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METHODS FOR STUDYING THE ASYMPTOTIC PROPERTIES AND STABILIZATION OF SOME SYSTEMS WITH LINEAR DELAY

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Methods for obtaining sufficient conditions of asymptotic stability and instability for systems of differential equations containing linear delay are proposed. Based on these conditions, some systems of linear differential equations are investigated, one of them is stabilized over an infinite period of time.

Keywords: asymptotic stability; linear delay; stabilization.

Introduction

In this paper, we study the asymptotic properties of some systems of differential equations with linear delay, as well as systems with constant delay, to which these systems are reduced by replacing the argument. The systems with constant delay obtained in this way contain an exponential multiplier on the right side, i.e. they are not Lipschitz systems.

The studied systems with linear delay have the form

$$\frac{dx(t)}{dt} = A(t)x(t) + B(t)x(\mu t), \;\; \mu = \text{const}, \;\; 0 < \mu < 1, \;\; t \geq t_0 > 0. \tag{1}$$

Here $A(t),\ B(t)$ are continuously differentiable matrices of dimension $m \times m$, $x(t)$ — m-dimensional vector function of time (argument) t, the delay has the form $(1-\mu)t$. The solution is determined at the initial moment of time t_0 by the vector function $\phi(\beta)$: $\eta \in [\mu t_0, t_0]$.

Systems with constant delay, to which the systems (1) are reduced by replacing the argument $\tau = \ln\left(\frac{t}{t_0}\right)$ have a delay $\sigma = -\ln(\mu)$ and an exponential multiplier e^{τ}

$$\frac{dz(\tau)}{d\tau} = t_0 e^{\tau}[\bar{A}(\tau)z(\tau) + \bar{B}(\tau)z(\tau - \sigma)],$$

$\sigma > 0,\ \ \tau \geq 0,\ \ z(\eta) = \phi(t_0 e^{\eta}),\ \ \eta \in [\mu t_0, t_0]$. Here $\bar{A}(\tau) = A(t_0 e^{\tau}),\ \ \bar{B}(\tau) = B(t_0 e^{\tau})$. In many cases, based on such systems, it is possible to obtain both sufficient conditions for stability and instability of the studied systems. Systems with linear delay are found in problems of mechanics, physics [1], biology. Consideration of the delay effect is important for the correct qualitative and quantitative description of processes containing unlimited delay.

A further generalization of such systems are neutral type systems

$$\frac{dx(t)}{dt} = A(t)x(t) + B(t)x(\mu t) + R(t)\frac{dx(\mu t)}{dt}, \ \ \mu = \text{const}, \ \ 0 < \mu < 1, \ \ t \geq t_0 > 0, \quad (2)$$

where $R(t)$ is a continuously differentiable matrix of dimension $m \times m$. Neutral type systems describe processes in which the speed at a given moment depends on the states and speeds at previous moments. Note that the derivative of the initial vector function $d\phi(\eta)/d\eta$ is assumed to be limited.

If we turn to a system with a constant delay, we obtain

$$\frac{dz(\tau)}{d\tau} = t_0 e^{\tau}[\bar{A}(\tau)z(\tau) + \bar{B}(\tau)z(\tau - \sigma)] + \mu\bar{R}(\tau)\frac{dz(\tau - \sigma)}{d\tau},$$

$\bar{R}(\tau) = R(t_0 e^{\tau})$. In this case, it is easy to verify that even for the simplest neutral type equations there are no continuously differentiable solutions under arbitrary initial conditions.

Linear systems of differential equations with linear delay, studied, for example, in [1], as a rule, have constant coefficients on the right side. When such (and more general) systems are unstable, the problem arises of their stabilization over an infinite period of time. The works [2] and [3] are devoted to the stabilization of such systems. Thus, the problem is to obtain sufficient conditions for asymptotic stability and to construct on this basis a stabilization algorithm for some systems with variable coefficients.

We will consider a linear normalized space $\mathbb{R}^m$, in which the norm of the vector $w = \{w_j\}^\top$ (here w_j $(j = 1, ..., m)$ are the components of the vector w, $\top$ is the transpose symbol) is defined, for example, by the equality $\|w\| = \sum\limits_{j=1}^{m} |w_j|$. The norm of the matrix $D = \{d_{ij}\}$ $(i, j = 1, ..., m)$ is defined according to the norm of the vector [4]: $\|D\| = \max\limits_{j} \sum\limits_{i} |d_{ij}|)$.

Here are some definitions that will be needed further [1].

Definition 1. *The solution of the system (1) $x(t)$ defined by a piecewise continuous initial vector function $\phi(\eta)$ is called stable if there exists a constant $\hat{C} > 0$, such that from the condition of boundedness of the quantity $\|\phi(\eta)\|$ it follows the inequality $\|x(t, \phi(\eta))\| < \hat{C}$.*

Definition 2. *If the solution $x(t, \phi(\eta))$, along with stability, has the property $\lim\limits_{t\to\infty} x(t, \phi(\eta)) = 0$, then the solution is asymptotically stable.*

Since we will also consider systems of a neutral type, we will introduce a definition of stability (and asymptotic stability) for such systems.

Definition 3. *A solution of the linear system (2) $x(t)$ defined by a continuously differentiable vector function $\phi(\eta)$, is called stable if there exists a constant $\hat{C}_0 > 0$, such that from the condition of boundedness of $\sup\limits_{\eta} \|\phi(\eta)\| + \sup\limits_{\eta} \|d\phi(\eta)/d\eta\|$ it follows the inequality $\|x(t, \phi(\eta))\| < \hat{C}_0$.*

Definition 4. *If the solution of a linear system of neutral type $x(t, \phi(\eta))$, along with stability, has the property $\lim\limits_{t\to\infty} x(t, \phi(\eta)) = 0$, then the solution is asymptotically stable.*

1. Obtaining Sufficient Conditions for Asymptotic Stability and Instability of Systems Using Lyapunov–Krasovskii Functionals

The asymptotic stability of some systems with linear delay, for example, for positive μ sufficiently close to unity, can be solved using the Lyapunov-Krasovskii type functionals of definite sign

$$V = \hat{W}(x) + \sum_{j=1}^{m} \nu_j \int_{\mu t}^{t} x_j^2(s)ds.$$

Here $\hat{W}(x)$ is a positive definite quadratic form, $\nu_j \neq 0$ are scalar quantities. Let's give the simplest example. Consider the first order equation

$$\frac{dx(t)}{dt} = ax(t) + b(t)x(\mu t). \tag{3}$$

Here $a = \text{const}$, $a < 0$, $b(t)$ is a scalar function of time (argument) t.

To obtain sufficient conditions for the stability of the solution of equation (3) we introduce the functional $V^0(t) = -ax^2(t) + \bar{\alpha} \int_{\mu t}^{t} x^2(s)ds$. Here the constant a is negative. Setting $\bar{\alpha} = -a$, we calculate (by virtue of equation (3)) the derivative $dV^0(t)/dt = ax^2 + 2b(t)x(t)x(\mu t) + \mu a(x(\mu t))^2$, requiring the negative definiteness of the resulting quadratic form of the variables $x(t)$ $x(\mu t)$, [5]. We obtain sufficient conditions for the asymptotic stability of the solution of equation (3)

$$a < 0, \;\; |b(t)| < \sqrt{\mu}(a - \varepsilon), \tag{4}$$

ε is a sufficiently small positive number [1]. If we study the behavior of the solution of equation (3), for example, for $b = \text{const}$ using more accurate methods [6], we obtain a set of inequalities

$$a < 0, \;\; |b| < |a| \tag{5}$$

valid for any $0 < \mu < 1$. Nevertheless, for $\mu \to 1$ the boundaries of the domain (4) are sufficiently close to the boundaries of the domain (5). To study the instability of such systems, we can consider the functional $V_\sigma(\tau) = e^{-\tau} z^2(\tau) + \alpha \int_{\tau-\sigma}^{\tau} z^2(\theta)d\theta$ and calculate $dV_\sigma(\tau)/d\tau$ by virtue of (3). In this way, we similarly obtain sufficient instability conditions for $a > 0, |b| < a$. Thus, the use of functionals of the type $V_\sigma(\tau)$ allows us to obtain sufficient instability conditions for some nonlinear systems.

2. Obtaining Sufficient Conditions for Asymptotic Stability for One System of Neutral Type

Let us now consider a neutral type system

$$\frac{dx(t)}{dt} = A(t)x(t) + B(t)x(\mu t) + R(t)\frac{dx(\mu t)}{dt}, \;\; t \geq 1, \;\; 0 < \mu < 1. \tag{6}$$

By replacing $\tau = \ln(t/t_0)$ let's reduce it to a system with constant delay

$$\frac{dz(\tau)}{d\tau} = t_0 e^{\tau}[\bar{A}(\tau)z(\tau) + \bar{B}(\tau)z(\tau-\sigma)] + \mu\bar{R}(\tau)\frac{dz(\tau-\sigma)}{d\tau}, \ \ \sigma = -\ln(\mu). \tag{7}$$

The system (7) is defined at the moment $\tau = 0$ by the initial vector function $\bar{\phi}(\eta)$, which has a bounded derivative. We assume that the matrices $\bar{A}(\tau), \bar{B}(\tau), \bar{R}(\tau)$ are periodic, of period σ, and differentiable a sufficient number of times. Let $\lambda(\tau)$ be the eigenvalues of the matrix $\bar{A}(\tau)$, $\rho(\tau)$ be the eigenvalues of the matrix $-\bar{A}^{-1}(\tau)\bar{B}(\tau)$ and $\nu(\tau)$ be the eigenvalues of the matrix $\bar{R}(\tau)$. The following conditions are true

$$\begin{array}{l} 1)\ \mathrm{Re}(\lambda(\tau)) < -2\beta, \ \ \beta = const, \ \ \beta > 0, \\ 2)\ |\rho(\tau)| < \delta, \ \ \delta = const, \ \ 0 < \delta < 1, \\ 3)\ |\nu(\tau)| < \gamma, \ \ \gamma = const, \ \ 0 < \gamma < 1. \end{array} \tag{8}$$

Assuming $z_{n+1}(\tau) = z(n\sigma + \tau)$, we pass to the counting system on a finite time interval $0 \le \tau \le \sigma$

$$\varepsilon_n \frac{dz_{n+1}(\tau)}{d\tau} = e^{\tau}[\bar{A}(\tau)z_{n+1}(\tau) + \bar{B}(\tau)z_n(\tau)] + \mu\bar{R}(\tau)\frac{dz_n(\tau)}{d\tau}, \ \ 0 \le \tau \le \sigma. \tag{9}$$

Here $\varepsilon_n = \mu^n/t_0$, $z_{n+1}(0) = z_n(\sigma)$. System (9) for sufficiently large n has a small parameter at the derivative. Let's transform it in the following form

$$\varepsilon_n \frac{d(z_{n+1}(\tau) - \mu\bar{R}(\tau)z_n(\tau))}{d\tau} = e^{\tau}\left[\bar{A}(\tau)z_{n+1}(\tau) + \left(\bar{B}(\tau) - \mu e^{-\tau}\varepsilon_n \frac{d\bar{R}(\tau)}{d\tau}\right) z_n(\tau)\right].$$

Under the conditions imposed on the matrices $\bar{A}(\tau)$, $\bar{A}^{-1}\bar{B}(\tau)$ the system without neutral terms

$$\varepsilon_n \frac{dy_{n+1}(\tau)}{d\tau} = e^{\tau}[\bar{A}(\tau)y_{n+1}(\tau) + \bar{B}(\tau)y_n(\tau)], \ \ 0 \le \tau \le \sigma$$

is exponentially stable [7, 8]. In addition, the quantity

$$\varepsilon_n^{-1} e^{\tau}\left[\bar{A}(\tau)y_{n+1}(\tau) + \left(\bar{B}(\tau) - \mu e^{-\tau}\frac{d\bar{R}(\tau)}{d\tau}\varepsilon_n\right) y_n(\tau)\right]$$

is dominant at $n \to \infty$ in a relationship of the form

$$d([z_{n+1}(\tau) - \mu\bar{R}(\tau)z_n(\tau))/d\tau = \varepsilon_n^{-1} e^{\tau}[\bar{A}(\tau)z_{n+1}(\tau) + (\bar{B}(\tau) - \mu e^{-\tau}\varepsilon_n d\bar{R}(\tau)/d\tau)z_n(\tau)].$$

We will obtain a solution to the system (9) by the method of successive approximations, setting $z_n^0(\tau) = y_n(\tau)$, and, further

$$\frac{d(z_{n+1}^1(\tau) - \mu\bar{R}(\tau)z_n^1(\tau))}{d\tau} = \frac{dz_{n+1}^0(\tau)}{d\tau}. \tag{10}$$

Obviously, due to the boundedness of the derivative of the initial vector function $\bar{\phi}(\eta)$, the initial data of the system (10) are bounded. The right-hand side of this relation is exponentially stable, as is the value $z_{n+1}^0(\tau)$. Having integrated both sides, we obtain from (10) an inhomogeneous difference system

$$z_{n+1}^1(\tau) = \mu\bar{R}(\tau)z_n^1(\tau) + z_{n+1}^0(\tau), \ \ \lim_{n\to\infty}\|z_{n+1}^0(\tau)\| = 0.$$

Let us first consider the asymptotic behavior of a homogeneous system

$$\bar{y}_{n+1}(\tau) = \bar{R}(\tau)\bar{y}_n(\tau). \tag{11}$$

Let us show that there exist constants $\bar{L}_0 > 1$, $\bar{q} : 0 < \bar{q} < 1$, such that the solution of the difference system (11) satisfies the estimate [9]

$$\sup_\tau \|\bar{y}_n(\tau)\| \leqslant \bar{L}_0(\bar{q})^n \sup_\tau \|y_0(\tau)\|, \;\; n = 1, 2, ... \tag{12}$$

Due to the uniform continuity (with respect to τ) of the matrix $\bar{R}(\tau)$, we can split the interval $[0, \sigma]$ into a finite number l of equal intervals of length less than $\bar{\delta}_1$ such that the inequality

$$\|\bar{R}(\tau) - \bar{R}(\tau_j)\| < \varepsilon, \;\; |\tau - \tau_j| < \delta_1,$$

$j = 1, 2, ..., l, \;\; 0 < \tau_1 < ... < \tau_l = \sigma$ is valid. But then (due to the inequality with respect to the eigenvalues of matrix $\bar{R}(\tau)$) for a sufficiently small ε for each interval $[\tau_j, \tau_{j+1}]$ there exist constants $\bar{L}_j > 1$, that for each fundamental matrix $\bar{Y}^0_{n,j}(\tau)$ and any specified interval the following estimate holds:

$$\|\bar{Y}^0_{n,j}(\tau)\| = \|\prod_{i=1}^{n} \bar{R}(\tau + i\sigma)\| < L_j\bar{\gamma}^n, \qquad \tau \in [\tau_j, \tau_{j+1}],$$

$$\bar{\gamma} = \text{const}, \;\; 0 < \gamma < \bar{\gamma} < 1, \;\; j = 1, 2, ..., l.$$

It follows that for any $\tau \in [0, \sigma]$ such an inequality is valid for constants $\bar{L}_0 = \max_j L_j$ $\bar{q} = \bar{\gamma}$, i.e. system (11) is exponentially stable.

Let us show that the solution of the inhomogeneous system (11) also tends to zero as $n \to \infty$. Let's write the solution of this inhomogeneous difference system using the formula of the variation of constants [9]

$$z_{n+1}(\tau) = \bar{R}(\tau)^{n+1}z_0(\eta) + \bar{R}(\tau)^n z^0_1(\tau) + \bar{R}(\tau)^{n-1}z^0_2(\tau) + ... + \bar{R}(\tau)z^0_n(\tau) + z^0_{n+1}(\tau).$$

From this follows

$$\|z_{n+k}(\tau)\|_1 \leq L_0\bar{\gamma}^{n+k}\|z_0(\eta)\|_1 + \bar{L}_0\bar{\gamma}^{n+k-1}\|z^0_1(\tau)\|_1 + ...+$$

$$+\bar{L}_0\bar{\gamma}^n\|z^0_k(\tau)\| + \bar{L}_0\bar{\gamma}^{n-1}\|z^0_{k+1}(\tau)\| + ... + \bar{L}_0\bar{\gamma}\|z^0_{n+k-1}(\tau)\| + \|z^0_{n+k}(\tau)\|.$$

3. Stabilization Algorithm for One Neutral Type System

Let's consider the stabilization algorithm of some controlled system with neutral type linear delay

$$\frac{dx(t)}{dt} = A(t)x(t) + B(t)x(\mu t) + R(t)\frac{dx(\mu t)}{dt} + Cu(t).$$

Here $u(t)$ is a vector function of the control action of dimension r, C is a matrix of dimension $m \times r$, $1 \leq r \leq m$. Now let the solution of this controlled system for $u(t) \equiv 0$ be unstable, or stable, but not asymptotically. Passing to the variable τ in the absence of control, we have the system

$$\frac{dz(\tau)}{d\tau} = t_0e^\tau[\bar{A}(\tau)z(\tau) + \bar{B}(\tau)z(\tau - \sigma)] + \mu\bar{R}(\tau)\frac{dz(\tau - \sigma)}{d\tau}.$$

Since the system without neutral terms is dominant, we first examine the behavior of the eigenvalues $\lambda(\tau)$ of the matrix $\bar{A}(\tau)$.

If for the roots $\lambda_j(\tau)$ of the characteristic equation

$$\det\{\bar{A}(\tau) - \lambda E_m\} = 0,$$

(here E_m is the identity matrix of dimension $m \times m$) the first of the asymptotic stability conditions (8) is violated, then first (in view of what was proved earlier in [8]) we stabilize the system without delay of the form

$$\varepsilon_n dy_{n+1}(\tau)/d\tau = e^{\tau}[\bar{A}(\tau)y_{n+1}(\tau) + C\bar{u}_{n+1}(\tau)], \ \ n = N, N+1, ...,$$

namely, we divide the segment $[0, \sigma]$ into k parts

$$0 = \tau_0 < \tau_0 + \Delta < \tau_0 + 2\Delta < ... < \tau_0 + k\Delta = \sigma$$

and correct the matrix $\bar{A}(\tau)$ in some way at some points τ_j, where the eigenvalues of these matrices $\bar{\lambda}_i^j$ do not satisfy the first of the inequalities. A fairly effective method of stabilization is proposed in [10], namely, we solve the nonlinear equation

$$\Gamma(\tau_j)\bar{A}(\tau_j) + \bar{A}^{\top}(\tau_j)\Gamma(\tau_j) - 2\Gamma(\tau_j)CC^{\top}\Gamma(\tau_j) = -\alpha\Gamma(\tau_j).$$

Here $\Gamma(\tau_j)$ is a symmetric matrix, α is a positive parameter that can be set. Finding the unknown matrix $\Gamma(\tau_j)$ due to the definition of the desired control in the form

$$\bar{u}(\tau) = -C^{\top}\Gamma(\tau_j)z(\tau) \tag{13}$$

leads to the fact that for the values of the stabilized matrix $A_s(\tau)$ (at the points τ_j) we obtain the equality $A_s(\tau_j) = \bar{A}(\tau_j) - CC^{\top}\Gamma_j$ and for its eigenvalues $\mathrm{Re}(\lambda_j^s(\tau_j)) \leq -0.5\alpha$; then, for a sufficiently large parameter α for each $\bar{\tau}$ from the intervals $[\tau_j, \tau_{j+1}]$ we have the estimate $\mathrm{Re}(\lambda_j^s(\bar{\tau})) \leq -0.25\alpha$. Note the following: if the matrix $\bar{A}^{-1}(\tau_j)$ exists, it is more convenient to find the corresponding matrix $\Gamma(\tau_j)$ by solving the linear equation

$$\Gamma^{-1}(\tau_j)\bar{A}(\tau_j)^{\top} + \bar{A}(\tau_j)\Gamma^{-1}(\tau_j) - 2CC^{\top} = -\alpha\Gamma^{-1}(\tau_j), \tag{14}$$

determine from it $\Gamma^{-1}(\tau_j)$ and then find $\Gamma(\tau_j)$, $u(\tau_j)$. At the same points where the first of the inequalities (8) is satisfied, we set the control $\bar{u}_{n+1}(\tau) \equiv 0$. Obviously, we obtain the value of the stabilized matrix at k points of the segment $[0, \sigma]$, namely, in the set of points $0, \Delta, 2\Delta, ..., \sigma$. Next, we construct a trigonometric polynomial $P_l(\tau)$, of period σ: $P_l(\tau_0 + j\Delta) = A_s(\tau_0 + j\Delta)$, $j = 0, ..., k$. Since the matrix $\bar{A}(\tau)$ has a bounded derivative, for a sufficiently large k this polynomial $P_l(\tau) \approx A_s(\tau)$ for all $\tau \in [0, \sigma]$, the eigenvalues $\bar{\lambda}$ of this matrix satisfy the inequality

$$\mathrm{Re}(\bar{\lambda}(\tau)) < -\bar{\delta}, \ \ \bar{\delta} = \mathrm{const}, \ \ \bar{\delta} > 0.$$

It is known [11], that when the last inequality is satisfied, the fundamental matrix $Y_{n+1}(\tau, s, \mu^n)$ of the stabilized system

$$\varepsilon_n \frac{dy_{n+1}(\tau)}{d\tau} = e^{\tau}[A_s(\tau)y_{n+1}(\tau)], \ \ n = N, N+1, ...$$

for a sufficiently small value of ε_n/t_0 admits the estimate

$$\|Y_{n+1}(\tau, s, \mu^n)\| \le \bar{C} \exp\left\{-\frac{t_0\bar{\delta}}{2\mu^n}(e^\tau - e^s)\right\}, \ \ \bar{C} = \text{const}, \ \ \bar{C} > 1, \ 0 \le s \le \tau \le \sigma.$$

This implies the asymptotic stability of the stabilized system without delay terms. Obviously, for a sufficiently small value of $\|\bar{B}(\tau)\|$ and the smallness of μ the solution of the system

$$\varepsilon_n \frac{dz_{n+1}(\tau)}{d\tau} = e^\tau[A_s(\tau)z_{n+1}(\tau) + \bar{B}(\tau)z_n(\tau)] + \mu R(\tau)\frac{dz(\tau-\sigma)}{d\tau}, \ \ \varepsilon_n = \frac{\mu^n}{t_0}, \ \ n = 0, 1, 2, ...$$

is asymptotically stable [8]. If the resulting system is unstable, then we consider the asymptotic behavior of the system without neutral terms and, in the case of its instability, we stabilize it with delayed terms. Namely, there exist regions $D_j \in [0, \sigma]$ such that inside them for $|\rho(\bar{\tau}_i)| \ge 1$ the second of the inequalities (8) is not satisfied (otherwise the differential-difference system without neutral terms would be asymptotically stable [7]). Therefore, further stabilization is necessary, namely, stabilization of the degenerate system

$$\bar{z}_{n+1}(\tau) = -A_s^{-1}(\tau)\Big[\bar{B}(\tau)\bar{z}_n(\tau) + Cw_n(\tau)\Big], \ \ n = 0, 1, 2, ... \tag{15}$$

We assume that $\det \bar{B}(\tau) \ne 0$. In this case, there is a matrix $\Big(A_s^{-1}(\tau)\bar{B}(\tau)\Big)^{-1}$ and the original system with constant delay is stabilized by methods using an algorithm similar to that given in relations (13), (14). We again divide the segment $[0, \sigma]$ into $\bar{k}$ equal parts $0 = \bar{\tau}_0 < \bar{\tau}_0 + \bar{\Delta} < \bar{\tau}_0 + 2\bar{\Delta} < ... < \bar{\tau}_0 + \bar{k}\bar{\Delta} = \sigma$, and correct, now the matrix $-A_s^{-1}(\tau)\bar{B}(\tau)$ at those points $\bar{\tau}_j$, where the eigenvalues of this matrix $\bar{\rho}_i^j$ are greater than or equal to one in absolute value. To do this, we first solve matrix equations of the form [12]

$$\begin{gathered}\bar{R}_{\bar{j}}^{-1} + \bar{C}_{\bar{j}}\bar{C}_{\bar{j}}^\top = \frac{1}{\bar{\beta}}\bar{A}_{\bar{j}}\bar{R}_{\bar{j}}^{-1}\bar{A}_{\bar{j}}^\top, \\ \bar{\beta} = \text{const}, \ \ 0 < \bar{\beta} < 1, \ \ \bar{A}_{\bar{j}} = -A_s^{-1}(\bar{\tau}_{\bar{j}})\bar{B}(\bar{\tau}_{\bar{j}}), \ \ \bar{C}_{\bar{j}} = -A_s^{-1}(\bar{\tau}_{\bar{j}})C_{\bar{j}},\end{gathered} \tag{16}$$

calculate matrices $\bar{R}_{\bar{j}}$ and set control at points $\bar{\tau}_j$

$$\bar{w}_n(\bar{\tau}_{\bar{j}}) = -\Big(E_r + \bar{C}_{\bar{j}}\bar{R}_{\bar{j}}\bar{C}_{\bar{j}}^\top\Big)^{-1}\bar{C}_{\bar{j}}^\top\bar{R}_{\bar{j}}\bar{A}_{\bar{j}}z_n(\bar{\tau}_{\bar{j}}) = P(\bar{\tau}_{\bar{j}})z_n(\bar{\tau}_{\bar{j}}). \tag{17}$$

Note that satisfactory solutions are obtained for small $\bar{\beta}$. At the remaining points $\tau_0 + i\bar{\Delta}$ we assume that the control is equal to zero. Again we obtain a set of values of the stabilized matrix $\bar{B}_s(\tau_0 + i\bar{\Delta})$, $i = 0, 1, ..., \bar{k}$. In this case, the eigenvalues of the stabilized matrix at these points do not exceed in absolute value the value $\bar{d}$, $\bar{d} = \text{const}$, $0 < \bar{d} < 1$. Obviously, the entire stabilized matrix can be approximated with a high degree of accuracy by the interpolation trigonometric polynomial $P_{\bar{n}}(\tau)$ [13], where $P_{\bar{n}}(\tau_0 + i\bar{\Delta}) = \bar{B}_s(\tau_0 + i\bar{\Delta})$ (or by approximation using splines). The the eigenvalues $\bar{\rho}(\tau)$ of the matrix $\bar{B}_s(\tau)$ will, at least, satisfy the inequality:

$$|\bar{\rho}(\tau)| < \bar{d} + \varepsilon < 1, \ \ 0 \le \tau \le \sigma,$$

ε is a small positive number. The system without neutral terms becomes exponentially stable. Now we check the obtained system again (already with neutral terms), i.e. we establish its asymptotic properties. In case of instability (or stability, but not asymptotic), we correct, the matrix $\mu\bar{R}(\tau)$ by methods similar to those used earlier for calming the discrete system (15).

4. Example

Let's consider a system with an exponential multiplier on the right-hand side

$$\frac{dz(\tau)}{d\tau} = e^{\tau}\left[\bar{A}_2(\tau)z(\tau) + \bar{B}_2(\tau)z\left(\tau - \frac{\pi}{4}\right)\right] + e^{-\frac{\pi}{4}}\bar{R}_2\frac{dz(\tau-\frac{\pi}{4})}{d\tau},$$
$$t_0 = 1,\ \ \tau \geq 0,\ \ \phi(\eta) = \{1,1\}^{\top},\ \ -\pi/4 \leq \eta \leq 0, \qquad (18)$$

$u(\tau)$ is a scalar, the matrices $\bar{A}_2(\tau)$, $\bar{B}_2(\tau)$, $\bar{R}_2$ and the vector $\bar{C}_2$ are defined as follows

$$\bar{A}_2(\tau) = \begin{pmatrix} 1.8 & 1.6 \\ 0.6\cos^2(4\tau) & 1.5\sin^2(4\tau) \end{pmatrix}, \qquad \bar{B}_2(\tau) = \begin{pmatrix} -1+0.5\cos(8\tau) & -0.5 \\ 0.5\sin(8\tau) & 1 \end{pmatrix},$$

$$\bar{R}_2 = \begin{pmatrix} 2.5 & 3 \\ 0.1 & 1 \end{pmatrix},\ \ \bar{C}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix},\ \ \bar{A}_2(\tau+\pi/4) = \bar{A}_2(\tau),\ \ \bar{B}_2(\tau+\pi/4) = \bar{B}_2(\tau).$$

The solution of a system without neutral terms in the absence of control is unstable. The system without neutral terms is dominant, let's stabilize it. To do this, first consider a controlled system without delayed components:

$$\frac{dy(\tau)}{d\tau} = e^{\tau}[\bar{A}_2(\tau)y(\tau) + \bar{C}_2u_1(\tau)],\ \ \tau \geq 0,\ \ y^0 = \{1,1\}^{\top}. \qquad (19)$$

Let the control be $u_1 \equiv 0$. In Figure 1, the graph of the system (19) without delay terms (shown as a dotted line) differs little from the graph of the solution of the system (18) without neutral terms (i.e. with the zero matrix $\bar{R}_2$) (shown as a solid line).

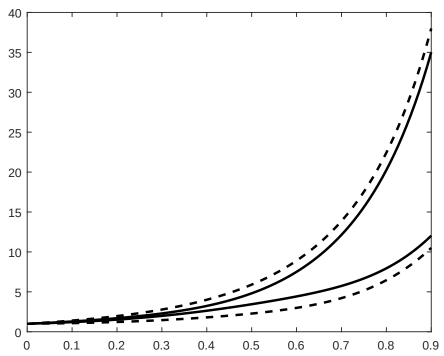

Fig. 1

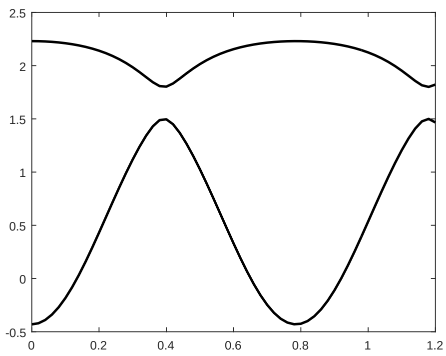

Fig. 2

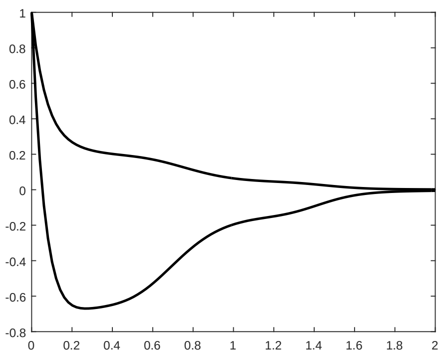

Fig. 3

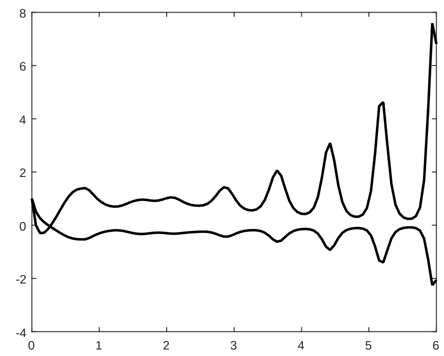

Fig. 4

The graph of the eigenvalues $\lambda(\tau)$ of the matrix $\bar{A}_2(\tau)$ is shown in Figure 2. Since there are regions where $\mathrm{Re}(\lambda(\tau)) > 0$, we stabilize the system without delay terms in accordance with the algorithm presented by equalities (8)–(10), for which we divide the interval $[0, \pi/4]$ into k parts. For each value of τ_j, we need to solve the calming problem, i.e. solve such equations and construct a control at these points. However, in our case, constructing a control for $\alpha = 3.5$ at the point $\tau_0 = 0$, we obtain that the corrected, matrix $A_s(\tau) = \bar{A}_2(\tau) - CC^\top\Gamma_0$ has all eigenvalues satisfying the first of the inequalities (8). Consequently, the system stabilized in this way only at the initial point is exponentially stable. The solution graph of the stabilized system without delay is shown in Figure 3.

This stabilized system is unstable in the presence of delay terms, which can be seen from the corresponding graph of the solution of the system shown in Figure 4. In this case, the eigenvalues of the degenerate matrix at some points are greater than one in absolute value. The graph of the eigenvalues of the singular matrix is shown in Figure 5. Further stabilization is necessary, which is carried out by the algorithm for stabilizing difference (singular) systems at the points $\tau_3 = 0.3$, $\tau_4 = 0.4$, $\tau_5 = 0.5$ in accordance with the rule given in (16)–(17).

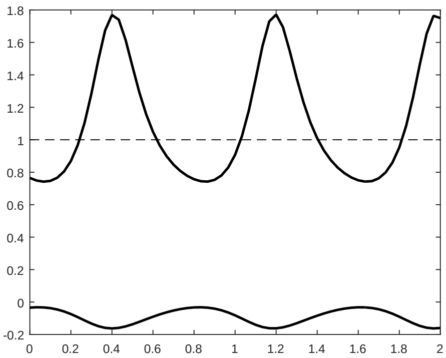

Fig. 5

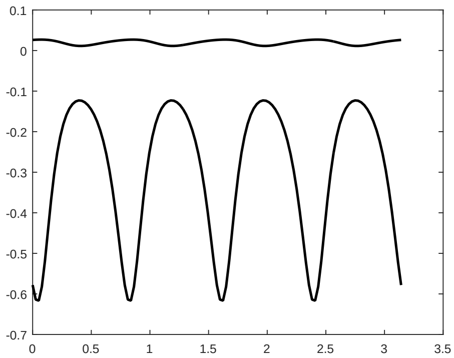

Fig. 6

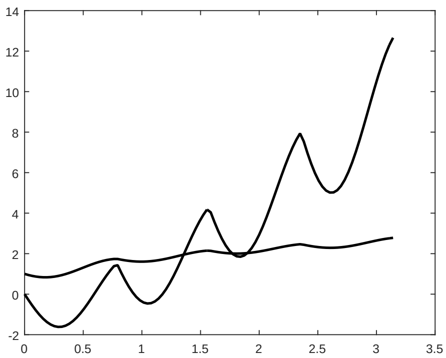

Fig. 7

After correcting the matrix $-A_s^{-1}(\tau)B(\tau)$ it is rather difficult to show the asymptotic properties using numerical calculation of the corresponding system without neutral terms for large τ (the system is stiff [13]). In order to establish the asymptotic stability of the solution of the stabilized system

$$\varepsilon_n \frac{dz^0_{n+1}(\tau)}{d\tau} = e^\tau [A_s(\tau) z^0_{n+1}(\tau) + B_s(\tau) z^0_n(\tau)], \;\; 0 \le \tau \le \pi/4, \;\; z_{n+1}(0) = z_n(\pi/4)$$

we have the following property: its asymptotic behavior (in the first approximation) depends on the asymptotic properties of the difference (perturbed) system [8]

$$\bar{z}^0_{n+1}(\tau) = -A_s(\tau)B_s(\tau)\bar{z}^0_n(\tau) + Y_{n+1}(\tau, s, \mu^n)\bar{z}_{n-1}(\tau+\sigma),$$
$$\sigma = \tfrac{\pi}{4},\ \ \bar{z}_{n+1}(0) = \bar{z}_n\left(\tfrac{\pi}{4}\right),\ \ 0 \le \tau \le \sigma,\ \ \mu = e^{-\frac{\pi}{4}}. \tag{20}$$

The graph of the eigenvalues of the matrix $-A_s^{-1}(\tau)B_s(\tau)$ is shown in Figure 6. We will show that for $\tau = \sigma$ the solution of homogeneous system is exponentially stable ($z_n(\sigma) \to 0$ at $n \to \infty$). Obviously, the first approximation of this system for $\tau = \sigma$ for sufficiently large $n \ge N$ is the homogeneous (unperturbed) system

$$\bar{y}^0_{n+1}(\sigma) = -A_s^{-1}(\sigma)B_s(\sigma)\bar{y}^0_n(\sigma),$$

$$\sigma = \frac{\pi}{4},\ \ \bar{z}_{n+1}(0) = \bar{z}_n\left(\frac{\pi}{4}\right),\ \ 0 \le \tau \le \sigma.$$

Its solution is exponentially stable, i.e. the following estimate is valid

$$\|\bar{y}^0_n(\sigma)\| \le \bar{L}(\bar{q})^n\|\bar{y}^0_0(\eta)\|_1,\ \ \bar{L} = \text{const},\ \ \bar{L} > 1,\ \ 0 < \bar{q} < 1.$$

Let us now consider the inhomogeneous system (20). Let's write its solution for $0 < \tau < \sigma$ using the formula of the variation of constants

$$\bar{z}^0_{n+k}(\tau) = \Big((-A_s^{-1}(\tau)B_s(\tau)\Big)^k\bar{z}^0_n(\tau) + \sum_{j=1}^{k}\Big((-A_s^{-1}(\tau)B_s(\tau)\Big)^{k-j}Y_{n+j}(\tau, s, \mu^n)\bar{z}_{n+j-1}(\tau+\sigma).$$

Since $\|\Big((-A_s^{-1}(\tau)B_s(\tau)\Big)^i\| < \bar{L}\bar{q}^i$ $(i = 1, 2, ...)$, and $\bar{q} = \alpha\hat{q}$, we obtain the estimate

$$\|\bar{z}^0_{n+k}(\tau)\| < \frac{\hat{L}\bar{L}}{1-\alpha}\hat{q}^k\|\bar{z}^0_n(\tau)\|_1.$$

The presence of an unstable neutral matrix $\mu\bar{R}$, having eigenvalues $\nu_1 = 0.614$, $\nu_2 = 1.38$, even under the condition of calming down the system of the form

$$\frac{dz^0(\tau)}{d\tau} = e^{\tau}[A_s(\tau)z^0(\tau) + B_s(\tau)z^0(\tau - \pi/4)]$$

still leads to the fact that the system stabilized in this way, having only neutral terms on the right-hand side, is unstable (for example, with the initial data $\left\{\frac{dz_0^1(\eta)}{d\eta} = \sin(\eta); \frac{dz_0^2(\eta)}{d\eta} = \cos(\eta)\right\}^{\top}$ (Fig. 7)). In this case, the matrix $\mu\bar{R}_2$ has eigenvalues $\nu_1 = 1.2213$, $\nu_2 = 0.3745$. Performing stabilization by neutral terms, i.e. assuming in addition the control

$$u_3(\tau) = -(E_1 + C^{\top}\bar{R}C)^{-1}C^{\top}R\mu\bar{R}_2\left\{\frac{dz^1(\tau)}{d\tau}; \frac{dz^2(\tau)}{d\tau}\right\}^{\top} =$$

$$= -0.1435\left\{\frac{dz^1(\tau)}{d\tau}\right\} - 0.2950\left\{\frac{dz^2(\tau}{d\tau}\right\}$$

we obtain that the stabilized neutral, matrix has components

$$\begin{pmatrix} 0.9963 & 1.0778 \\ -0.2414 & -0.1240 \end{pmatrix},$$

its eigenvalues $\rho_1 = 0.66764$, $\rho_2 = 0.2047$. We have a stabilized system of neutral type. Finally, we obtain control $u(\tau) = u_1(\tau) + u_2(\tau) + u_3(\tau)$, where $u_1(\tau)$ linearly depends on the values $z_1(\tau), z_2(\tau)$; $u_2(\tau)$ linearly depends on the values $z_1(\tau - \sigma), z_2(\tau - \sigma)$; $u_3(\tau)$ linearly depends on the values $\dfrac{dz_1(\tau-\sigma)}{d\tau}, \dfrac{dz_2(\tau-\sigma)}{d\tau}$.

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МЕТОДЫ ИССЛЕДОВАНИЯ АСИМПТОТИЧЕСКИХ СВОЙСТВ И УСПОКОЕНИЯ НЕКОТОРЫХ СИСТЕМ С ЛИНЕЙНЫМ ЗАПАЗДЫВАНИЕМ

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В работе предлагаются методы получения достаточных условий асимптотической устойчивости и неустойчивости для систем дифференциальных уравнений, содержащих линейное запаздывание. На основании этих условий исследуются некоторые системы линейных дифференциальных уравнений, при этом для одной из них произведена стабилизация на бесконечном промежутке времени.

Ключевые слова: асимптотическая устойчивость; линейное запаздывание; стабилизация.

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