MSC 35G15, 65N30 DOI: 10.14529/jcem250203

INFORMATION PROCESSING IN STATE ANALYSIS FOR STOCHASTIC DYNAMIC AND EVOLUTIONARY SYSTEMS OF WENTZELL EQUATIONS

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The paper studies stochastic dynamical and evolutionary Wentzell systems in the domain and on its boundary. In particular, a structural system analysis of various aspects of the study of Wentzell equation systems is carried out for the above Wentzell equation systems and an algorithm is constructed to process the information obtained from computational experiments and analyze the state of the stochastic dynamic and evolutionary Wentzell system at different values of their parameters.

Keywords: Wentzell's dynamical system; Wentzell's evolutionary system; Wentzell system of equations; information processing; three sigma rule; Nelson – Glicklich derivative.

Introduction

Let $\Omega = \{(\theta,\varphi) : \theta \in [0, \frac{\pi}{2}], \varphi \in [0, 2\pi)\}$ be a hemisphere in $\mathbb{R}^3$, and $\Gamma = \{\varphi : \varphi \in [0, 2\pi)\}$ is the edge of the hemisphere. For the sake of simplicity, we introduce real separable Hilbert spaces $\mathfrak{U} = \{u \in W_2^2(\Omega) \oplus W_2^2(\Gamma) : \partial_\theta u = 0\}$, $\mathfrak{F} = L_2(\Omega) \oplus L_2(\Gamma)$ and construct the spaces of *random* $\mathbf{K}$*-values. The random* $\mathbf{K}$*-values* $\eta, \kappa \in \mathbf{U_K L_2}$ have the following form

$$\xi = \sum_{j=1}^{\infty} \lambda_j \xi_j \phi_j, \quad \chi = \sum_{j=1}^{\infty} \lambda_j \chi_j \psi_j, \tag{1}$$

where $\{\phi_k\}$ is the family of eigenfunctions of the modified Laplace – Beltrami operator $\Delta_{\theta,\varphi} \in \mathcal{L}(\mathfrak{U};\mathfrak{F})$ orthonormalized in the sense of the scalar product $\langle\cdot,\cdot\rangle$ of $L_2(\Omega)$; $\{\psi_k\}$ is the family of eigenfunctions of the modified Laplace – Beltrami operator $\Delta_\varphi \in \mathcal{L}(\mathfrak{U};\mathfrak{F})$ orthonormalized in the sense of the scalar product $\langle\cdot,\cdot\rangle$ of $L_2(\Gamma)$. Let us consider the linear stochastic dynamic Wentzell system (see, e.g., [1]) on a hemisphere and on its edge

$$(\lambda - \Delta_{\theta,\varphi}) \overset{\circ}{\xi} (t) = \alpha \Delta_{\theta,\varphi} \xi + \beta \xi, \ \ \xi \in C^\infty(\mathbb{R}_+; \mathbf{U_K L_2}), \tag{2}$$

$$(\lambda - \Delta_\varphi) \overset{\circ}{\chi} (t) = \gamma \Delta_\varphi \chi + \partial_\theta \xi + \delta \chi, \ \chi \in C^\infty(\mathbb{R}_+; \mathbf{U_K L_2}), \tag{3}$$

where

$$\Delta_{\theta,\varphi} = \frac{1}{\sin\theta} \frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2},$$
$$\Delta_\varphi = \frac{\partial^2}{\partial\varphi^2}, \quad \partial_\theta = \frac{\partial}{\partial\theta}\bigg|_{\theta=\frac{\pi}{2}}.$$

Here, the symbol ∂_θ, denotes the external normal to Ω; the symbols $\overset{\circ}{\xi} (t)$ and $\overset{\circ}{\chi} (t)$ denote the Nelson – Glicklich derivative for the corresponding stochastic process. The parameters

$\alpha, \gamma, \lambda, \beta, \delta \in \mathbb{R}$ characterize the medium. Let us add to this system (2)–(3) a matching condition, which guarantees the uniqueness of the obtained solution (see, e.g., [2]) and equip it with initial conditions

$$\xi(0) = \xi_0, \ \chi(0) = \chi_0. \tag{4}$$

Let us call the solution of the problem (2)–(4) the solution of the stochastic dynamic Wentzell system.

In addition, on the $\Omega \cup \Gamma$ compact, we consider stochastic evolutionary Wentzell system [3], which equations modeling the evolution of the free surface of a filtering fluid,

$$(\lambda - \Delta_{\theta,\varphi}) \overset{\circ}{\xi} (t) = \alpha_0 \Delta_{\theta,\varphi} \xi - \beta_0 \Delta^2_{\theta,\varphi} \xi - \gamma_0 \xi, \ \xi \in C^\infty(\mathbb{R}_+; \mathbf{U_K L}_2), \tag{5}$$

$$(\lambda - \Delta_{\varphi}) \overset{\circ}{\chi} (t) = \alpha_1 \Delta_{\varphi} \chi - \beta_1 \Delta^2_{\varphi} \chi + \partial_\theta \eta - \gamma_1 \chi, \ \chi \in C^\infty(\mathbb{R}_+; \mathbf{U_K L}_2), \tag{6}$$

with a matching condition, which guarantees the uniqueness of the obtained solution and equip it with initial conditions

$$\xi(0) = \xi_0, \ \chi(0) = \chi_0. \tag{7}$$

Here, the symbol ∂_θ denotes the external normal to Ω; the symbols $\overset{\circ}{\xi} (t)$ и $\overset{\circ}{\chi} (t)$ denote the Nelson – Glicklich derivative for the corresponding stochastic process. The parameters $\alpha_0, \alpha_1, \lambda, \beta_0, \beta_1, \gamma_0, \gamma_1 \in \mathbb{R}$ characterize the medium. The solution of the problem (5)–(7) is called the solution of the stochastic evolutionary Wentzell system.

The paper, in addition to the introduction and the references, consists of two parts. The first part deals with the analysis of various aspects of investigating the solvability of stochastic systems of Wentzell equations. The second part contains a method of processing information obtained from computational experiments, which allows us to analyze the states of stochastic, dynamic and evolutionary Wentzell systems at different parameter values.

1. Analytical Solution for the Stochastic Wentzell System of Fluid Filtration Equations

We give an analytical study for the corresponding system (2)–(4) following the results in [2]. The [4] boundary conditions introduced by A.D. Wentzell in 1956, in which the order of derivatives on spatial variables at the boundary of the domain is not lower than the order on the same variables in the considered differential equation in the domain, allow describing the behavior of the solution at the boundary of the domain, which is of key importance for understanding the physics of processes occurring in the domain. In particular, in [4] it has arisen at construction of the Feller semigroup generator for multivariate diffusion processes in a bounded region.

To date, the number of scientific papers devoted to the study of this type of conditions is rapidly increasing, and it is impossible to list all the studies here. In the school of O.A. Ladyzhenskaya, the study of problems in which boundary conditions are interpreted not as limit values of the desired function and its derivative, but as a description of the processes occurring on the boundary of the region was carried out. This made it possible to solve single-phase (when the boundary of the region is covered by a film) and two-phase (when the medium is divided into two parts by a film) Wentzell problems using repeated double and simple layer potentials. In particular, D.E. Apushkinskaya and A.I. Nazarov obtained a priori estimates of the gradient of the solution of the initial boundary value problem for non-divergent parabolic equations [5], as well as for solutions of quasilinear two-phase parabolic and elliptic equations with degenerate and nongenerate Wentzell conditions at the interface [6]. Among other important works we can mention studies on the solvability of the Wentzell problem for the Laplace and Hemholtz equations by A.I. Nazarov and V.V. Lukyanov [6]. Lukyanov [7], and results on the strong solvability of quasilinear Wentzell problems for parabolic equations with discontinuous senior coefficients by D.E. Apushkinskaya, A.I. Nazarov, D.K. Palagachev and L.G. Softova [8], [9].

On the other hand, in A. Favini, G.R. Goldstein, J.A. Goldstein, S. Romanelli, E. Obrecht, S. Romanelli, K. J. Engel, G. Fragnelli [**?**, 10–16] results in the framework of semigroup theory of operators were obtained. In particular, in [11] it is shown for the first time that an operator including the Laplace operator Δ inside the region Ω and the Laplace – Beltrami operator Δ on its boundary $\partial\Omega$ is a generator of a C_0-semigroup; in [10] was found analyticity conditions for solving C_0-continuous semigroups of operators; in [15] the physical interpretation for the heat conduction equation and the string vibration equation is described. Among the applied problems considered in the context of Wentzell boundary conditions, we also note the works of A.A. Amosov, N. Krymov [17] and J.L. Diaz, L. Tello [18], in which a system of equations describing complex processes of climate evolution over a relatively long period of time obtained with the help of the global energy balance for the atmospheric surface temperature and a system of equations arising from the homogenization of complex heat transfer problems (radiative and convective), in a periodic system are considered. Note that the above works were studied only in the deterministic case. It is important to note that the considered systems reflect complex processes and require the application of methods of system analysis.

Since Wentzell systems are an important part of optimal control problems, we note that the presence of processes on the boundary of the domain affecting the dynamics of the whole system complicates optimization problems and optimal control problems (see, e.g., Luo Y. [19]). The application of optimal control theory to elliptic equations with Wentzell conditions covers a wide range of problems. For example, in the field of heat transfer, one can consider the problem of optimal control of the temperature in some object, where the boundary conditions may describe the heat transfer with the environment. Nevertheless, the use of Wentzell conditions is also associated with certain difficulties. First, their mathematical formulation may be more complex than that of classical conditions, which requires additional effort in analyzing and solving problems. Second, in some cases it can be difficult to determine which derivatives should be considered and how they should be related to the values of the function on the boundary. This may lead to uncertainty in the formulation of the problem and, as a consequence, to difficulties in finding solutions. Note that the above optimal control problems have also been studied only in the deterministic case.

In the course of synthesis on the basis of aspectual evaluations, it was concluded that, under conditions of certainty, deterministic systems of Wentzell equations are partially studied (the question of the existence of a solution is investigated). This creates prerequisites for the beginning of the system study under stochastic uncertainty (Fig. 1).

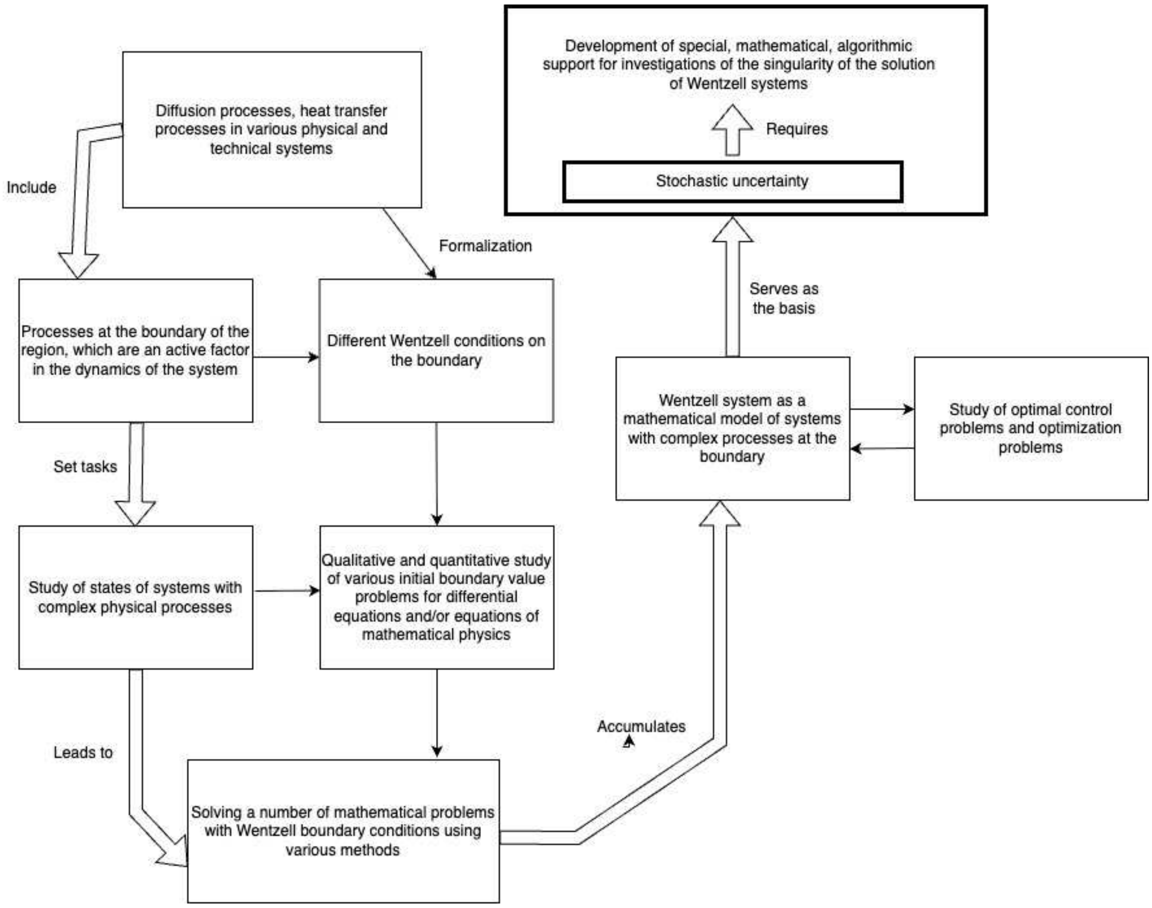

Fig. 1. The contex diagram

2. Method of Information Processing State Analysis for Stochastic Dynamic and Evolutionary Systems of Wentzell Equation

Let us present the algorithm of the program (Fig. 2 – 5), which implements information processing.

The program is written in Python language, operated on a personal computer of Intel platform (UX64), running under Microsoft Windows. Let's describe its operation by steps.

Step 1. We enter the coefficients of the stochastic Wentzell system of fluid filtration equations α, β, γ, the number of Galerkin approximations N, the parameter of the region under consideration R, the total number of experiments M, the number of experiments in each group N_m, where

$$\sum_{m=1}^{10} N_m = M,$$

N_{10} – rare events.

Step 2. A check for statistical data is performed. If the data are not statistical, an expert estimate of the mathematical expectations $\mathbf{M}\xi$, $\mathbf{M}\chi$ and variance $\mathbf{D}\xi$, $\mathbf{D}\chi$ from which the random variables ξ, χ are determined. If the data are statistical, a statistical relevance check is performed.

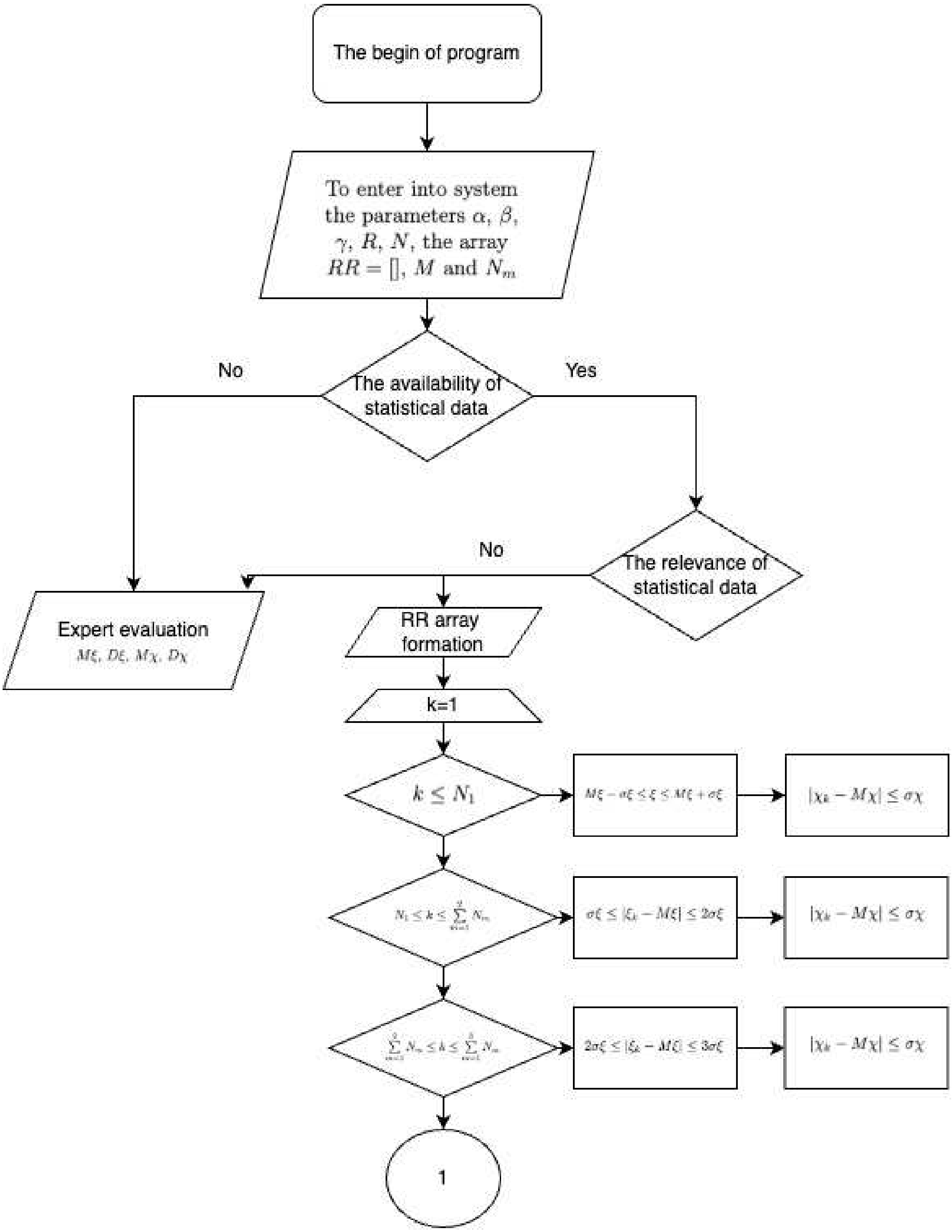

Fig. 2. Beginning of the block diagram of information processing for the system of Wentzell equations

Step 3. The statistical data are checked for relevance. If the data are not relevant, the expert estimation of mathematical expectations $\mathbf{M}\xi$, $\mathbf{M}\chi$ and dispersions $\mathbf{D}\xi$, $\mathbf{D}\chi$ from which the random variables ξ, χ are determined. If the data are relevant, the mathematical expectations $\mathbf{M}\xi$, $\mathbf{M}\chi$ and variance $\mathbf{D}\xi$, $\mathbf{D}\chi$ on the input data.

Step 4. The array RR of size $9 \times N$ is formed according to the following rule:

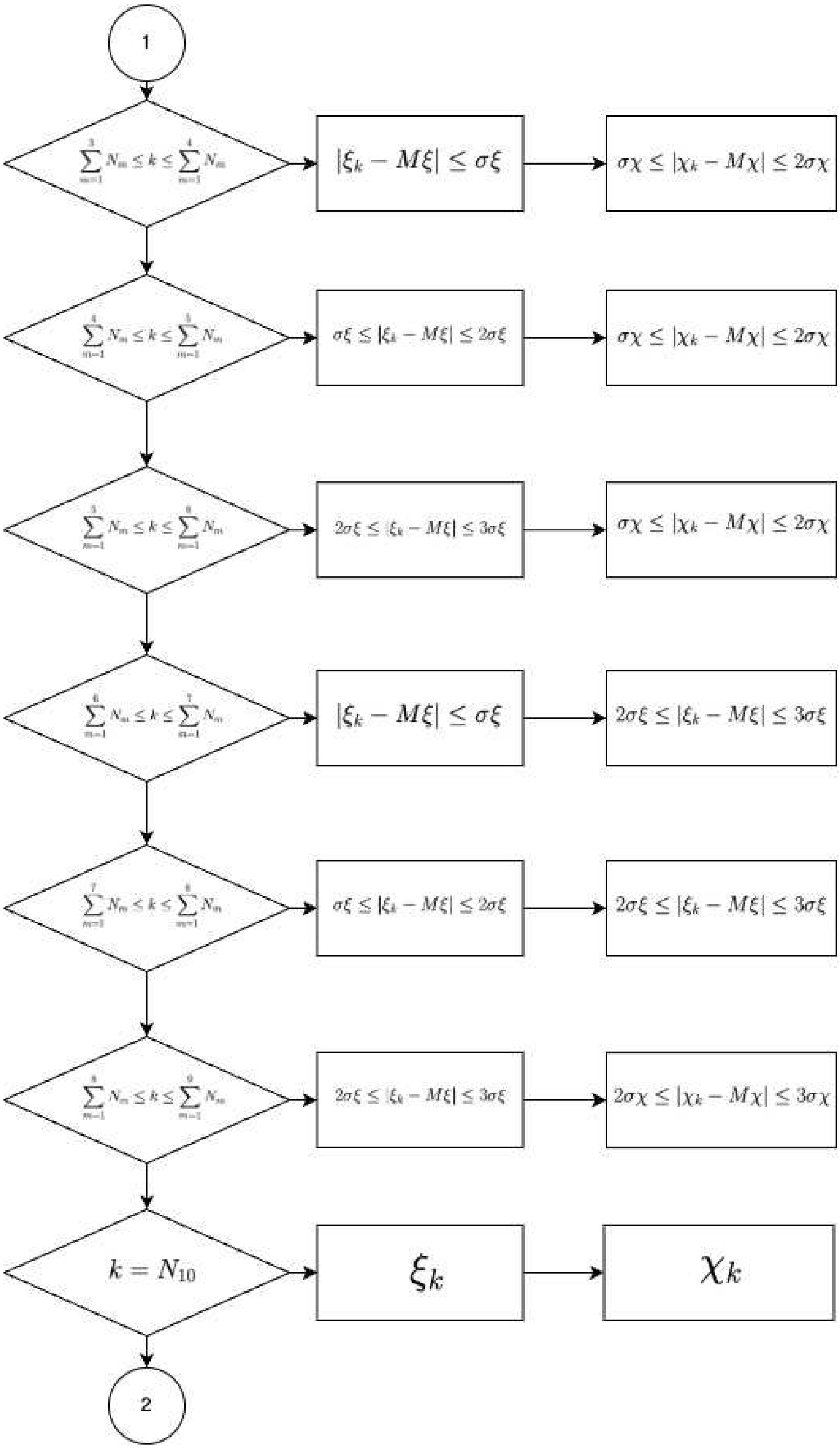

Fig. 3. Continuation of the information processing flowchart for the system of Wentzell equations

Step 4.1. For the first group of experiments N_1, N_2, N_3 for random variables in the considered region ξ_k and on its boundary χ_k, the numbering of the obtained data is fixed in the array R as the first three lines r_{1j}, r_{2j}, r_{3j} taking into account the specified confidence

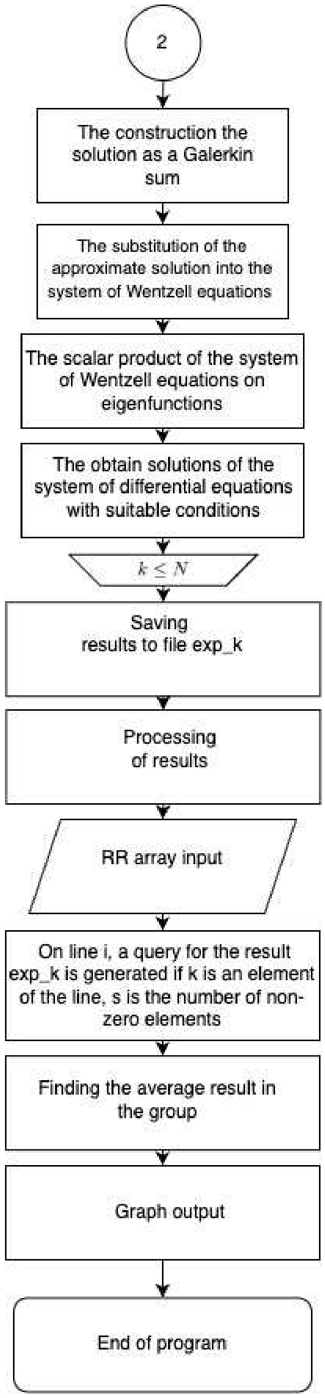

Fig. 4. End of information processing flowchart for the system of Wentzell equations

intervals $\mathbf{M}\xi \pm \sigma\xi$ and $\mathbf{M}\chi \pm \sigma\chi$; $\mathbf{M}\xi \pm \sigma\xi$ and $\mathbf{M}\chi \pm 2\sigma\chi$; $\mathbf{M}\xi \pm \sigma\xi$ and $\mathbf{M}\chi \pm 3\sigma\chi$, respectively.

Step 4.2. For the second group of experiments N_4, N_5, N_6 for random variables in the considered region ξ_k and on its boundary χ_k, the numbering of the obtained data is fixed in the array R as the next three lines r_{4j}, r_{5j}, r_{6j} taking into account the specified confidence intervals $\mathbf{M}\xi \pm 2\sigma\xi$ and $\mathbf{M}\chi \pm \sigma\chi$; $\mathbf{M}\xi \pm 2\sigma\xi$ and $\mathbf{M}\chi \pm 2\sigma\chi$; $\mathbf{M}\xi \pm 2\sigma\xi$ and $\mathbf{M}\chi \pm 3\sigma\chi$, respectively.

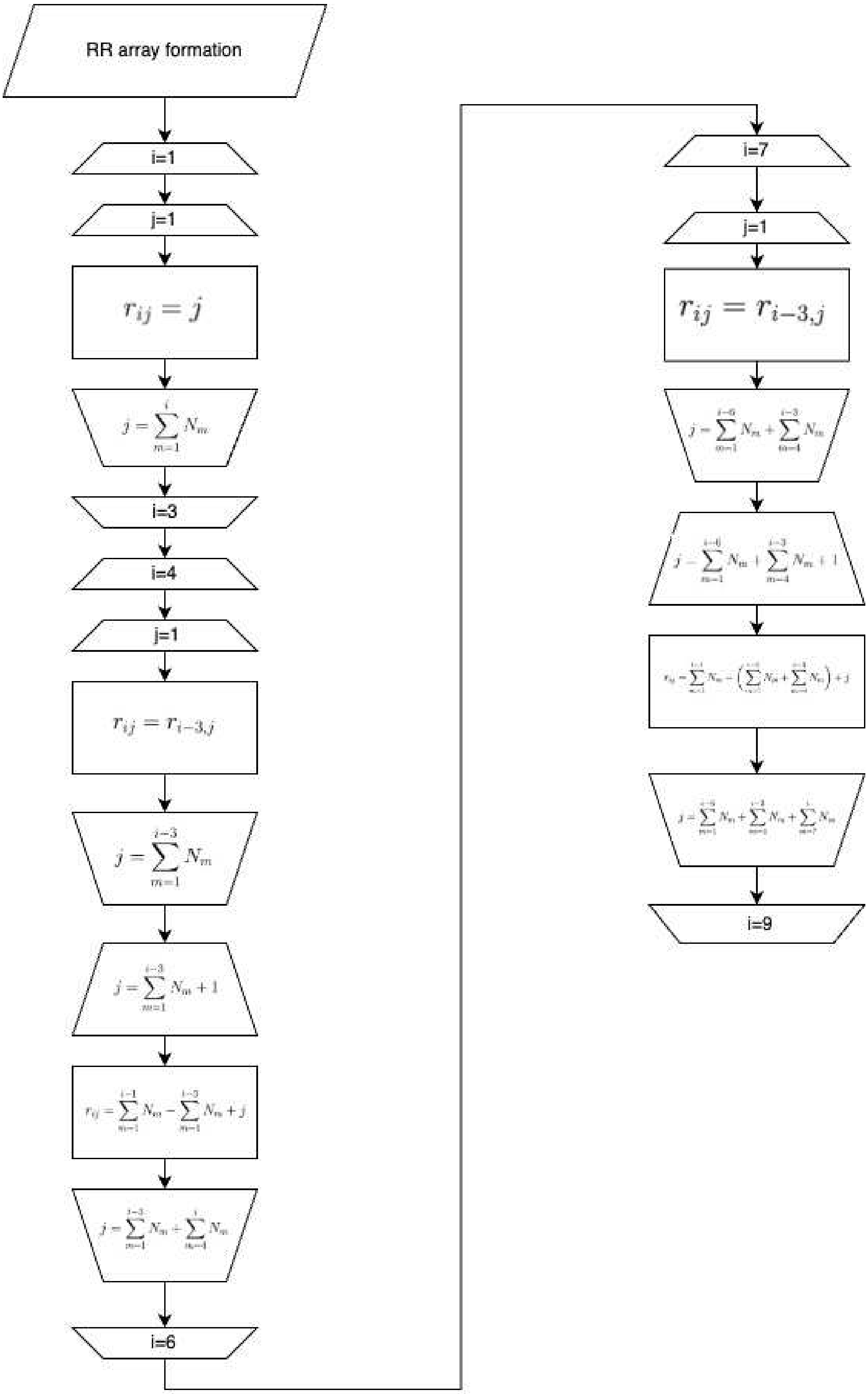

Fig. 5. Block diagram for forming the RR array

Step 4.3. For the third group of experiments N_7, N_8, N_9 for random variables in the considered region ξ_k and on its boundary χ_k, the numbering of the obtained data is fixed in the array R as the next three lines r_{7j}, r_{8j}, r_{9j}, taking into account the specified confidence intervals $\mathbf{M}\xi \pm 3\sigma\xi$ and $\mathbf{M}\chi \pm \sigma\chi$; $\mathbf{M}\xi \pm 3\sigma\xi$ and $\mathbf{M}\chi \pm 2\sigma\chi$; $\mathbf{M}\xi \pm 3\sigma\xi$ and $\mathbf{M}\chi \pm 3\sigma\chi$, respectively.

Step 5. The array of random variables ξ_k, χ_k is constructed taking into account the rule of three sigma by the previously selected number of experiments M and the number of experiments in each group N_m. When the number of experiments k falls within N_{10}, the expert values ξ_k and χ_k are entered.

Step 6. An approximation of the solution of the problem in the form of the Galerkin sum for each experiment is constructed.

Step 7. The approximate solution obtained at the previous step is substituted into the system of Wentzell equations under consideration for each experiment.

Step 8. A separate procedure generates a system of differential equations for each experiment.

Step 9. The system of differential equations is solved separately.

Step 10. The solution of the required problem is formed, taking into account step 9 and step 8, the three-dimensional graph and the animated three-dimensional graph taking into account the time change are output, the results are written to a separate file exp_k.

Step 11. The results are processed. A query of the result exp_k is generated for the line i, if k is an element of the line, s is the number of non-zero elements in RR. The standard deviation for each group is found.

Step 12. The overall graph is plotted.

Remark 1. Since the formation of the RR array plays a special role, let us give a general view of RR on the example of test data. In particular, initially the user knows the following table

	$\mathbf{M}\chi \pm \sigma\chi$	$\mathbf{M}\chi \pm \sigma\chi$	$\mathbf{M}\chi \pm 2\sigma\chi$
$\mathbf{M}\xi \pm \sigma\xi$	N_1	N_2	N_3
$\mathbf{M}\xi \pm 2\sigma\xi$	N_4	N_5	N_6
$\mathbf{M}\xi \pm 3\sigma\xi$	N_7	N_8	N_9

,

with the property

$$\sum_{m=1}^{10} N_m = M,$$

where M – total number of experiments, N_{10} – rare events (basically expert judgment is given).

Here, for the first group of experiments N_1, N_2, N_3 the data are filled in by the following formulas

$$r_{ij} = j, \text{где } 1 \leq j \leq \sum_{m=1}^{i} N_m,$$

the remaining elements are filled with zeros.

Here, for the second group of experiments N_4, N_5, N_6, the data are filled with the following formulas considering the previous experiments,

$$r_{ij} = r_{i-3,j}, \text{где } 1 \leq j \leq \sum_{m=1}^{i-3} N_m,$$

$$r_{ij} = \sum_{m=1}^{i-1} N_m - \sum_{m=1}^{i-3} N_m + j, \text{где } \sum_{m=1}^{i-3} N_m + 1 \le j \le \sum_{m=1}^{i-3} N_m + \sum_{m=4}^{i} N_m,$$

the remaining elements are filled with zeros.

Here, for the third group of experiments N_7, N_8, N_9, the data are filled with the following formulas considering the previous experiments,

$$r_{ij} = r_{i-3,j}, \text{где } 1 \le j \le \sum_{m=4}^{i-3} N_m + \sum_{m=1}^{i-6} N_m,$$

$$r_{ij} = \sum_{m=1}^{i-1} N_m - \Big(\sum_{m=1}^{i-6} N_m + \sum_{m=4}^{i-3} N_m) + j \Big),$$

where

$$\sum_{m=1}^{i-6} N_m + \sum_{m=4}^{i-3} N_m + 1 \le j \le \sum_{m=1}^{i-6} N_m + \sum_{m=4}^{i-3} N_m + \sum_{m=7}^{i} N_m$$

the remaining elements are filled with zeros.

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Received April 30, 2025

УДК 517.9, 519.216.2 DOI: 10.14529/jcem250203

ОБРАБОТКА ИНФОРМАЦИИ ПРИ АНАЛИЗЕ СОСТОЯНИЯ СТОХАСТИЧЕСКИХ ДИНАМИЧЕСКИХ И ЭВОЛЮЦИОННЫХ СИСТЕМ УРАВНЕНИЙ ВЕНТЦЕЛЯ

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В работе исследуются стохастические динамическая и эволюционная система Вентцеля в области и на ее границе. В частности, для указанных систем уравнений Вентцеля проводится структурный системный анализ различных аспектов исследования систем уравнений Вентцеля и строится алгоритм для обработки информации, получаемой в результате вычислительных экспериментов, и анализа состояния стохастической динамической и эволюционной системы Вентцеля при различных значениях их параметров.

Ключевые слова: динамическая система Вентцеля; эволюционная система Вентцеля; обработка информации; правило трех сигм; производная Нельсона – Гликлиха.

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Поступила в редакцию 30 апреля 2025 г.