \equiv \Gamma_{Tf}, \quad (5)$$

$$\mathbf{w}_l|_{\Gamma_f} \equiv 0, \qquad \forall (t, x) \in \Gamma_{Tf}, \quad (6)$$

$$u \equiv w_t, \qquad \forall (t, x) \in (0, T] \times \Gamma_s \equiv \Gamma_{Ts}, \quad (7)$$

$$\frac{\partial u}{\partial \nu} - \frac{\partial w}{\partial \nu} = p\nu, \qquad \forall (t, x) \in \Gamma_{Ts} \quad (8)$$

and the initial value condition

$$(w(0, \cdot), w_t(0, \cdot), \mathbf{w}_1(0, \cdot), \ldots, \mathbf{w}_K(0, \cdot), u(0, \cdot)) = (w_0, w_1, \mathbf{w}_{10}, \ldots, \mathbf{w}_{K0}, u_0) \in \mathbf{H}, \quad (9)$$

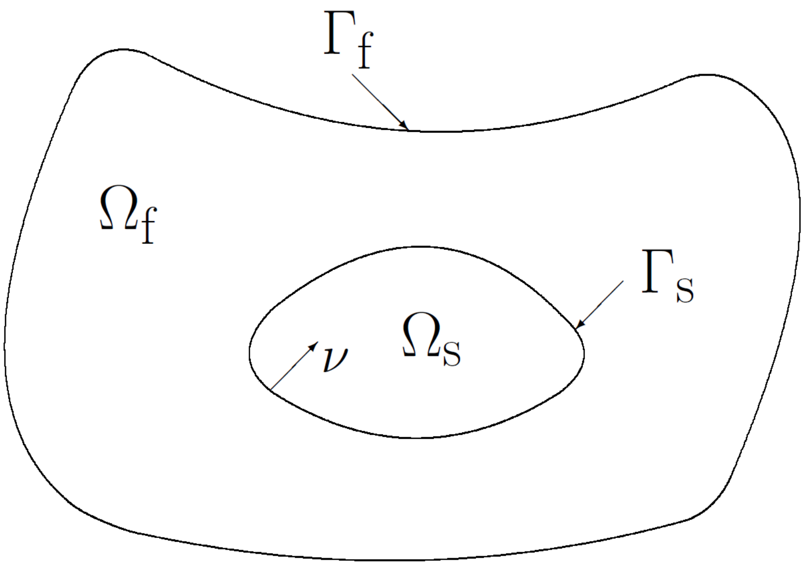

Fig. 1. Physical model

where $\mathbf{H} = (H^1(\Omega_s))^n \times (L^2(\Omega_s))^n \times \mathcal{H}_1 \times \ldots \times \mathcal{H}_K \times \mathcal{H}_f$ and $\mathcal{H}_l = (L^2(\Omega_s))^n, l = \overline{1,\ K}$, $\mathcal{H}_f = \{f \in (L^2(\Omega_f))^n : \nabla \cdot f = 0 \text{ in } \Omega_f \text{ and } [f \cdot \nu]|_{\Gamma_f} = 0\}$.

In the system (1)–(9), the parameters λ and η characterize the elastic and viscous properties of the fluid, respectively, ν is a unit normal vector, the parameters $\beta_l,\ l = \overline{1,\ K}$ determine the time of pressure retardation (delay). In the case of $\lambda = 0$, $K = 0$ problem (1)–(9) without clauses containing $\tilde{u}$ was investigated in [3, 4]. The AT problem for the linear Oskolkov system and a system of wave equations for $\lambda \neq 0$ was considered in [5, 6], and the AT problem for the linear Oskolkov system of non-zero order and a system of wave equations is investigated in [7, 8] . The AT problem for the linearized Oskolkov system and a system of wave equations was considered in [9]. The AT problem for the linearized Oskolkov system of non-zero order and a system of wave equations is considered for the first time and in this article the results [9] are generalized.

2. Reduction Method

Following [3, 4], we assume that $p(t)$ satisfies the following elliptic problem:

$$\begin{aligned} &\Delta p = 0 \quad \text{in} \quad \Omega_{Tf}, \\ &p = \frac{\partial u}{\partial \nu} \cdot \nu - \frac{\partial w}{\partial \nu} \cdot \nu \quad \text{on} \quad \Gamma_{Ts}, \\ &\frac{\partial p}{\partial \nu} = \Delta u \cdot \nu \quad \text{on} \quad \Gamma_{Tf}. \end{aligned} \tag{10}$$

Then due to (10) the pressure p can be represented as follows:

$$p(t) = D_s\left\{\left(\frac{\partial u(t)}{\partial \nu} \cdot \nu - \frac{\partial w(t)}{\partial \nu} \cdot \nu\right)_{\Gamma_{Ts}}\right\} + N_f((\Delta u(t) \cdot \nu)_{\Gamma_{Tf}}) \quad \text{in} \quad \Omega_{Tf};$$

where the Dirichlet map D_s is defined by the relations

$$h = D_s(g) \Leftrightarrow \begin{cases} \Delta h = 0 \quad \text{in} \quad \Omega_f, \\ h = g \quad \text{on} \quad \Gamma_s, \\ \dfrac{\partial h}{\partial \nu} = 0 \quad \text{on} \quad \Gamma_f, \end{cases}$$

and the Neumann map N_f is defined by the relations

$$h = N_f(g) \Leftrightarrow \begin{cases} \Delta h = 0 \quad \text{in} \quad \Omega_f, \\ h = 0 \quad \text{on} \quad \Gamma_s, \\ \dfrac{\partial h}{\partial \nu} = g \quad \text{on} \quad \Gamma_f. \end{cases}$$

Then original system (1)–(7), which describes the interaction of the fluid and the body immersed in the fluid, takes the form

$$(1-\lambda\nabla^2)u_t - \eta\nabla^2 u - (\tilde{u}\cdot\nabla)u - (u\cdot\nabla)\tilde{u} - \sum_{l=1}^{K}\beta_l\nabla^2\mathbf{w}_l - G_1 w - G_2 u = 0, \quad \forall (t,x)\in\Omega_{Tf}, \tag{11}$$

$$\frac{\partial \mathbf{w}_l}{\partial t} = u + \alpha_l \mathbf{w}_l, \qquad \alpha_l \in \mathbb{R}_-, \qquad \beta_l \in \mathbb{R}_+, \qquad l = \overline{1,\ K}, \qquad \forall (t,x)\in\Omega_{Tf}, \tag{12}$$

$$\nabla \cdot u = 0, \tag{13}$$

$$w_{tt} - \nabla^2 w + w = 0, \qquad \forall (t,x)\in\Omega_{Ts} \tag{14}$$

with the boundary value conditions

$$u|_{\Gamma_f} \equiv 0, \qquad \forall (t,x)\in\Gamma_{Tf}, \tag{15}$$

$$\mathbf{w}_l|_{\Gamma_f} \equiv 0, \qquad \forall (t,x)\in\Gamma_{Tf}, \tag{16}$$

$$u \equiv w_t, \qquad \forall (t,x)\in\Gamma_{Ts}, \tag{17}$$

where

$$G_1 w \equiv \nabla\left\{D_s\left\{\left(\frac{\partial w(t)}{\partial \nu}\cdot\nu\right)_{\Gamma_{Ts}}\right\}\right\} \quad \text{in} \quad \Omega_{Tf},$$

$$G_2 u \equiv -\nabla\left\{D_s\left\{\left(\frac{\partial u(t)}{\partial \nu}\cdot\nu\right)_{\Gamma_{Ts}}\right\} + N_f((\Delta u(t)\cdot\nu)_{\Gamma_{Tf}})\right\} \quad \text{in} \quad \Omega_{Tf}.$$

Let us rewrite problem (11)–(17), in which pressure is excluded, in the form of an abstract Cauchy problem:

$$L\dot{v} = Mv, \quad v(0) = v_0, \tag{18}$$

where the operators L and M are defined by the matrices respectively

$$\begin{pmatrix} I & O & O & O & \dots & O & O \\ O & I & O & O & \dots & O & O \\ O & O & I & O & \dots & O & O \\ O & O & O & I & \dots & O & O \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ O & O & O & O & \dots & I & O \\ O & O & O & O & \dots & O & A_\lambda \end{pmatrix}, \begin{pmatrix} O & I & O & O & \dots & O & O \\ \Delta - I & O & O & O & \dots & O & O \\ O & O & \alpha_1 & O & \dots & O & I \\ O & O & O & \alpha_2 & \dots & O & I \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ O & O & O & O & \dots & \alpha_K & I \\ G_1 & O & \beta_1\Delta & \beta_2\Delta & \dots & \beta_K\Delta & B + G_2 \end{pmatrix}.$$

Here $v = \mathrm{col}(w, w_t, \mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_K, u)$, $\quad A_\lambda = 1 - \lambda\nabla^2, B : u \to \eta\nabla^2 u - (\tilde{u}\cdot\nabla)u - (u\cdot\nabla)\tilde{u}$ [9]. I is a unit operator. It's domain is clear out of context.

3. Solvability of the Problem

We study problem (18) based on the results citing in [9].

Lemma 1. *Let $\lambda \in \mathbb{R}$, $\eta \in \mathbb{R}_+$, the operators L and M be linear continuous operators from $\boldsymbol{G}$ to $\boldsymbol{H}$ ($L, M \in \mathcal{L}(\boldsymbol{G}, \boldsymbol{H})$), then there exists $L^{-1} \in \mathcal{L}(\boldsymbol{H})$. Here the space $\boldsymbol{G} = (H^2(\Omega_s))^n \times (H^2(\Omega_s))^n \times \mathcal{G}_1 \times \dots \times \mathcal{G}_K \times \mathcal{G}_f$, where $\mathcal{G}_l = (H^2(\Omega_s))^n, l = \overline{1,\ K}$, $\mathcal{G}_f$ is closure according to the norm of the space $(H^2(\Omega_s))^n$ spaces of infinitely differentiable solenoid functions such that (15)–(17) are fulfilled.*

Theorem 1. *For any $\lambda \in \mathbb{R}, \eta \in \mathbb{R}_+$ and $v_0 \in \boldsymbol{G}$, there is a unique solution to problem (18) $v \in C^\infty(\mathbb{R}, \boldsymbol{G})$.*

Remark 1. Received results generalize results of the article [9] and can be generalized to the AT problem with the linearized Oskolkov system of the highest order [1]. We intend to develop our research in the direction indicated in [10, 11].

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ЛИНЕАРИЗОВАННАЯ СИСТЕМА ОСКОЛКОВА НЕНУЛЕВОГО ПОРЯДКА В ЗАДАЧЕ АВАЛОС – ТРИДЖИАНИ

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Исследуется задача Авалос–Триджиани для линеаризованной системы Осколкова ненулевого порядка и системы волновых уравнений. Математическая модель содержит линеаризованную систему Осколкова ненулевого порядка и волновое уравнение, соответствующее некоторой структуре, погруженной в несжимаемую вязкоупругую жидкость Кельвина–Фойгта. С помощью метода, предложенного авторами этой задачи, доказывается теорема существования единственного решения указанной задачи Авалос–Триджиани. Результаты данной статьи обобщают результаты, полученные ранее.

Ключевые слова: задача Авалос – Триджиани, несжимаемая вязкоупругая жидкость, линеаризованная система Осколкова.

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