MSC 35Q99 DOI: 10.14529/jcem250202

NUMERICAL STUDY OF THE NON-UNIQUITY SOLUTIONS PHENOMENON TO THE SHOWALTER–SIDOROV PROBLEM FOR THE MODEL OF NERVE IMPULSE PROPAGATION IN A RECTANGULAR MEMBRANE

O. V. Gavrilova[1]**,** gavrilovaov@susu.ru**,**
V. A. Bryzgalova[1]**,** torkaart@gmail.com
[1]South Ural State University, Chelyabisk, Russian Federation

This article presents a numerical study of a model of nerve impulse propagation in a rectangular area based on the Fitz Hugh–Nagumo system of equations. This model belongs to the class of reaction-diffusion systems describing a wide range of physico-chemical processes, including chemical reactions with diffusion and transmission of nerve impulses. Given the asymptotic stability of the model and a significant difference in the rates of change of its components, the initial problem can be reduced to a problem for a semilinear Sobolev type equation.The study touches upon the issues of non-uniqueness of the solution of the Showalter – Sidorov problem. The paper develops a computational algorithm implemented in the Maple environment based on the Galerkin method, which correctly takes into account the degeneracy of one of the equations of the system. The article provides an example of a numerical experiment for the Fitz Hugh–Nagumo system on a rectangle, illustrating the behavior of solutions depending on the parameters of the problem.

Keywords: Showalter–Sidorov problem; Fitz Hugh–Nagumo system of equations; uniqueness of solutions.

Introduction

The study of the dynamics of nerve impulses in biological systems, both numerically and analytically, is of fundamental interest for understanding the processes in cells of striated muscles, intestinal smooth muscles and the cardiovascular system [1–3]. One of the models of these processes is based on the Fitz Hugh–Nagumo system of equations [4,5]:

$$\begin{cases} \varepsilon_1 v_t = \alpha_1 \Delta v + \beta_1 w - \varkappa_1 v, \\ \varepsilon_2 w_t = \alpha_2 \Delta w + \beta_2 w - \varkappa_2 v - w^3, \end{cases} \tag{1}$$

which describes the dynamics of ion currents Na_+ and K_+ and the generation of nerve impulses in response to external influences. In this article, we will consider the degenerate case $\varepsilon_2 = 0$, when the system takes the form:

$$\begin{cases} v_t = \alpha_1 (v_{s_1 s_1} + v_{s_2 s_2}) + \beta_{12} w - \beta_{11} v, \\ 0 = \alpha_2 (w_{s_1 s_1} + w_{s_2 s_2}) + \beta_{22} w - \beta_{21} v - w^3 \end{cases} \tag{2}$$

in a rectangular area, $\Pi = ((0, l_1) \times (0, l_2)) \times \mathbb{R}_+$ with boundary conditions:

$$v(s_1, s_2, t) = w(s_1, s_2, t) = 0, \quad (s_1, s_2) \in \partial\Pi, \quad t \in (0, T). \tag{3}$$

It is in the case when $\varepsilon_2 = 0$ that the phenomenon of non-uniqueness of solutions to the Showalter – Sidorov problem arises, which was first discovered in the research of T.A. Bokareva, G.A. Sviridyuk [6] in the study of reaction-diffuse systems. As it was shown later in the works [7–12], this phenomenon appears if the phase space of the system contains features such as assembly and Whitney folds. In this paper, a study has been conducted that allows us to identify the conditions for the existence of multiple solutions, analyze the dependence of the number of solutions on the parameters, and visualize the spatiotemporal dynamics of solutions in the case of a rectangular area. The results of the study expand the understanding of the mechanisms of propagation of nerve impulses and can be applied in neurophysiology.

1. Mathematical Model in the Case of $\varepsilon_2 = 0$

In the rectangle, $\Pi = ((0, l_1) \times (0, l_2)) \times \mathbb{R}_+$ consider the Showalter – Sidorov problem

$$v(s_1, s_2, 0) = v_0(s_1, s_2). \tag{4}$$

for a system of equations (2) with a boundary condition (3). We are interested in the solvability of the problem (2) – (4) for any $x_0 = (v_0, w_0) \in \mathfrak{C}_\alpha$, where $\mathfrak{C}_\alpha = \mathfrak{C}_1^0 \oplus \mathfrak{C}_\alpha^1$, $\mathfrak{C}_1^0 = \{0\} \times W_2^1(\Pi)$, $\mathfrak{C}_\alpha^1 = \mathfrak{C}^\alpha \times \{0\}$, $\mathfrak{C}_\alpha^1 = L_4(\Pi)$.

Definition 1. *Vector function $x \in C^1((0,\tau); L_2(\Pi) \times L_2(\Pi)) \cap C((0,\tau); \mathfrak{C}_\alpha$,satisfying a system of equations (2) with a boundary condition (3), we will call the solution of the problem (2), (3). The solution $x = x(t)$ of the problem (2), (3) is called the solution of the problem (2) – (4) if $\lim\limits_{t\to 0+} ||v(s_1, s_2, t) - v_0(s_1, s_2)||_{\mathfrak{C}_\alpha} = 0$.*

The phase manifold $\mathfrak{M}$ will take the form

$$\mathfrak{M} = \left\{ u \in \mathfrak{C}_\alpha : -\langle v, \eta \rangle = \left\langle -\frac{\beta_2}{\varkappa_2} w + \frac{1}{\varkappa_2} w^3, \eta \right\rangle + \left\langle \frac{\alpha_2}{\varkappa_2} w_{s_i}, \eta_{s_i} \right\rangle \right\} \tag{5}$$

and note that all solutions of the system of equations (2) satisfying the boundary conditions (3) will lie in this set.

Theorem 1. *For any $\alpha_2, \varkappa_2 \in \mathbb{R}_+$,$\beta_2 \in (0, \alpha_2((\frac{\pi}{l_1})^2 + (\frac{\pi}{l_2})^2))$ and $x_0 \in \mathfrak{C}_\alpha$ there is a unique solution to the problem (2) – (4).*

Consider the case $\beta_{22} = \alpha_2((\frac{\pi}{l_1})^2 + (\frac{\pi}{l_2})^2)$, where $\varphi_{1,1} = \sqrt{\frac{l_1 l_2}{2}} \sin(\frac{\pi s_1}{l_1}) \sin(\frac{\pi s_2}{l_2})$, let's say $\mathfrak{C}^{\alpha\perp} = \{v^\perp \in \mathfrak{C}^\alpha : \langle v^\perp, \sqrt{\frac{l_1 l_2}{2}} \sin(\frac{\pi s_1}{l_1}) \sin(\frac{\pi s_2}{l_2}) \rangle = 0\}$, $\mathfrak{Y}_2^\perp = \{w^\perp \in \overset{\circ}{W}{}_2^1(\Pi) : \langle w^\perp, \sqrt{\frac{l_1 l_2}{2}} \sin(\frac{\pi s_1}{l_1}) \sin(\frac{\pi s_2}{l_2}) \rangle = 0\}$.

If $v \in \mathfrak{C}^{\alpha\perp}$ and $w \in \overset{\circ}{W}{}_2^1(\Pi)$ represent as $v = v^\perp + r$ and $w = w^\perp + q\sqrt{\frac{l_1 l_2}{2}} \sin(\frac{\pi s_1}{l_1}) \sin(\frac{\pi s_2}{l_2})$, where $r, q \in \mathbb{R}$, then the set $\mathfrak{P}_{\varepsilon_2}$ takes the following form:

$$\mathfrak{P}_{\varepsilon_2} = \left\{ \begin{array}{c} x \in \mathfrak{C}_\alpha : \\ \left\{ \begin{array}{l} \int\limits_\Pi -v^\perp \eta^\perp ds = \int\limits_\Pi \Big(\frac{\beta_{22}}{\beta_{21}} w^\perp \eta^\perp \frac{\alpha_2}{\beta_{21}} w_{s_i}^\perp \eta_{s_i}^\perp + \\ + \frac{1}{\beta_{21}} (w^\perp + q\sqrt{\frac{l_1 l_2}{2}} \sin(\frac{\pi s_1}{l_1}) \sin(\frac{\pi s_2}{l_2}))^3 \eta^\perp \Big) ds, \\ -\beta_{21} r = \int\limits_\Pi (w^\perp + q\sqrt{\frac{l_1 l_2}{2}} \sin(\frac{\pi s_1}{l_1}) \sin(\frac{\pi s_2}{l_2}))^3 \sqrt{\frac{l_1 l_2}{2}} \sin(\frac{\pi s_1}{l_1}) \sin(\frac{\pi s_2}{l_2}) ds \end{array} \right. \end{array} \right\}. \tag{6}$$

If we substitute instead of $\beta_{22} = \alpha_2((\frac{\pi}{l_1})^2 + (\frac{\pi}{l_2})^2)$ into the system of equations defining the set(5), then in order to to get the first equation of the system (6) instead of η in (5), substitute $\eta = \eta^{\perp}$, and then to get the second equation (6) instead of η in (5) substitute $\eta = \sqrt{\frac{l_1 l_2}{2}} \sin(\frac{\pi s_1}{l_1}) \sin(\frac{\pi s_2}{l_2})$.

Let's move on to the second equation of the system defining the set (6). Converting the resulting equation, we get:

$$
\begin{gathered}
q^3 ||\varphi_{1,1}||^4_{L_4(\Pi)} + 3q^2 \int\limits_{\Pi} w^{\perp} \sqrt{\frac{l_1 l_2}{2}} \sin(\frac{\pi s_1}{l_1}) \sin(\frac{\pi s_2}{l_2})^3 ds + \\
+ 3q \int\limits_{\Pi} (w^{\perp})^2 \sqrt{\frac{l_1 l_2}{2}} \sin(\frac{\pi s_1}{l_1}) \sin(\frac{\pi s_2}{l_2})^2 ds + \int\limits_{\Pi} \sqrt{\frac{l_1 l_2}{2}} \sin(\frac{\pi s_1}{l_1}) \sin(\frac{\pi s_2}{l_2}) (w^{\perp})^3 ds + \beta_{21} r = 0.
\end{gathered}
\tag{7}
$$

Equation (7) is a cubic equation of the general form $aq^3 + bq^2 + cq + d = 0$ with respect to q. According to the Cardano formulas, any cubic equation of the general form can be reduced to the canonical form $y^3 + py + e = 0$ with coefficients by replacing $q = y - \frac{b}{3a}$

$$
\begin{gathered}
a = ||\varphi_{1,1}||^4_{L_4(\Pi)}, \\
b = 3 \int\limits_{\Pi} w^{\perp} (\sqrt{\frac{l_1 l_2}{2}} \sin(\frac{\pi s_1}{l_1}) \sin(\frac{\pi s_2}{l_2}))^3 ds, \\
c = 3 \int\limits_{\Pi} (w^{\perp})^2 (\sqrt{\frac{l_1 l_2}{2}} \sin(\frac{\pi s_1}{l_1}) \sin(\frac{\pi s_2}{l_2}))^2 ds, \\
d = \int\limits_{\Pi} \sqrt{\frac{l_1 l_2}{2}} \sin(\frac{\pi s_1}{l_1}) \sin(\frac{\pi s_2}{l_2}) (w^{\perp})^3 ds - \beta_{21} r, \; p = \frac{3ac - b^2}{9a^2}, \\
e = \frac{1}{2} \left(\frac{2b^3}{27a^3} - \frac{bc}{3a^2} + \frac{d}{a} \right), \\
\mathfrak{Res}(q, w^{\perp}) = p^3 + e^2, \\
R(q, w^{\perp}) = q^2 \|\varphi_{1,1}\|^4_{L_4(\Pi)} + 2q \int\limits_{\Pi} w^{\perp} (\sqrt{\frac{l_1 l_2}{2}} \sin(\frac{\pi s_1}{l_1}) \sin(\frac{\pi s_2}{l_2}))^3 ds + \\
+ \int\limits_{\Pi} (w^{\perp})^2 (\sqrt{\frac{l_1 l_2}{2}} \sin(\frac{\pi s_1}{l_1}) \sin(\frac{\pi s_2}{l_2}))^2 ds.
\end{gathered}
\tag{8}
$$

For the convenience of further consideration, we introduce sets $\mathfrak{Y}^0_2 = \{w \in \overset{\circ}{W}{}^1_2(\Pi) : R(q, w^{\perp}) = 0\}$, $\mathfrak{Y}^+_2 = \{w \in \overset{\circ}{W}{}^1_2(\Pi) : \mathfrak{Res}(q, w^{\perp}) > 0\}$, $\mathfrak{Y}^-_2 = \{w \in \overset{\circ}{W}{}^1_2(\Pi) : \mathfrak{Res}(q, w^{\perp}) < 0\}$.

Theorem 2. *For any $\alpha_1, \beta_{11}, \beta_{12} \in \mathbb{R}$, $\alpha_2, \beta_{21} \in \mathbb{R}_+$, $\beta_{22} = \alpha_2((\frac{\pi}{l_1})^2 + (\frac{\pi}{l_2})^2)$, $n \leq 4$, and*
(i) for any $v_0 \in \mathfrak{C}^{\alpha} \bigcap \mathfrak{Y}^-$ there is one solution to the problem (2) – (4);
(ii) for any $v_0 \in \mathfrak{C}^{\alpha} \bigcap \mathfrak{Y}^+$, there are three solutions to the problem (2) – (4).

2. Numerical Algorithm

We describe an algorithm for numerically solving the problem (2) – (4) on the domain Π, based on the modified Galerkin–Petrov method and the phase space method. The algorithm allows you to find approximate solutions on a given area Π for given initial values $v_0(s_1,s_2)$ and coefficient values $\alpha_1,\alpha_2,\beta_{12},\beta_{22},\beta_{11},\beta_{21}$ to model the propagation of a nerve impulse in the membrane membrane, as well as to obtain graphs of approximate solutions.

Stage 0. We find the eigenfunctions $\{\varphi_{k_1,k_2}(s_1,s_2)\} = \sqrt{\frac{l_1l_2}{4}}\sin(\frac{\pi k_1 s_1}{l_1})\sin(\frac{\pi k_2 s_2}{l_2})$ homogeneous Dirichlet problem of the Laplace operator $(-\Delta)$ in the domain Π.

Stage 1. Following the Galerkin-Petrov method, we will look for an approximate solution $\tilde{x}=(\tilde{v},\tilde{w})$ of the problem in question in the form of sums

$$\begin{aligned}\tilde{v}(s_1,s_2,t) &= \sum_{k_1=1}^{m_1}\sum_{k_2=1}^{m_2} v_{k_1,k_2}(t)\sqrt{\tfrac{l_1l_2}{4}}\sin(\tfrac{\pi k_1 s_1}{l_1})\sin(\tfrac{\pi k_2 s_2}{l_2}),\\ \tilde{w}(s_1,s_2,t) &= \sum_{k_1=1}^{m_1}\sum_{k_2=1}^{m_2} w_{k_1,k_2}(t)\sqrt{\tfrac{l_1l_2}{4}}\sin(\tfrac{\pi k_1 s_1}{l_1})\sin(\tfrac{\pi k_2 s_2}{l_2}).\end{aligned} \tag{9}$$

Stage 2. To find the unknowns $v_{i,j}(t), w_{i,j}(t)$, substitute the Galerkin sums (9) into the system of equations (2), and then multiply the right and left sides of the resulting system of equations scalar in $L_2(\Pi)$ on eigenfunctions $\varphi_{i,j}(s_1,s_2) = \sqrt{\frac{l_1l_2}{4}}\sin(\frac{\pi i s_1}{l_1})\sin(\frac{\pi j s_2}{l_2})$, $i=\overline{1,m_1}$, $j=\overline{1,m_2}$, thus obtaining a system of algebraic differential equations of the form:

$$\begin{aligned}&-\alpha_1\sum_{k_1=1}^{m_1}\sum_{k_2=1}^{m_2} v_{k_1,k_2}(t)(\frac{\pi k_1}{l_1})^2+(\frac{\pi k_2}{l_2})^2\langle(\frac{\pi k_1}{l_1})^2+(\frac{\pi k_2}{l_2})^2,\sqrt{\frac{l_1l_2}{4}}\sin(\frac{\pi i s_1}{l_1})\sin(\frac{\pi j s_2}{l_2})\rangle+\\ &+\beta_{12}\sum_{k_1=1}^{m_1}\sum_{k_2=1}^{m_2} w_{k_1,k_2}(t)\langle\sqrt{\frac{l_1l_2}{4}}\sin(\frac{\pi k_1 s_1}{l_1})\sin(\frac{\pi k_2 s_2}{l_2}),\sqrt{\frac{l_1l_2}{4}}\sin(\frac{\pi i s_1}{l_1})\sin(\frac{\pi j s_2}{l_2})\rangle-\\ &-\beta_{11}\sum_{k_1=1}^{m_1}\sum_{k_2=1}^{m_2} v_{k_1,k_2}(t)\langle\sqrt{\frac{l_1l_2}{4}}\sin(\frac{\pi k_1 s_1}{l_1})\sin(\frac{\pi k_2 s_2}{l_2}),\sqrt{\frac{l_1l_2}{4}}\sin(\frac{\pi i s_1}{l_1})\sin(\frac{\pi j s_2}{l_2})\rangle=0,\end{aligned} \tag{10}$$

$$\begin{aligned}&\sum_{k_1=1}^{m_1}\sum_{k_2=1}^{m_2}\frac{d}{dt}w_{k_1,k_2}(t)\langle\sqrt{\frac{l_1l_2}{4}}\sin(\frac{\pi k_1 s_1}{l_1})\sin(\frac{\pi k_2 s_2}{l_2}),\sqrt{\frac{l_1l_2}{4}}\sin(\frac{\pi i s_1}{l_1})\sin(\frac{\pi j s_2}{l_2})\rangle+\\ &\alpha_2\sum_{k_1=1}^{m_1}\sum_{k_2=1}^{m_2} w_k(t)((\frac{\pi k_1}{l_1})^2+(\frac{\pi k_2}{l_2})^2)\langle\sqrt{\frac{l_1l_2}{4}}\sin(\frac{\pi k_1 s_1}{l_1})\sin(\frac{\pi k_2 s_2}{l_2}),\\ &\sqrt{\tfrac{l_1l_2}{4}}\sin(\tfrac{\pi i s_1}{l_1})\sin(\tfrac{\pi j s_2}{l_2})\rangle-\\ &-\beta_{22}\sum_{k_1=1}^{m_1}\sum_{k_2=1}^{m_2} w_{k_1,k_2}(t)\langle\sqrt{\frac{l_1l_2}{4}}\sin(\frac{\pi k_1 s_1}{l_1})\sin(\frac{\pi k_2 s_2}{l_2}),\sqrt{\frac{l_1l_2}{4}}\sin(\frac{\pi i s_1}{l_1})\sin(\frac{\pi j s_2}{l_2})\rangle+\\ &\beta_{21}\sum_{k_1=1}^{m_1}\sum_{k_2=1}^{m_2} v_{k_1,k_2}(t)\langle\sqrt{\tfrac{l_1l_2}{4}}\sin(\tfrac{\pi k_1 s_1}{l_1})\sin(\tfrac{\pi k_2 s_2}{l_2}),\sqrt{\tfrac{l_1l_2}{4}}\sin(\tfrac{\pi i s_1}{l_1})\sin(\tfrac{\pi j s_2}{l_2})\rangle+\\ &+\langle\left(\sum_{k_1=1}^{m_1}\sum_{k_2=1}^{m_2} w_{k_1,k_2}(t)\sqrt{\tfrac{l_1l_2}{4}}\sin(\tfrac{\pi k_1 s_1}{l_1})\sin(\tfrac{\pi k_2 s_2}{l_2})\right)^3,\sqrt{\tfrac{l_1l_2}{4}}\sin(\tfrac{\pi i s_1}{l_1})\sin(\tfrac{\pi j s_2}{l_2})\rangle=0.\end{aligned} \tag{11}$$

Stage 3. Find $v_{k_1,k_2}(0)$ by scalar multiplying by $L_2(\Pi)$ initial condition (4) for eigenfunctions $\sqrt{\frac{l_1l_2}{4}}\sin(\frac{\pi i s_1}{l_1})\sin(\frac{\pi j s_2}{l_2})$, $i=\overline{1,m_1}$, $j=\overline{1,m_2}$, i.e. $\langle v(0)-v_0,\sqrt{\frac{l_1l_2}{4}}\sin(\frac{\pi i s_1}{l_1})\sin(\frac{\pi j s_2}{l_2})\rangle=0$.

Stage 4. Having solved the system of algebraic equations (10) with respect to $w_{k_1,k_2}(0)$, we obtain the values of $w_{k_1,k_2}(0)$.

Stage 5. Using the Runge–Kutta method, we find the solution of the system of algebraic differential equations (10), (11) with $v_{k_1,k_2}(0)$ found in stage 3.

Let's describe the logical structure of the program in more detail.

Step 1. The coefficients of the equations of the system α_1, α_2, β_{12}, β_{22}, β_{11}, β_{21}, the initial value function $v_0(s_1, s_2)$ for the case $\varepsilon_2 = 0$ for the initial Showalter–Sidorov condition, the parameters of the region Π, as well as the number of Galerkin approximations m_1, m_2.

Step 2. From a separate file **eigenfunction.mw** using the built-in procedure **read**, the found normalized system of functions $\varphi_{m_1,m_2}(s_1, s_2) = \sqrt{\frac{l_1 l_2}{4}} \sin(\frac{\pi k_1 s_1}{l_1}) \sin(\frac{\pi k_2 s_2}{l_2})$ for the area under consideration Π.

Step 3. The procedure **unapply** allows you to present the desired approximate solutions in the form of sums

$$v := unapply(v_{1,1}(t)\varphi_{1,1}(s_1, s_2) + v_{1,2}(t)\varphi_{1,2}(s_1, s_2) + \ldots + v_{m_1,m_2}(t)\varphi_{m_1,m_2}(s_1, s_2)),$$
$$w := unapply(w_{1,1}(t)\varphi_{1,1}(s_1, s_2) + w_{1,2}(t)\varphi_{1,2}(s_1, s_2) + \ldots + w_{m_1,m_2}(t)\varphi_{m_1,m_2}(s_1, s_2)).$$

Step 4. The expressions compiled in step 3 are substituted into the algebraic equation of the system and in the cycles **for** i **to 1 do** m_1 **end do**, **for** j **to 1 do**m_2 **end do** the resulting equation is multiplied It is based on the eigenfunctions $\varphi_{i,j}$ and is integrated into the considered domain Π using the procedure **int**. Using the built-in procedures **subs** and **solve**, with the setting **RealDomain**, we solve the resulting system of algebraic equations with respect to unknowns $v_{i,j}(0)$, $i = \overline{1, m_1}$, $j = \overline{1, m_2}$. To implement the possibility of finding solutions using the built-in procedure **save**, the initial conditions are saved in the file **usl.mw** , set of (v_0, w_{01}) is saved in the **file resh.mw**.

Step 5. Using the built-in procedure **read**, the initial conditions and one of the sets of (v_0, w_{01}) or (v_0, w_{02}) or (v_0, w_{03}) stored in the file are read **resh.mw**. In the cycles **for** i **to 1 do** m_1 **end do**, **for** j **to1 do** m_2**end do**, the left and right sides of the differential equation obtained in the third step are multiplied by their own function $\varphi_{i,j}(s_1, s_2) = \sqrt{\frac{l_1 l_2}{4}} \sin(\frac{\pi i s_1}{l_1}) \sin(\frac{\pi j s_2}{l_2})$ and integrate (**int**) in the domain under consideration Π. As a result of steps 4 and 5, we obtain a system of algebra-differential equations for determining the coefficients of approximation $v_{i,j}, w_{i,j}$.

Step 6. The system obtained in step 5 is solved with initial conditions using the built-in procedure **dsolve**.

Step 7. A solution is created and displayed on the screen as a graph using the built-in procedures **plot** or **plot3d**.

3. Numerical Experiment

Example 1. It is necessary to find a solution to the problem

$$\begin{cases} v_t = (v_{s_1 s_1} + v_{s_2 s_2}) - w + v, \\ 0 = (w_{s_1 s_1} + w_{s_2 s_2}) - 2w + v + w^3 \end{cases} \tag{12}$$

with a boundary condition

$$v(s_1, s_2, t) = w(s_1, s_2, t) = 0, (s_1, s_2) \in \partial\Pi, t \in (0, T), \tag{13}$$

and the initial condition of Showalter – Sidorov

$$v(s_1, s_2, 0) = v_0(s_1, s_2). \tag{14}$$

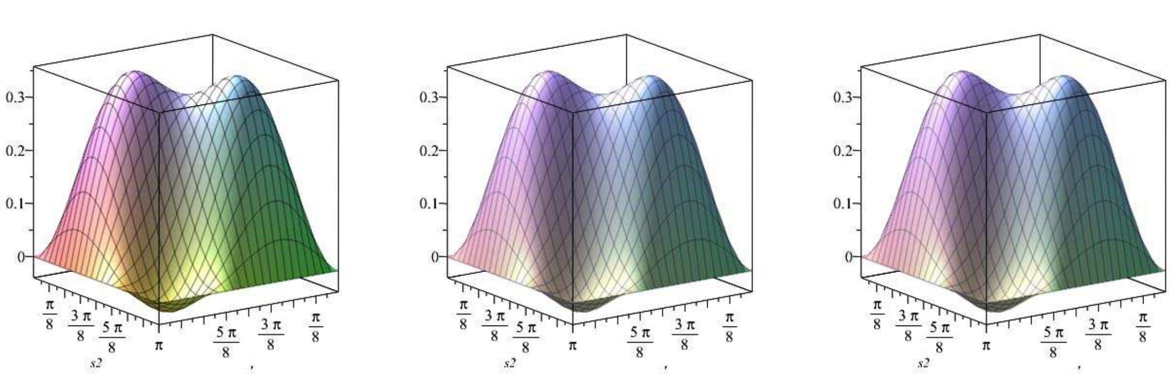
when $\Pi = (0, \pi)\times(0, \pi)$, $t = (0, T)$, $T = 1$, $\varepsilon_2 = 0$ и $v_0 = \dfrac{\sin(s1)\sin(s2)}{\pi} - \dfrac{\sin(2s1)\sin(2s2)}{2\pi}$.

According to algorithm 1, approximate solutions to the problem (12) – (14) can be represented as $\tilde{v}(s_1, s_2, t) = v_{1,1}(t)\varphi_{1,1}(s_1, s_2) + v_{2,1}(t)\varphi_{2,1}(s_1, s_2) + v_{1,2}(t)\varphi_{1,2}(s_1, s_2) + v_{2,2}(t)\cdot \varphi_{2,2}(s_1, s_2)$, $\tilde{w}(s_1, s_2, t) = w_{1,1}(t)\varphi_{1,1}(s_1, s_2) + w_{2,1}(t)\varphi_{2,1}(s_1, s_2) + w_{1,2}(t) \cdot \varphi_{1,2}(s_1, s_2) + w_{2,2}(t)\varphi_{2,2}(s_1, s_2)$. In the case of $\varepsilon_2 = 0$, according to Theorem 1.1.2, the problem (12) – (14) will have several solutions. According to the second point of the algorithm, we obtain a system of algebro-differenqialth equations.

$$-v_{1,1}(t) + w_{1,1}(t) = 0, -v_{1,2}(t) + w_{1,2}(t) = 0,$$
$$-v_{2,1}(t) + w_{2,1}(t) = 0, -v_{2,2}(t) + w_{2,2}(t) = 0.$$

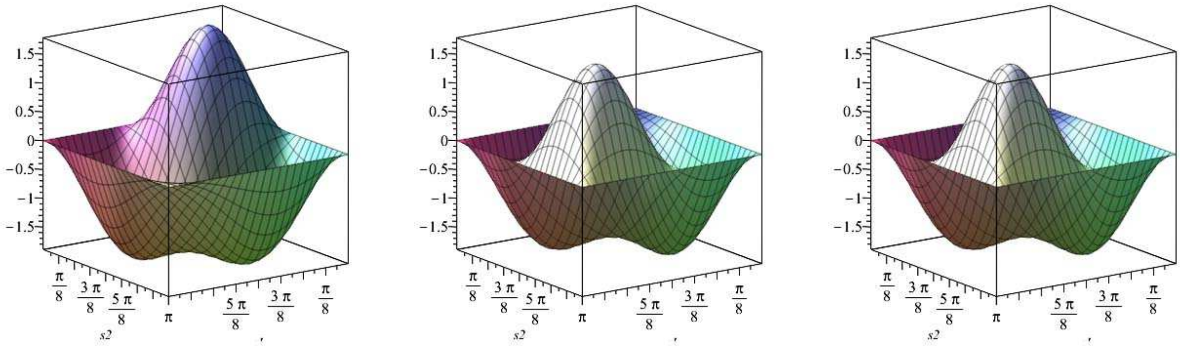
$$\frac{9w_{1,1}^3 + (-8\pi^2 + 18w_{1,2}^2 + 18w_{2,1}^2 + 12w_{2,2}^2)w_{1,1} + 4\pi^2 v_{1,1} + 24w_{1,2}w_{2,1}w_{2,2}}{4\pi^2} = 0,$$
$$\frac{9w_{1,2}^3 + (-8\pi^2 + 18w_{1,1}^2 + 12w_{2,1}^2 + 18w_{2,2}^2)w_{1,2} + 4\pi^2 v_{1,2} + 24w_{1,1}w_{2,1}w_{2,2}}{4\pi^2} = 0,$$
$$\frac{9w_{2,1}^3 + (-8\pi^2 + 18w_{1,1}^2 + 12w_{1,2}^2 + 18w_{2,2}^2)w_{2,1} + 4\pi^2 v_{2,1} + 24w_{1,1}w_{1,2}w_{2,2}}{4\pi^2} = 0,$$
$$\frac{9w_{2,2}^3 + (-8\pi^2 + 12w_{1,1}^2 + 18w_{1,2}^2 + 18w_{2,1}^2)w_{2,2} + 4\pi^2 v_{2,2} + 24w_{1,1}w_{1,2}w_{2,1}}{4\pi^2} = 0.$$

According to algorithm clause 5, solving the algebraic equation of the system with respect to v_{k_1,k_2},w_{k_1,k_2} at the initial moment of time, we obtain three solutions. Conditionally, we number the first one with the initial conditions $v_{1,1}(0) = 1/2$, $v_{1,2}(0) = 0$, $v_{2,1}(0) = 0$, $v_{2,2}(0) = -1/4, w_{1,1}(0) = -1.788274398$, $w_{1,2}(0) = 1.426880739$, $w_{2,1}(0) = 1.426880739$, $w_{2,2}(0) = 1.676481996$, the second one with initial conditions is $v_{1,1}(0) = 1/2$, $v_{1,2}(0) = 0$, $v_{2,1}(0) = 0$, $v_{2,2}(0) = -1/4, w_{1,1}(0) = -1.788274398$, $w_{1,2}(0) = -1.426880739$, $w_{2,1}(0) = -1.426880739$, $w_{2,2}(0) = 1.676481996$ and the third with initial conditions $v_{1,1}(0) = 1/2$, $v_{1,2}(0) = 0$, $v_{2,1}(0) = 0$, $v_{2,2}(0) = -1/4, w_{1,1}(0) = -1.788274398$, $w_{1,2}(0) = -1.426880739$, $w_{2,1}(0) = -1.426880739$, $w_{2,2}(0) = 1.676481996$. Solving the system of algebraic-differential equations, we obtain numerical solutions, which are shown in Fig. 1 – 4 and Tabl. 1 – 6.



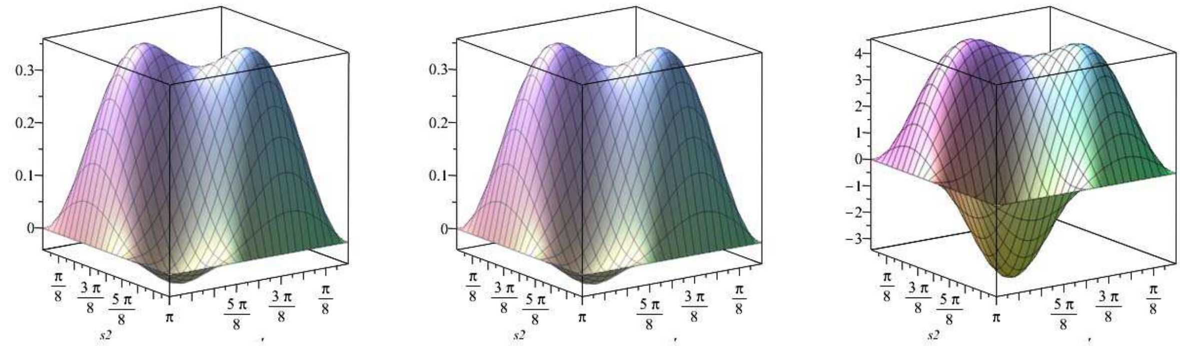
a) b) c)

Fig. 1. Solutions for the component $v(s_1, s_2, 0)$ for $k_1 = \overline{1, 2}$,$k_2 = \overline{1, 2}$, $t = 0$, where a) the first solution; b) the second solution; c) the third solution



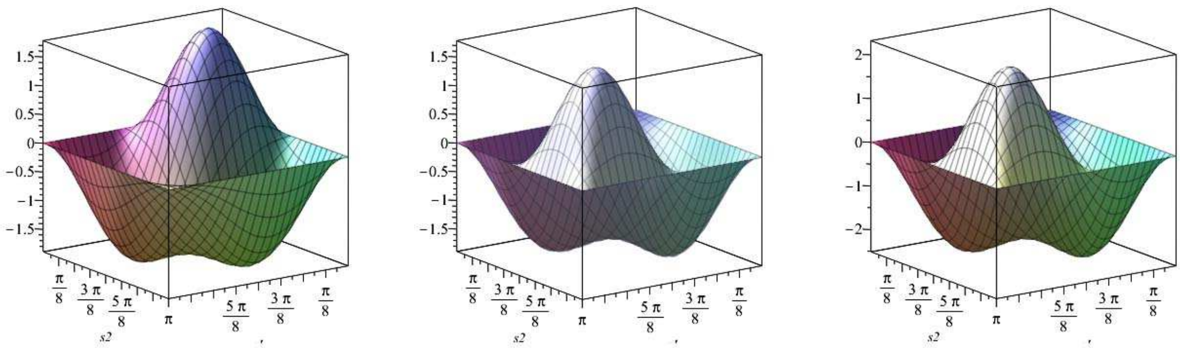
a) **b)** **c)**

Fig. 2. Solutions for the component $w(s_1, s_2, 0)$ for $k_1 = \overline{1,2}$,$k_2 = \overline{1,2}$, $t = 0$, where a) the first solution; b) the second solution; c) the third solution



a) **b)** **c)**

Fig. 3. Solutions for the component $v(s_1, s_2, 1)$ for $k_1 = \overline{1,2}$,$k_2 = \overline{1,2}$, $t = 1$, where a) the first solution; b) the second solution; c) the third solution



a) **b)** **c)**

Fig. 4. Solutions for the component $w(s_1, s_2, 1)$ for $k_1 = \overline{1,2}$,$k_2 = \overline{1,2}$, $t = 1$, where a) the first solution; b) the second solution; c) the third solution

Table 1

The first numerical solution of the problem (2) – (4)
for $k_1 = \overline{1,2}$, $k_2 = \overline{1,2}$, for the component v

t	$v_{1,1}(t)$	$v_{1,2}(t)$	$v_{2,1}(t)$	$v_{2,2}(t)$
0.0000	0.5000000000	0.0000000000	0.0000000000	-0.2500000000
0.0001	0.5002288414	-0.0001426968	-0.0001426968	-0.2501926600
0.0002	0.5004577107	-0.0002854112	-0.0002854112	-0.2503853438
0.0003	0.5006866080	-0.0004281431	-0.0004281431	-0.2505780513
0.0004	0.5009155331	-0.0005708926	-0.0005708926	-0.2507707824
0.0005	0.5011444862	-0.0007136596	-0.0007136596	-0.2509635373
0.0006	0.5013734673	-0.0008564442	-0.0008564442	-0.2511563159
0.0007	0.5016024762	-0.0009992463	-0.0009992463	-0.2513491182
0.0008	0.5018315131	-0.0011420660	-0.0011420660	-0.2515419442
0.0009	0.5020605779	-0.0012849032	-0.0012849032	-0.2517347939
0.0010	0.5022896706	-0.0014277580	-0.0014277580	-0.2519276674

Table 2

The first numerical solution of the problem (2) – (4)
for $k_1 = \overline{1,2}$, $k_2 = \overline{1,2}$, for the component w

t	$w_{1,1}(t)$	$w_{1,2}(t)$	$w_{2,1}(t)$	$w_{2,2}(t)$
0.0000	-1.7882743980	1.4268807390	1.4268807390	1.6764819960
0.0001	-1.7883247148	1.4269134426	1.4269134426	1.6765263490
0.0002	-1.7883750335	1.4269461484	1.4269461484	1.6765707038
0.0003	-1.7884253541	1.4269788563	1.4269788563	1.6766150606
0.0004	-1.7884756765	1.4270115663	1.4270115663	1.6766594193
0.0005	-1.7885260007	1.4270442784	1.4270442784	1.6767037799
0.0006	-1.7885763269	1.4270769926	1.4270769926	1.6767481423
0.0007	-1.7886266549	1.4271097089	1.4271097089	1.6767925067
0.0008	-1.7886769848	1.4271424273	1.4271424273	1.6768368729
0.0009	-1.7887273165	1.4271751478	1.4271751478	1.6768812410
0.0010	-1.7887776501	1.4272078704	1.4272078704	1.6769256111

Table 3

The second numerical solution of the problem (2) – (4)
for $k_1 = \overline{1,2}$,$k_2 = \overline{1,2}$, for the component v

t	$v_{1,1}(t)$	$v_{1,2}(t)$	$v_{2,1}(t)$	$v_{2,2}(t)$
0.00000	0.5000000000	0.0000000000	0.0000000000	-0.2500000000
0.00001	0.5000228829	0.0000142689	0.0000142689	-0.2500192649
0.00002	0.5000457660	0.0000285380	0.0000285380	-0.2500385301
0.00003	0.5000686495	0.0000428072	0.0000428072	-0.2500577955
0.00004	0.5000915332	0.0000570766	0.0000570766	-0.2500770612
0.00005	0.5001144172	0.0000713462	0.0000713462	-0.2500963271
0.00006	0.5001373015	0.0000856160	0.0000856160	-0.2501155932
0.00007	0.5001601860	0.0000998859	0.0000998859	-0.2501348595
0.00008	0.5001830709	0.0001141561	0.0001141561	-0.2501541261
0.00009	0.5002059560	0.0001284264	0.0001284264	-0.2501733930
0.00010	0.5002288414	0.0001426968	0.0001426968	-0.2501926600

Table 4

The second numerical solution of the problem (2) – (4)
for $k_1 = \overline{1,2}$, $k_2 = \overline{1,2}$, for the component w

t	$w_{1,1}(t)$	$w_{1,2}(t)$	$w_{2,1}(t)$	$w_{2,2}(t)$
0.00000	-1.7882743980	-1.4268807390	-1.4268807390	1.6764819960
0.00001	-1.7882794296	-1.4268840093	-1.4268840093	1.6764864312
0.00002	-1.7882844612	-1.4268872796	-1.4268872796	1.6764908664
0.00003	-1.7882894929	-1.4268905499	-1.4268905499	1.6764953017
0.00004	-1.7882945245	-1.4268938202	-1.4268938202	1.6764997369
0.00005	-1.7882995562	-1.4268970906	-1.4268970906	1.6765041722
0.00006	-1.7883045879	-1.4269003609	-1.4269003609	1.6765086075
0.00007	-1.7883096196	-1.4269036313	-1.4269036313	1.6765130429
0.00008	-1.7883146513	-1.4269069017	-1.4269069017	1.6765174782
0.00009	-1.7883196831	-1.4269101722	-1.4269101722	1.6765219136
0.00010	-1.7883247148	-1.4269134426	-1.4269134426	1.6765263490

Table 5

The third numerical solution to the problem (2) – (4)
for $k_1 = \overline{1,2}$, $k_2 = \overline{1,2}$, for the component v

t	$v_{1,1}(t)$	$v_{1,2}(t)$	$v_{2,1}(t)$	$v_{2,2}(t)$
0.0	0.5000000000	0.0000000000	0.0000000000	-0.2500000000
0.1	0.7432939844	0.1517932576	0.1517932576	-0.4549360384
0.2	1.0176673266	0.3232077736	0.3232077736	-0.6862921189
0.3	1.3265990914	0.5165238550	0.5165238550	-0.9470569953
0.4	1.6739431050	0.7342598678	0.7342598678	-1.2405402121
0.5	2.0639668778	0.9791984294	0.9791984294	-1.5704055540
0.6	2.5013949902	1.2544153099	1.2544153099	-1.9407083446
0.7	2.9914573206	1.5633113249	1.5633113249	-2.3559369049
0.8	3.5399414822	1.9096468196	1.9096468196	-2.8210576129
0.9	4.1532517924	2.2975803455	2.2975803455	-3.3415655900
1.0	4.8384753995	2.7317119101	2.7317119101	-3.9235415245

Table 6

The third numerical solution to the problem (2) – (4)
for $k_1 = \overline{1,2}$, $k_2 = \overline{1,2}$, for the component w

t	$w_{1,1}(t)$	$w_{1,2}(t)$	$w_{2,1}(t)$	$w_{2,2}(t)$
0.0	-1.7882743980	-1.4268807390	-1.4268807390	1.6764819960
0.1	-1.8395383935	-1.4606311852	-1.4606311852	1.7217950657
0.2	-1.8927598192	-1.4964546916	-1.4964546916	1.7690746985
0.3	-1.9480255045	-1.5343281526	-1.5343281526	1.8183839513
0.4	-2.0054174782	-1.5742410131	-1.5742410131	1.8697829419
0.5	-2.0650148141	-1.6161932935	-1.6161932935	1.9233305936
0.6	-2.1268950002	-1.6601940353	-1.6601940353	1.9790858458
0.7	-2.1911350005	-1.7062601030	-1.7062601030	2.0371085327
0.8	-2.2578121631	-1.7544153212	-1.7544153212	2.0974601005
0.9	-2.3270047674	-1.8046896491	-1.8046896491	2.1602039799
1.0	-2.3987925102	-1.8571185683	-1.8571185683	2.2254059088

References

1. Hodgkin A.L., Huxley A.F. A Quantitative Description of Membrane Current and Its Application to Conduction and Excitation in Nerve. *The Journal of Physiology*, 1952, no. 117 (4), pp. 500–544.
2. Lefever R., Prigogin I. Symmetry-Breaking Instabilities in Dissipative System. II. *American Journal of Physics*, 1968, vol. 48, pp. 1695–1700.
3. Gubernov V. V., Kolobov A. V., Polezhaev A. A. et. al. Stabilization of Combustion Wave Through the Competitive Endothermic Reaction. *Proceeding of the Royal Society A*, 2015, vol. 471, no. 2180, article ID: 20150293.
4. Fitz Hugh R. Mathematical Models of Threshold Phenomena in the Nerve Membrane. *Bulletin of Mathematical Biology*, 1955, vol. 17, no. 4, pp. 257–278.
5. Nagumo J., Arimoto S., Yoshizawa S. An Active Pulse Transmission Line Simulating Nerve Axon. *Proceedings of the IRE*, 1962, vol. 50, no. 10, pp. 2061–2070.
6. Bokarieva T.A., Sviridiuk G.A. Whitney Folds in Phase Spaces of Some Semilinear Sobolev-Type Equations.*Mathematical Notes*, 1994, vol. 55, no. 3, pp. 237–242. DOI: 10.1007/BF02110776.
7. Manakova, N.A., Gavrilova O.V., Perevozchikova K.V. Semi-linear models of the Sobolev type. Non - uniqueness of the solution of the Showalter - Sidorov problem. *Bulletin of the South Ural State University. Series: Mathematical Modelling, Programming and Computer Software*, 2022, vol. 15, no. 1, pp. 84–100. (in Russian)
8. Gilmutdinova A.F. On Nonuniqueness of Solutions to the Showalter–Sidorov Problem for the Plotnikov Model. *Vestnik of Samara University. Natural Science Series*, 2007, no 9, pp. 85–90. (in Russian).
9. Manakova, N.A., Gavrilova O.V. About Nonuniqueness of Solutions of the Showalter–Sidorov Problem for One Mathematical Model of Nerve Impulse Spread in Membrane. *Bulletin of the South Ural State University. Series: Mathematical Modelling, Programming and Computer Software*, 2018, vol. 11, no. 4, pp. 161–168.
10. Sviridyuk G.A., Kazak V.O. The Phase Space of an Initial-Boundary Value Problem for the Hoff Equation. *Mathematical Notes*, 2002, vol. 71, no. 1-2, pp. 262–266. DOI: 10.1023/A:1013919500605
11. Sviridyuk G.A., Trineeva I.K. A Whitney Fold in the Phase Space of the Hoff Equation. *Russian Mathematics (Izvestiya VUZ. Matematika)*, 2005, vol. 49, no. 10, pp. 49-55.
12. Gavrilova O.V., Nikolaeva N.G., Manakova N.A. On the Non-Uniqueness of Solutions of the Showalter–Sidorov Problem for One Mathematical Model of the Deformation of an I-Beam. *South Ural Youth School of Mathematical Modelling*, 2021, pp. 67–71. (in Russian)

Olga V. Gavrilova, PhD (Math), Associate Professor, Department of Mathematical Physics Equations, South Ural State University (Chelyabinsk, Russian Federation), gavrilovaov@susu.ru

Victoria A. Bryzgalova, Student, South Ural State University (Chelyabinsk, Russian Federation), torkaart@gmail.com

Received May 30, 2025

УДК 517.9 DOI: 10.14529/jcem250202

ЧИСЛЕННОЕ ИССЛЕДОВАНИЕ ФЕНОМЕНА НЕЕДИНСТВЕННОСТИ РЕШЕНИЙ ЗАДАЧИ ШОУОЛТЕРА – СИДОРОВА ДЛЯ МОДЕЛИ РАСПРОСТРАНЕНИЯ НЕРВНОГО ИМПУЛЬСА В МЕМБРАНЕ ПРЯМОУГОЛЬНОЙ ФОРМЫ

О. В. Гаврилова[1]***, В. А. Брызгалова***[1]
[1]Южно-Уральский государственный университет, г. Челябинск, Российская Федерация

В данной статье представлено численное исследование модели распространения нервного импульса в прямоугольной области, основанной на системе уравнений Фитц Хью – Нагумо. Эта модель относится к классу реакционно-диффузионных систем, описывающих широкий спектр физико-химических процессов, включая химические реакции с диффузией и передачу нервных импульсов. При условии асимптотической устойчивости модели и значительного различия в скоростях изменения её компонент исходная задача может быть сведена к задаче для полулинейного уравнения соболевского типа.Исследование затрагивает вопросы неединственности решения задачи Шоуолтера – Сидорова. В работе разработан вычислительный алгоритм, реализованный в среде Maple, основанный на методе Галёркина, который корректно учитывает вырожденность одного из уравнений системы. В статье приведен пример численного эксперимента для системы Фитц Хью – Нагумо на прямоугольнике, иллюстрирующий особенности поведения решения в зависимости от параметров задачи.

Ключевые слова: уравнения соболевского типа; задача Шоуолтера – Сидорова; система уравнений Фитц Хью – Нагумо; неединственность решений.

Литература

1. Hodgkin, A.L. A Quantitative Description of Membrane Current and Its Application to Conduction and Excitation in Nerve / A.L. Hodgkin, A.F. Huxley // The Journal of Physiology. – 1952. – № 117 (4). – P. 500–544.

2. Lefever, R. Symmetry-Breaking Instabilities in Dissipative System. II / R. Lefever, I. Prigogin // The Journal of Chemical Physics. – 1968. – V. 48, № 4. – P. 1695–1700.

3. Gubernov, V.V. Stabilization of Combustion Wave Through the Competitive Endothermic Reaction / V.V. Gubernov, A.V. Kolobov, A.A. Polezhaev et. al. // Proceeding of the Royal Society A. – 2015. – V. 471, №2180. – P. 20150293.

4. Fitz Hugh, R. Mathematical Models of Threshold Phenomena in the Nerve Membrane / R. Fitz Hugh // Bulletin of Mathematical Biology. – 1955. – V. 17, № 4. – P. 257–278.

5. Nagumo, J. An Active Pulse Transmission Line Simulating Nerve Axon / J. Nagumo, S. Arimoto, S. Yoshizawa // Proceedings of the IRE. – 1962. – V. 50, № 10. – P. 2061–2070.

6. Бокарева, Т.А. Сборки Уитни фазовых пространств некоторых полулинейных уравнений типа Соболева / Т.А. Бокарева, Г.А. Свиридюк // Математические заметки. – 1994. – Т. 55, № 3. – С. 3–10.

7. Манакова, Н.А. Полулинейные модели соболевского типа. Неединственность решения задачи Шоуолтера - Сидорова/ Н.А. Манакова, О.В. Гаврилова, К.В. Перевозчикова // Вестник ЮУрГУ. Серия: Математическое моделирование и программирование. – 2022. – Т. 15, № 1. – С. 84–100.
8. Гильмутдинова, А.Ф. О неединственности решений задачи Шоуолтера – Сидорова для одной модели Плотникова / А.Ф. Гильмутдинова // Вестник СамГУ. – 2007. – № 9. – С. 85–90.
9. Manakova, N.A. About Nonuniqueness of Solutions of the Showalter – Sidorov Problem for One Mathematical Model of Nerve Impulse Spread in Membrane / N.A. Manakova, O.V. Gavrilova // Вестник ЮУрГУ. Серия: Математическое моделирование и программирование. – 2018. – Т. 11, № 4. – С. 161–168.
10. Свиридюк, Г.А. Фазовое пространство начально-краевой задачи для уравнения Хоффа / Г.А. Свиридюк, В.О. Казак // Математические заметки. – 2002. – Т. 71, № 2. – С. 292–297.
11. Свиридюк, Г.А. Сборка Уитни в фазовом пространстве уравнения Хоффа / Г.А. Свиридюк, И.К. Тринеева // Известия Вузов. Математика. – 2005. – № 10. – С. 54–60.
12. Гаврилова, О.В. О неединственности решений задачи Шоуолтера – Сидорова для одной математической модели деформации двутавровой балки / О.В. Гаврилова, Н.Г. Николаева, Н.А. Манакова // Южно-Уральская молодежная школа по математическому моделированию. – 2021. – С. 67–71.

Гаврилова Ольга Витальевна, кандидат физико-математических наук, доцент, кафедра уравнений математической физики, Южно-Уральский государственный университет (г. Челябинск, Российская Федерация), gavrilovaov@susu.ru

Брызгалова Виктория Андреевна, студент, кафедра уравнений математической физики, Южно-Уральский государственный университет (г. Челябинск, Российская Федерация), torkaart@gmail.com

Поступила в редакцию 30 мая 2025 г.