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THE ACCURACY OF THE APPROXIMATE SOLUTIONS TO A BOUNDARY VALUE INVERSE PROBLEM WITH FINAL OVERDETERMINATION

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The aim of the paper is to investigate the accuracy of the methods for approximate solving a boundary value inverse problem with final overdetermination for a parabolic equation. We use the technique of the continuation to the complex domain and the expansion of the unknown function into a Dirichlet series (exponential series) to formulate the inverse problem as a linear operator equation of the first kind in the appropriate linear normed spaces. This allows us to estimate the continuity module for the inverse problem by means of classical spectral technique and investigate the order-optimal approximate methods for the boundary value inverse problem under study.

Keywords: parabolic equation; boundary value inverse problem; module of continuity of the inverse operator; exponential series.

Introduction

We study a boundary value inverse problem with final overdetermination (the problem of the most accurate heating of a rod). Namely, we should recover the boundary condition in a mixed boundary value problem for the heat transfer equation from knowledge of the solution at the final time moment. Originally, the problem was formulated in [1] in the form of an optimization problem. In applications, a large number of optimization problems associated with parabolic equations arise. One of such problems in thermophysical terms can be formulated as follows. Consider a homogeneous rod with a thermally insulated lateral surface, the left end of which is thermally insulated, and the given temperature $h(t)$ is maintained at the right end. We need, by controlling the temperature at the right end of the rod, make the temperature distribution in the rod as close as possible to the specified distribution $g(x)$ by a given point in time. Namely, let $u(x,t) = u(x,t,h)$ define the distribution of the temperature in the rod at the time moment t. It is required, by controlling the function $h \in L_2[0,T]$, to minimize the function

$$J(h) = \|u(x,T) - g(x)\|^2$$

under the condition that $u(x,t) = u(x,t,h)$ solves the boundary value problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad t \in (0,T), \quad x \in (0,l),$$

$$u(x,0) = 0 \quad (0 < x < l), \; u(0,t) = 0, \;\; u(l,t) = h(t) \quad (0 < t < T).$$

$h(t)$ satisfies the conditions $h(0) = 0$, $\|h'(t)\|_{L_2[0,T]} \le r$.

The problem of the most accurate heating of the rod and the related optimization problems for the heat equation is investigated, for example, in [2].

Aviation and rocket and space technology, energy, metallurgy use experimental studies, bench and field studies of thermal conditions, the creation of effective diagnostic methods and the results of heat exchange processes based on the results of experiments and tests. The basis of these methods may lie in the problems associated with inverse problems of different types but the boundary value inverse problems are among the most important classes of the inverse problems of thermal conductivity. Various statements of boundary value inverse problems was studied also in [3].

To solve the improperly posed problem, we use the quasi-reversibility method, which consists of replacing the boundary value problem for a parabolic equation with the corresponding boundary value problem for a hyperbolic equation with a small parameter. The quasireversibility method is efficient for solving inverse problems by reducing computational complexity compared to optimization-based approaches. That is the main reason why the quasirevesibility method and its applications are widely used for solving differfnt types of inverse problems. For example, the paper [4] develops the second generation elliptic systems method (ECM) for solving parabolic inverse problems. It demonstrates improved accuracy in imaging unknown coefficients and localization of targets using fewer iterations compared to optimization-based methods.

In the case of severely ill-posed problems, the issues of assessing the accuracy of the obtained approximate solutions and studying the optimality of the proposed method for approximate solution play an important role. For linear ill-posed problems classical spectral technique is widely used to obtain estimates of the error for the approximate solutions on compact sets (correctness classes) and the error of the optimal approximate methods. For linear ill-posed problems, the technique of calculating the error of the optimal method of the is based on the relation between the error of the optimal method and the module of continuity of the inverse operator, which can be calculated for specific operators and correctness classes. The continuity module for some classes of nonlinear inverse problems was estimated, for evample, in [5–7].

In the classical spectral technique, the commuting of the operator of the problem with the operator defining the correctness class (reflecting a priori information about the exact solution of the inverse problem) plays the main role but for some important inverse problems in the classical formulation, these operators do not commute. We use the technique of the continuation to the complex domain and the expansion of the unknown function into a Dirichlet series (exponential series) to formulate the inverse problem as an operator equation of the first kind in the space isometric to the space of the initial data and the space of the solutions. This allows us to calculate the module of continuity and investigate the accuracy of optimal and order-optimal approximate methods for the inverse problem under study. The obtained estimate for the modulus of continuity makes it possible to investigate the optimal methods for the approximate solving of the inverse problem and construct the order-optimal methods.

The paper is organized as follows. Section 1 formulates the boundary value inverse problem and the corresponding direct problem and introduces the functional spaces which we use to investigate the inverse problem. Section 2 is intended to obtain the linear operator equation which gives the equivalent formulation of the inverse problem and the corresponding direct problem in l_2 space. Section 3 is aimed to estimate the function which plays the basic role for the investigation the accuracy of the approximate solution to the inverse problem - the continuity module for the inverse problem. Section 4, a method for an approximate solution of the original inverse problem is formulated, and an estimate for the accuracy of the constructed approximate solution is obtained.

1. The Inverse Problem with Final Overdetermination

Direct problem. Consider the following boundary value problem for a heat conductivity equation. We have to determine the function $u = u(x,t)$,

$$u \in C([0,l]; W_2^1[0;T]) \cap C^2((0,l); L_2[0;T])$$

which meets the conditions

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad t \in (0,T), \quad x \in (0,l), \tag{1}$$

$$u(x,0) = 0, \quad (0 < x < l),$$

$$u(0,t) = 0, \quad u(l,t) = h(t), \quad (0 < t < T).$$

(we will consider the case $T = 2\pi$).

Inverse problem. We study the following inverse problem for a parabolic equation. Suppose that $u(x,t)$ satisfies the conditions (1) and the additional condition

$$u(x,T) = g(x). \tag{2}$$

We have to recover the function $h \in L_2[0,T]$ (the boundary condition), $g \in C[0,l]$ is the given function. The problem was formulated and studied in [1, 2] as an optimization problem. Recall the well known theorem on approximation of continuous functions by polynomials.

Theorem 1. *Let $0 < \lambda_1 < \lambda_2 < \ldots$, $\lambda_i \to \infty$, $0 < a < b$. For any function $f \in C[a,b]$and for any $\varepsilon > 0$ there exists a linear combination*

$$P_n(x) = \sum_{i=1}^{n} c_i x^{\lambda_i}$$

such that

$$\|f - P_n\|_{C[a,b]} < \varepsilon,$$

if and only if the condition

$$\sum_{i=1}^{\infty} \frac{1}{\lambda_i} = \infty$$

is satisfied.

The next lemma follows directly from the theorem and gives the possibility to approximate continuous functions by exponential polynomials [8].

Lemma 1. *Suppose that the functions $\{e_i(x)\}$, $e_0(x) = 1$, $e_i(x) = e^{-\lambda_i x}$, $0 < \lambda_1 < \lambda_2 < \ldots$, $\lambda_i \to \infty$ satisfy the condition*

$$\sum_{i=1}^{\infty} \frac{1}{\lambda_i} = \infty.$$

Then the system of functions $\{e_i(x)\}_{n=0}^{\infty}$ *is closed in the space* $C[0,l]$.

Denote $g(x)$ a continuous function on $[0,l]$ which we can expand into the uniformly converging exponential series

$$G(x) = \sum_{n=1}^{\infty} b_n e^{-\sqrt{n}x}. \tag{3}$$

Let the series (3) converge for $x \geq -l$, then the odd continuation of $g(x)$ defined on the segment $[-l,l]$ can be expanded into the uniformly converging series

$$g(x) = \sum_{n=1}^{\infty} b_n \operatorname{sh} \sqrt{n}x. \tag{4}$$

We conclude from the elementary properties of the exponential series that the function [10]

$$G_0(z) = \sum_{n=1}^{\infty} b_n \operatorname{sh} \sqrt{n}z \tag{5}$$

is analytical and bounded in the domain $-l < \mathrm{Re}z < l$. Let a function $G(z)$ have a Dirichlet series expansion

$$G(z) = \sum_{n=1}^{\infty} b_n \operatorname{sh} \sqrt{n}z,$$

convergent uniformly in the strip $-l < \mathrm{Re}\ z < l$, i.e. is an analytic almost periodic function in the strip. Then the function $p(y) = G(x+iy)$ for any $-l < v < l$ is a uniform almost periodic function, and its norm can be defined by the formula

$$\|p\|_0^2 = \lim_{Y\to\infty} \frac{1}{2Y} \int_{-Y}^{Y} \|p(iy)\|^2 dy.$$

Let X denote the linear space of functions $G(z)$ analytical and bounded in the domain $0 < \mathrm{Re}z < l$, equipped with the norm

$$\|G\|^2 = \lim_{Y\to\infty} \frac{1}{2Y} \int_{-Y}^{Y} \|G(iy)\|^2 dy.$$

Suppose that for the given continuous function $g(x)$ there is the exact solution $h(t)$ to the inverse problem (1) which belongs to the set

$$M = \{h(t) : \|h''(t)\|_{L_2[0,\infty)} \leq r\},$$

but the values of $g(x)$ are unknown. Instead of the exact values, we know approximate values of the given function. That is we know the function $g_\delta \in C[0,l]$ such that for the analytical continuation of the exact and the given functions in the domain $0 < \mathrm{Re}z < l$ the inequality $\|g - g_\delta\|_X < \delta$ holds. We have to determine an approximate solution h_δ to the boundary value inverse problem and estimate its deviation from the exact solution.

2. Reducing the Boundary Value Inverse Problem to the Operator Equation

We will find the solution to the direct problem in the form of complex Fourier series

$$u(x,t) = \sum_{n=-\infty}^{\infty} c_n(x)e^{int} = 2\mathrm{Re}U(x,t),$$

where

$$U(x,t) = \frac{c_0(x)}{2} + \sum_{n=1}^{\infty} c_n(x)e^{int}.$$

Consider the expansion into Fourier series

$$h(t) = \sum_{n=-\infty}^{\infty} h_n e^{int} = 2\mathrm{Re}H(t).$$

Here

$$H(t) = \frac{h_0}{2} + \sum_{n=1}^{\infty} h_n e^{int}.$$

Consider the particular solution $u_n(x,t)$ to the mixed boundary value problem which corresponds to the boundary condition $u_n(l,t) = e^{int}$. We find the particular solution in the form $u_n(x,t) = w_n(x,t) + v_n(x,t)$ (see, for example, [11, 12])

$$w_n(x,t) = \frac{\mathrm{sh}\,\mu_0\sqrt{n}x}{\mathrm{sh}\,\mu_0\sqrt{n}l}e^{int},$$

where $\mu_0 = \frac{1}{\sqrt{2}}(1+i)$ and the function $v_n(x,t)$ satisfies the conditions

$$\frac{\partial v_n}{\partial t} = \frac{\partial^2 v_n}{\partial x^2}, \quad t \in (0,T), \quad x \in (0,l), \tag{6}$$

$$v_n(x,0) = -\frac{\mathrm{sh}\,\mu_0\sqrt{n}x}{\mathrm{sh}\,\mu_0\sqrt{n}l}, \quad (0 < x < l),$$

$$v_n(0,t) = 0, \; v_n(l,t) = 0 \quad (0 < t < T).$$

Make sure that the function

$$Y(x,t) = \sum_{n=1}^{\infty} h_n \frac{\mathrm{sh}\,\mu_0\sqrt{n}x}{\mathrm{sh}\,\mu_0\sqrt{n}l}e^{int} \tag{7}$$

satisfies the conditions

$$\frac{\partial Y}{\partial t} = \frac{\partial^2 Y}{\partial x^2}, \quad t \in (0,T), \quad x \in (0,l), \tag{8}$$

$$Y(x,0) = \Phi(x),$$

$$Y(0,t) = 0, \quad Y(l,t) = H(t), \quad (0 < t < T),$$

where
$$\Phi(x)=\sum_{n=1}^{\infty}h_n\frac{\operatorname{sh}\mu_0\sqrt{n}x}{\operatorname{sh}\mu_0\sqrt{n}l}.$$

Actually, we have for $0<x<l-\delta$
$$\left|\frac{\operatorname{sh}\mu_0\sqrt{n}x}{\operatorname{sh}\mu_0\sqrt{n}l}\right|\leq e^{-\sqrt{\frac{n}{2}(l-x)}},$$

so for the series (7) the majorizing series is
$$\sum_{n=1}^{\infty}|h_n|e^{-\sqrt{\frac{n}{2}(l-x)}} \tag{9}$$

and for the series which we obtain by termwise differentiation of (7), the majorizing series is
$$\sum_{n=1}^{\infty}n|h_n|e^{-\sqrt{\frac{n}{2}(l-x)}}. \tag{10}$$

The series (9) and (10) are converging, so the series (7) is uniformly converging on any interval $0<x<l-\delta$, and the function $Y(x,t)$ satisfies the conditions (8).

Hence, we can represent the solution of the boundary value problem in the form
$$U(x,t)=\sum_{n=1}^{\infty}h_n\frac{\operatorname{sh}\mu_0\sqrt{n}x}{\operatorname{sh}\mu_0\sqrt{n}l}e^{int}-Z(x,t),$$

$Z(x,t)$ satisfies the conditions
$$\frac{\partial Z}{\partial t}=\frac{\partial^2 Z}{\partial x^2},\quad t\in(0,T),\quad x\in(0,l), \tag{11}$$
$$Z(x,0)=-\Phi(x),\quad x\in(0,l),$$
$$Z(0,t)=0,\quad Z(l,t)=0\quad(0<t<T).$$

Here
$$\Phi(x)=\sum_{n=1}^{\infty}h_n\frac{\operatorname{sh}\mu_0\sqrt{n}x}{\operatorname{sh}\mu_0\sqrt{n}l}.$$

We have to recover the function $G(x)=\Phi(x)+Z(x,T)$. Denote
$$a_n=\frac{h_n}{\operatorname{sh}\mu_0\sqrt{n}l}$$

the coefficients of the expansion of $\Phi(x)$ in the Dirichlet series. Considering the expansion of a hyperbolic sine in the Fourier series, we find the Fourier coefficients of the function $\Phi(x)$:
$$\Phi(x)=\sum_{k=1}^{\infty}\varphi_k\sin\frac{\pi kx}{l},$$

where

$$\varphi_k = 2(-1)^{k+1}\sum_{n=1}^{\infty} a_n \frac{\pi k \operatorname{sh} \mu_0 \sqrt{n} l}{(\pi k)^2 + inl^2}. \tag{12}$$

Consider the function

$$F(\lambda) = 2\sum_{n=1}^{\infty} a_n \frac{\lambda \operatorname{sh} \mu_0 \sqrt{n} l}{\lambda^2 + inl^2}.$$

It is evident that

$$F(\pi k) = (-1)^{k+1}\varphi_k.$$

We show that the function $F(\lambda)$ is analytic in the domain

$$D_k = \left\{0 < |\lambda - \mu_1\sqrt{k}l| < \frac{1}{k}\right\}, \quad \mu_1 = \frac{1}{\sqrt{2}}(1-i).$$

Consider the series

$$\sum_{n=1}^{\infty} a_n \frac{\lambda \operatorname{sh} \mu_0 \sqrt{n} l}{\lambda^2 + inl^2} = \sum_{n=1}^{\infty} h_n \frac{\lambda}{\lambda^2 + inl^2}.$$

For $n \neq k$ the inequalities hold

$$|\lambda| \le \sqrt{k} + \frac{1}{k} \le \sqrt{k}l + 1,$$

$$|\lambda - \mu_1\sqrt{n}l| \ge |\mu_1\sqrt{n}l - \mu_1\sqrt{k}l| - |\lambda - \mu_1\sqrt{k}l| \ge$$
$$\ge l|\sqrt{n} - \sqrt{k}| - \frac{1}{k}.$$

Similarly,

$$|\lambda - \mu_1\sqrt{n}l| \ge l|\sqrt{n} - \sqrt{k}| - \frac{1}{k}.$$

Then,

$$|\lambda^2 + inl^2| = |\lambda - \mu_1\sqrt{n}l||\lambda + \mu_1\sqrt{n}l| \ge$$
$$\ge \left(|\sqrt{n} - \sqrt{k}| - \frac{1}{k}\right)^2$$

and

$$|\frac{h_n\lambda}{\lambda^2 + inl^2}| \le \frac{|h_n|(\sqrt{k}l + 1)}{(|\sqrt{n} - \sqrt{k}| - \frac{1}{k})^2}.$$

Note that

$$\frac{|h_n|(\sqrt{k}l + 1)}{(\sqrt{n} - \sqrt{k}| - \frac{1}{k})^2} \simeq \frac{|h_n|(\sqrt{k}l + 1)}{l^2 n}$$

for $n \to \infty$, so the majorizing series

$$\sum_{n=1}^{\infty} \frac{|h_n|(\sqrt{k}l + 1)}{(\sqrt{n} - \sqrt{k}| - \frac{1}{k})^2}$$

uniformly converges in every domain D_k $(k = 1, 2, ...)$. Hence, the function $F(\lambda)$ is analytical in the domain

$$D = \bigcup_{n=1}^{\infty} D_k.$$

The function $F(\lambda)$ has simple poles at the points $\lambda_n = \mu_1\sqrt{n}l$. Hence, the equality (12) imply

$$\lim_{\lambda\to\mu_1\sqrt{n}l} F(\lambda)(\lambda - \mu_1\sqrt{n}l) = a_n \operatorname{sh}\mu_0\sqrt{n}l. \tag{13}$$

Consider the expansion in Dirichlet series of the function $G(x)$:

$$G(x) = \sum_{n=1}^{\infty} b_n \operatorname{sh}\mu_0\sqrt{n}x.$$

Taking into account the expansion in Fourier series of the function $G(x)$ we can conclude that

$$G(x) = \sum_{k=1}^{\infty} g_k \sin\frac{\pi k x}{l},$$

where

$$g_k = \varphi_k(1 + e^{-(\frac{\pi k}{l})^2 T}).$$

Consider the function

$$P(\lambda) = F(\lambda)(1 + e^{-\lambda^2 T}).$$

The function $G(\lambda)$ has simple poles at the points $\lambda_n = \mu_1\sqrt{n}l$ if the condition $nl^2T \neq \pi + 2\pi m$ is met. Similarly to the equality (13) we obtain

$$\lim_{\lambda\to\mu_1\sqrt{n}l} P(\lambda)(\lambda - \mu_1\sqrt{n}l) = b_n \operatorname{sh}\mu_0\sqrt{n}. \tag{14}$$

The equalities (13) and (14) imply

$$\frac{b_n}{a_n} = \lim_{\lambda\to\mu_1\sqrt{n}l}(1 + e^{-\lambda^2 T}) = 1 + e^{-inT}.$$

We set the correspondence between the function $G(z) \in X$. and the sequence of coefficients of its Dirichlet series $\hat{G} = \{b_n\} \in l_1 \cap l_2$; set the correspondence between the function $h(t) \in L_2[0,T]$ and the sequence of coefficients of its Fourier series $\hat{H} = \{h_n\} \in l_2$.

The mean value theorem [9] shows that

$$\|G\|^2 = \lim_{Y\to\infty}\frac{1}{2Y}\int_{-Y}^{Y}\|G(iy)\|^2 dy = \sum_{n=1}^{\infty} b_n^2.$$

That is, the operator $E : X \to l_2$ which acts according to the rule

$$EG = \{b_n\}$$

is an isometry. Denote $A : l_2 \to l_2$ a linear operator acting according to the rule

$$A\hat{H} = \left\{\frac{h_n}{2\operatorname{sh}\mu_0\sqrt{n}l}\right\}_{n=1}^{\infty}.$$

Thus, the inverse problem with final overdetermination can be formulated as an operator equation

$$A\hat{H} = \hat{G}.$$

3. The Continuity Module for the Inverse Problem

Denote

$$\hat{\omega}(M,\delta) = \sup\{\|h_1 - h_2\| \colon h_1, h_2 \in M, \|\hat{g}_1 - \hat{g}_2\| \le \delta\}$$

is the continuity module for the boundary value inverse problem with final overdetermination. We use the scheme proposed in [13, 14] to estimate the continuity module. The following theorem holds.

Theorem 2. *There exists $\delta_0 > 0$, such that for all $0 < \delta < \delta_0$ the inequalities*

$$C_1 \frac{rl^2}{(\ln\delta)^2} \le \omega(M,\delta) \le C_2 \frac{rl^2}{(\ln\delta)^2}$$

are true.

Proof. Let $H = \{h_n\}_{n=1}^{\infty} \in l_2$. Consider the linear bounded operator A acting in the space l_2 according to the rule

$$A\hat{H} = \left\{ \frac{h_n}{2\operatorname{sh}\mu_0\sqrt{n}l} \right\}_{n=1}^{\infty}.$$

B is a linear bounded operator acting in the space l_2 according to the rule

$$B\hat{H} = \left\{ \frac{h_n}{n^2 l} \right\}_{n=1}^{\infty}.$$

Denote $B_1 = B^*B$, that is the operator B_1 acts in l_2 according to the rule

$$B_1\hat{H} = \left\{ \frac{h_n}{n^4 l} \right\}_{n=1}^{\infty}.$$

Denote $C = AB$, $C_1 = C^*C$, that is C_1 is the operator acting in l_2 according to the rule

$$C_1\hat{H} = \left\{ \frac{h_n}{4n^2|\operatorname{sh}\mu_0\sqrt{n}l|^2} \right\}_{n=1}^{\infty}.$$

It follows from the definitions of the operators that C_1 is a function of the operator B_1. Namely,

$$C_1 = \lambda(B_1),$$

where

$$\lambda(\sigma) = \frac{\sigma^2}{4\left|\operatorname{sh}\frac{\mu_0 l}{\sigma^{1/4}}\right|^2}. \tag{15}$$

Calculating the module of a hyperbolic sine and denoting

$$s = \frac{1}{\sigma^{1/4}},$$

we write the equality (15) as

$$\lambda(s) = \frac{1}{s^8(\operatorname{sh}^2\frac{l}{2}s + \sin^2\frac{l}{2}s)}. \tag{16}$$

Further,

$$\lim_{s\to 0}\lambda(s)=+\infty,\quad \lim_{s\to\infty}\lambda(s)=0,$$

the function $\lambda(s)$ is continuous and monotone on $(0,\infty)$. Thus, the equation (16) has a unique solution $s=s(\lambda)$ for every $\lambda>0$.

Put to use the elementary inequalities

$$\mathrm{sh}^2\,\frac{l}{2}s+\sin^2\frac{l}{2}s\le e^{\sqrt{2}ls},\tag{17}$$

$$\mathrm{sh}^2\,\frac{l}{2}s+\sin^2\frac{l}{2}s\ge \frac{e^{\sqrt{2}ls}}{4}-\frac{1}{2}.\tag{18}$$

For $s>\frac{2\sqrt{2}}{l}\ln 2$ we obtain

$$e^{\sqrt{2}ls}>4$$

and the inequality (18) imply

$$\mathrm{sh}^2\,\frac{l}{2}s+\sin^2\frac{l}{2}s\ge \frac{e^{\sqrt{2}ls}}{8}.\tag{19}$$

Applying the inequalities (17) and (19) for $s>\frac{2\sqrt{2}}{l}\ln 2$ we obtain

$$\frac{e^{\sqrt{2}ls}}{8}\le\lambda(s)\le e^{\sqrt{2}ls.}.\tag{20}$$

Taking the logarithm of both sides in the inequality (20) and considering that $s>\frac{2\sqrt{2}}{l}\ln 2$, we get the inequality

$$\frac{\sqrt{2}}{2}l-8\varepsilon\le\frac{\ln\lambda}{s}\le\sqrt{l}+8\varepsilon,\tag{21}$$

where

$$\varepsilon(s)=\frac{\ln s}{s},\quad \varepsilon(s(\lambda))\to 0$$

as $\lambda\to 0$. Consiquently,

$$s\simeq\frac{\ln\lambda}{\sqrt{2}l}$$

for $\lambda\to 0$. Finally,

$$\sigma=p(\lambda)\simeq\frac{4l^4}{(\ln\lambda)^4}\tag{22}$$

as $\lambda\to 0$. The relation (20) by Theorem which was proved in [13], imply

$$s\simeq\frac{\ln\lambda}{\sqrt{2}l}$$

as $\lambda\to 0$. Hence,

$$\omega(r,\delta)\simeq r\sqrt{p\left(\frac{r^2}{\delta^2}\right)}\simeq\frac{rl^2}{(\ln\delta)^2}\tag{23}$$

as $\delta\to 0$. That is, the statement of the theorem holds.

□

4. An Approximate Solution to the Inverse Problem

To construct a stable approximate solution to the original inverse problem, we use the quasi-inversion method, which consists of replacing the unstable original problem with a stable problem for an equation with a small parameter.

Let us consider an auxiliary hyperbolic equation with a small parameter

$$\varepsilon \frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \tag{24}$$

and the following initial and boundary conditions

$$u(x,0) = 0, \quad u(0,t) = 0, u_x(0,t) = \varphi_\delta(t).$$

Here $u(x,\cdot) \in C^2(0,1) \bigcap C[0,1], u(\cdot,t) \in W_2^2[0,\infty), \varepsilon > 0$ is the time constant (the thermal stress relaxation time). As an approximate solution to the problem, we will consider the function $u_\delta(t) = u_\delta^\varepsilon(1,t)$, where $u_\delta^\varepsilon(x,t)$ satisfies conditions (24) and $\varepsilon = \varepsilon(\delta)$. The quantity used as a characteristic of the accuracy for the constructed approximate solution is

$$\Delta(\varepsilon,\delta) = \sup\left\{\|v_\delta^\epsilon - v\| : v \in M; \|\varphi - \varphi_\delta\| \le \delta\right\}.$$

Let us choose the dependence $\varepsilon = \varepsilon(\delta)$ according to M.M. Lavrentiev's scheme, i.e. the dependence $\varepsilon = \varepsilon(\delta)$ is chosen from the condition

$$\delta e^{\frac{1}{2\sqrt{\varepsilon}}} = c_0 r \delta$$

Finally, we obtain

$$\varepsilon \le \frac{c}{\ln^2(r/\delta)}.$$

Therefore, there exist numbers $C > 0, \delta_1 > 0$, such that for all $\delta < \delta_1$ the inequality holds

$$\Delta(\varepsilon(\delta),\delta) \le \frac{C}{\ln^2(r/\delta)}. \tag{25}$$

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ОБ ОЦЕНКЕ ТОЧНОСТИ МЕТОДА ПРИБЛИЖЕННОГО РЕШЕНИЯ ГРАНИЧНОЙ ОБРАТНОЙ ЗАДАЧИ ДЛЯ ПАРАБОЛИЧЕСКОГО УРАВНЕНИЯ С ФИНАЛЬНЫМ ПЕРЕОПРЕДЕЛЕНИЕМ

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Целью статьи является исследование точности методов приближенного решения граничной обратной задачи с финальным переопределением для параболического уравнения. С помощью техники продолжения в комплексную область и разложения неизвестной функции в ряд Дирихле (ряд экспонент) обратная задача формулируется в виде линейного операторного уравнения первого рода в соответствующих линейных нормированных пространствах. Это позволяет оценить модуль непрерывности обратной задачи с помощью классической спектральной техники и исследовать оптимальные по порядку приближенные методы для изучаемой краевой обратной задачи.

Ключевые слова: параболическое уравнение; граничная обратная задача; модуль непрерывности обратного оператора; ряды экспонент.

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