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NUMERICAL SOLUTION OF THE STARTING CONTROL PROBLEM FOR THE BOUSSINESQ – LOVE MODEL

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The paper presents numerical studies of the starting control problem using a compromise objective functional. It develops an algorithm to find a numerical solution to the control problem using an adapted projection method for degenerate equations. Computational experiments are set up to demonstrate the effectiveness of the proposed algorithms.

Keywords: Sobolev type equations of higher order; starting control; adapted projection method; numerical solution.

Introduction

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with $\partial\Omega$ boundary of class C^∞. In the cylinder $\Omega \times \mathbb{R}$ Rlet us consider the Boussinesq – Love equation

$$(\lambda - \Delta)x_{tt} = \alpha(\Delta - \lambda')x_t + \beta(\Delta - \lambda'')x + y, t \in (0,\tau), s \in \Omega, \tag{1}$$

with the following boundary condition

$$x(s,t) = 0, \quad (s,t) \in \partial\Omega \times (0,\tau). \tag{2}$$

The coefficients of the equation are real numbers, while $u(x,t)$ is the desired function, which has a diverse physical meaning depending on the model. For example, with a certain choice of coefficients, equation (1) is transformed into an equation named after Sergey L'vovich Sobolev and will describe small oscillations of a rotating fluid [1]. (1) is also a more general case of the equation

$$\frac{\partial^2\rho}{\partial t^2} - c_0^2\frac{\partial^2\rho}{\partial x^2} = mc_0^2\tau\frac{\partial^3\rho}{\partial t\partial x^2} - mc_0^2\frac{\partial^4\rho}{\partial t^2\partial x^2}, \tag{3}$$

where ρ is density, c_0 is sound velocity, τ is relaxation time, wherein the first term on the right-hand side is responsible for the attenuation of the sound wave due to thermal conductivity and viscosity, and the second term regulates dispersion effects [2]. Equation (3) describes the propagation of gravitational and gyroscopic waves in dispersive media, for example, surface acoustic waves. B.A. Iskanderov studied internal gravitational and gyroscopic waves in an unbounded cylindrical domain [3]. A private case of the equation (1) is the Boussinesq – Love equation

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} - \frac{\partial^4 u}{\partial x^2\partial t^2} = f(u), \tag{4}$$

which describes the propagation of longitudinal waves in a thin elastic rod under an external load. We will also call equations (1) and (4) the Boussinesq – Love equations.

The Boussinesq – Love equation for $n = 1$ 1describes longitudinal oscillations of an elastic rod taking into account inertia. In [4], the equation of form (1) is called «damped generalized IMBq equation» used to study damped small disturbances of the free liquid surface, where α is the hydrodynamic damping coefficient.

Problem (1) – (2) in the appropriate Hilbert spaces $\mathfrak{X}$ and $\mathfrak{Y}$, $\mathfrak{U}$ can be reduced to the operator-differential Sobolev type equation,

$$A\ddot{x} = B_1\dot{x} + B_0x + Cy, \tag{5}$$

with operators $A, B_1, B_0 \in \mathcal{L}(\mathfrak{X};\mathfrak{Y})$, $C \in \mathcal{L}(\mathfrak{Y})$, and functions $y : [0,\tau) \subset \mathbb{R}_+ \to \mathfrak{Y}$ $(\tau < \infty)$.

Let us consider the Showalter – Sidorov problem [5]

$$P(x(0) - u_0) = 0,\ P(\dot{x}(0) - u_1) = 0. \tag{6}$$

We are interested in the starting control problem, which consists in finding the pair $(\hat{x}, \hat{u})$, where $\hat{x}$ is a solution to problem (5), (6), and $\hat{u} = (u_0, u_1) \in \mathfrak{U}_{ad}$ is control for which

$$J(\hat{x}, \hat{u}) = \min_{(x,u)\in\mathfrak{X}\times\mathfrak{U}_{ad}} J(x, u). \tag{7}$$

Here, $J(x, u)$ is a specially constructed functional, $\mathfrak{U}_{ad}$ is a closed and convex set in the control space $\mathfrak{U}$.

The optimal control of linear first-order Sobolev type equations was first studied by G.A. Sviridyuk together with his follower A.A. Efremov [6]. They considered the Cauchy problem for linear Sobolev-type equations with relatively bounded or relatively sectorial operators, which resulted in the investigation of various formulations of linear control problems, including their study with the Cauchy problem, the Showalter – Sidorov problem, the initial-final problem, or even with a multipoint problem. The first studies of control problems for semilinear Sobolev type equations were carried out in [7]. An abstract theory was constructed for studying optimal control problems to solve to the Showalter – Sidorov problem, as well as the initial-final problem with an s-monotone and p-coercive operator. Alena Aleksandrovna Zamyshlyaeva and Olga Nikolaevna Tsyplenkova investigated various optimal control problems for Sobolev type equations of higher order [8]. This work will study the starting control problem with a compromise objective functional.

1. Numerical Algorithm

The obtained theoretical conclusions were used to build a computational algorithm for determining the initial control action to solve to the Showalter – Sidorov problem as applied to the Boussinesq – Love model. The studied problem is distinguished by the type of control action, namely $u = u(s)$ control as an unknown state is included in the initial condition of the problem.

Let us use σ to denote the spectrum of the operator $(-\Delta)$ with a homogeneous Dirichlet condition, $\{\lambda_k\}$ the set of eigenvalues, and $\{\varphi_k\}$ the family of corresponding eigenfunction that are orthonormal with respect to the scalar product $< \cdot, \cdot >$ from $L_2(\Omega)$. Let us find a numerical solution to the problem in the form of Galerkin approximations

$$\tilde{x}(s,t) = \sum_{k=1}^{K} x_k(t)\varphi_k(s). \tag{8}$$

In the starting control problem, the right-hand side of the equation is a known function. To form the system of differential or algebra-differential equations, let us represent the right-hand side of equation (1) as follows

$$\tilde{y}(s,t) = \sum_{k=1}^{K} y_k(t)\varphi_k(s). \tag{9}$$

We obtain the initial state of the system from the modified Showalter – Sidorov condition

$$(\lambda' - \Delta)(\dot{x}(s,0) - u_1(s)) = 0, \quad (\lambda - \Delta)(x(s,0) - u_0(s)) = 0, \ s \in \Omega. \tag{10}$$

We also present the function $u(s)$ in the following form

$$\tilde{u} = \sum_{k=1}^{K} u_k \varphi_k(s). \tag{11}$$

We apply the decomposition method and linearize the original equation (1).

The algorithm for finding the starting control problem is reduced to the following steps:

Step 1. Input the parameters of the Boussinesq – Love equation, the right-hand side of the equation $y(s,t)$, the eigenvalues and eigenfunctions of the Sturm – Liouville problem, as well as the number of Galerkin approximations K. These data are used to forman approximate solution $\tilde{x}(s,t)$ and control $\tilde{u}(s)$ in the form of Galerkin sums. This approximate solution is substituted into the Boussinesq – Love equation and the Showalter – Sidorov condition.

Step 2. Use the given λ to check (whether λ belongs to the spectrum of the operator $(-\Delta)$) to which of the two cases the mathematical model belongs (degenerate or non-degenerate case). After that, we substitute sums (8), (9) into the first equation (1). We multiply the obtained equation scalarly by the eigenfunctions $\varphi_k(s),\ k = 1, ..., K$, to obtain the following system of equations

$$< (\lambda - \Delta)x_{tt}, \varphi_k > = < \alpha(\Delta - \lambda')x_t, \varphi_k > + < \beta(\Delta - \lambda'')x, \varphi_k > + < y, \varphi_k >, \tag{12}$$

with the following condition

$$< (-\lambda - \Delta)(x(0) - u_0), \varphi_k > = 0, \quad < (-\lambda - \Delta)(x_t(0) - u_1), \varphi_k > = 0. \tag{13}$$

Depending on the parameter λ, the equations in this system can be differential or algebraic. Let us analyze these cases:

(i) If $\lambda \notin \sigma$, all the equations of system (12) are ordinary differential equations of the second order. To solve this system for $x_k(t)$, $k = 1, ..., K$, we use the initial conditions (13), to find K initial conditions $x_k(0) = u_k$, $k = 1, .., K$. We express the unknown coefficients $x_k(t)$ in the approximate solution $\tilde{x}(s,t)$ through $f_k(t)$, u_k, $k = 1, ..., K$.

(ii) If $\lambda \in \sigma$, one equation of the system degenerates. We use condition (13) to find $(K-1)$ initial condition. We solve the system of algebraic differential equations and express the unknown coefficients $x_k(t)$, $k = 2, .., K$ in the approximate solution $\tilde{x}(s,t)$ through $f_k(t)$, u_k, $k = 1, ..., K$. We find $a_1(t)$ from the algebraic equation. Let $u_1 = x_1(0)$.

Step 3. Find the minimum of the functional and substitute the obtained expansions into the corresponding penalty objective functional

$$J(x,u) = \mu \sum_{q=0}^{2} \int_0^{\tau} ||x^{(q)} - \tilde{x}^{(q)}||^2_{\mathfrak{X}} dt + \nu \|u\|^2_{\mathfrak{U}} \to \inf, \tag{14}$$

where $\tilde{x} = \tilde{x}(s,t)$ is a fixed state of the system to be achieved with a minimum initial impact $u = (u_0, u_1)$.

Step 4. Substitute the obtained functions $x_i(t)$, $k = 1, ..., K$, $f_i(t)$, $k = 1, ..., K$, into the corresponding penalty functional. After integration, we obtain the function of many variables with respect to the unknowns.

Step 5. Find the coefficients u_{kn} so that the function u_k provides a minimum to the penalty functional of the problem. Thus, the problem is reduced to finding the extremum of the function of several variables. We find the minimum of the functional and set up the functions $\tilde{x}(s,t)$, $\tilde{u}(s)$.

2. Program Description

The algorithm described in the previous paragraph is used as a basis to develop a program for finding an approximate solution to the control problem. The implementation is done in the Maple programming language and integrated into the Maple computer algebra system, version 17.

The program consists of the following successive processing stages:

– Input stage: reading input data.
– Decomposition: decomposition of the original equation into its constituent parts.
– Formation of an approximate solution: approximate framing of the desired solution.
– Quality assessment: construction of the functional to assess the accuracy of the solution.
– Optimization: minimization of the obtained quality functional.
– Calculation of the result: obtaining the final approximate solution.
– Presentation of results: display of the found solution and visualization of the graph.

Step 1. Set the initial data: We determine the coefficients of the equation, the number of terms of the approximate solution, the eigenfunctions and eigenvalues of the homogeneous Dirichlet problem for the Laplace operator $(-\Delta)$ and set the interval length.

Step 2. Form approximate solutions: Approximate solutions and control functions are constructed as Galerkin expansions using **for()** type cycle from 1 to K.

Step 3. Substitute expansions: The expanded terms are substituted into the main equation using the **subs** command to form an expression for subsequent analysis.

Step 4. Generate the system of equations: The previously obtained expression is multiplied by the eigenfunctions $\varphi_k(s)$ and integrated over a given interval from 0 to π. We form a system of equations to be checked for degeneracy using the **if()** command.

Step 5. Solve the system of differential or algebra-differential equations: We use

the **dsolve()** command to solve the resulting system of equations with the Showalter – Sidorov initial conditions, which contain the control function (starting control) with respect to the sought variables $a_k(t)$.
Step 6. Determine the quality functional: We use the **subs()** procedure to construct the quality functional. The control functions are approximated by polynomials.
Step 7. Minimize the quality functional: We use the **Optimization** suite and **NPLSolve()** minimization procedure to find the minimum value of the functional and then form the final solution to the problem.
Step 8. Present the result: We display the final solution on the screen and additionally construct an appropriate graph.

3. Computational Experiment

Let us give several examples to demonstrate how the program operates. The calculations are made on the $(0, \pi)$ interval for clarity, since in this range the eigenfunctions of the Dirichlet problem for the Laplacian take the simplest form. Let us first consider the situation when λ is not included in the spectrum of the Laplace operator in the considered region Ω. The most interesting is the opposite case – when $\lambda \in \sigma(\Delta)$, as considered in the second example.

Example 1. Let us consider starting control problem (1), (2), (10), (14) on $(0, \pi)$ at the fixed values of the coefficients $\lambda = 1$, $\lambda_1 = -1$, $\lambda_2 = -1$, $\alpha = 3$, $\beta = 2$, $N = 3$, $M = 3$.

The Showalter – Sidorov conditions will take the form

$$a_1(0) = u_{10}, a_2(0) = u_{20}, a_3(0) = u_{30},$$

$$(D(a_1))(0) = u_{11}, (D(a_2))(0) = u_{12}, (D(a_3))(0) = u_{13}.$$

Since the eigenfunctions of the Dirichlet problem are the functions $\varphi_k = \sin ks$, the solution and control can be presented as follows:

$$x(s,t) = \sum_{k=1}^{3} a_k(t) \cdot \sin ks,$$

$$u_0(s) = u_{10} \cdot \sin(s) + u_{20} \cdot \sin(2s) + u_{30} \cdot \sin(3s), \quad u_1(s) = u_{11} \cdot \sin(s) + u_{12} \cdot \sin(2s) + u_{13} \cdot \sin(3s).$$

We obtain the following system of differential equations

$$\begin{cases} \dfrac{d^2 a_1(t)}{dt^2} = 0, \\ 5\dfrac{d^2 a_3(t)}{dt^2} + 12\dfrac{da_2(t)}{dt} + 8a_3(t) = 0, \\ \dfrac{9}{2}\dfrac{da_2(t)}{dt} + 3a_2(t) - t + \dfrac{5}{2}\dfrac{d^2 a_2(t)}{dt^2} = 0. \end{cases}$$

This system contains only second-order differential equations, i.e. λ is not included in the spectrum of the Laplace operator in the considered region.

We construct functional (14) and set the required state of the system

$$x_d = \frac{2}{\pi}\left(\frac{\sin s}{3} + \sin 2s\right).$$

We find the minimum of the functional as a function of several variables on a closed and convex subset of admissible controls and obtain $J_{min} = 0.10136077$, as well as control coefficients $u_{10} = 0.25093857$, $u_{11} = -0.47183122$, $u_{12} = 0.11827694$, $u_{13} = 0.14073235$, $u_{20} = 0.37262641$, $u_{30} = 0.8742964$. We substitute the found control components into the solution and obtain

$$\tilde{x}(t,s) = (-0.47183122t + 0.25093857)\sin s + (0.91322267e^{-0.9t}\sin(0.62449980t)+ \\ +0.87262641e^{0.9t}\cos(0.62449980t) - 0.5 + 0.(3)t)\sin 2s+ \\ +e^{-1.2t}(0.61411980\sin(0.4t) + 0.8742964\cos(0.4t))\sin 3s.$$

Figure 1 shows the obtained approximate solutions $\tilde{x}$ together with the required state x_d at time $t = T$; Figure 2 shows the control function $\tilde{u}$, where the solid line shows the required state x_d and the dotted line shows $\tilde{x}$. Figure 3 shows the obtained approximate solution $\tilde{x}$ for $t \in (0,1)$.

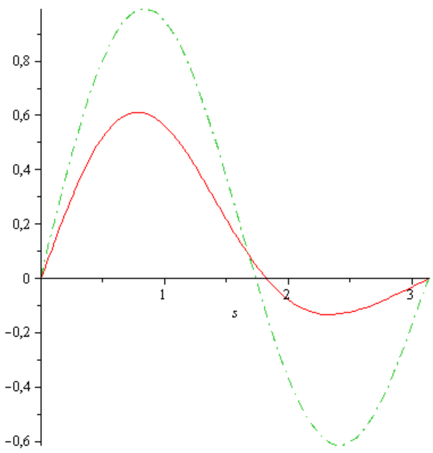

Fig. 1 Numerical solution graph

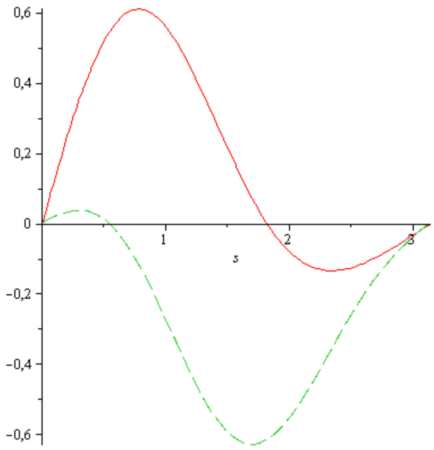

Fig. 2 Problem solution control

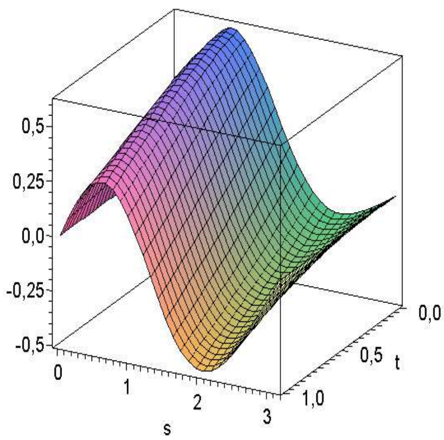

Fig. 3 Numerical solution graph

Example 2. Let us consider starting control problem (1), (2), (10), (14) on the interval $(0, \pi)$ at given coefficients $\lambda = -1$, $\lambda_1 = 2$, $\lambda_2 = -1$, $\alpha = 2$, $\beta = 1$, $N = 3$, $M = 3$.

Similarly to the first example, we obtain the following system of differential equations

$$\begin{cases} 3\dfrac{da_1(t)}{dt} - \dfrac{t}{2} = 0, \\ \dfrac{3}{2}\dfrac{d^2a_2(t)}{dt^2} + 6\dfrac{da_2(t)}{dt} + \dfrac{3}{2}a_2(t) = 0, \\ 4\dfrac{d^2a_3(t)}{dt^2} + 11\dfrac{da_3(t)}{dt} + 4a_3(t) = 0. \end{cases}$$

The distinctive feature of the presented system is that, along with second-order differential equations, it also includes one first-order differential equation, i.e. one of the control action components should be zero.

Let us compose the functional according to formula (14) and determine the desired state of the system

$$x_d = \sqrt{\frac{2}{\pi}} \sum_{i=1}^{3} \sin is.$$

We minimize the functional on a limited and convex set of possible control actions to obtain $J_{min} = 1.80879089761$, as well as control coefficients $u_{10} = 0.385053391513$, $u_{11} = 0$, $u_{12} = 0.0759422532479$, $u_{13} = 0.235950748458$, $u_{20} = 0.390454928053$, $u_{30} = 0.430260855425$. We substitute the found control components into the solution to obtain

$$\begin{aligned} &\tilde{x}(t,s) = (0.8333333t^2 + 0.38505339)\sin s + \\ &+(0.44257936e^{-0.26794919t} - 0.5212443e^{-3.73205081t})\sin 2s + \\ &+(0.65358213e^{-0.43127070t} - 0.22332127e^{-2.31872930t})\sin 3s. \end{aligned}$$

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ЧИСЛЕННОЕ РЕШЕНИЕ ЗАДАЧИ СТАРТОВОГО УПРАВЛЕНИЯ ДЛЯ МОДЕЛИ БУССИНЕСКА – ЛЯВА

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В статье проведено численно исследования задачи стартового управления с использованием компромиссного целевого функционала. Разработан алгоритм нахождения численного решения задачи управления посредством адаптированного проекционного метода для вырожденных уравнений. Выполнены вычислительные эксперименты, демонстрирующие эффективность предложенных алгоритмов.

Ключевые слова: уравнения соболевского типа высокого порядка; стартовое управление; адаптированный проекционный метод; численное решение.

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