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ON A PROBLEM IN THE THEORY OF RELAXATION OSCILLATIONS

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We consider a model of a braking device described by a differential equation relating the brake shoe rotation angle and its relative angular velocity. The dry friction torque depends on the rotation angle and angular velocity as a piecewise function, while the moment of inertia of the brake shoe device under consideration is a small quantity. From a mathematical standpoint, this equation reduces to a system of two differential equations, one of which contains a small parameter at the highest derivative, a so-called Tikhonov system. The system under consideration has a single equilibrium state, but it is unstable. It is self-excited, and relaxation self-oscillations will set in. Our goal was to provide an example of such a right-hand side of the equation of motion for which experimental phenomena are sufficiently accurately explained, and to obtain an asymptotic expansion of the solution as a function of time. To find the asymptotic expansion of an arbitrary-order solution to our problem, we used the method of constructing boundary functions. The justification of the asymptotic expansion can be carried out as in classical theory.

Keywords: singularly perturbed equations; degenerate equation; asymptotic expansion; boundary function method; relaxation oscillations.

Introduction

A real phenomenon, such as the operation of an object or the course of a process, is described in theory using differential equations. Unlike problems arising from practice, mathematical idealization often neglects small factors. While discarding some information that has a negligible effect on the nature of the process, we should simultaneously include quantities in the equations that, although small, can significantly alter the picture of the phenomenon. Thus, the research result will be more realistic if it takes into account the dependence on small parameters. This concept is inherent in singular perturbations, as opposed to regularly perturbed problems. Examples of numerical solutions of systems of differential equations can be found in [1], the application of asymptotic methods see [2].

1. Problem Statement

Let us consider the mechanical oscillations that can occur under certain conditions in bodies experiencing high friction but having a small mass [3]. For concreteness, let us assume that we are talking about a brake pad characterized by the following equation of motion

$$\varepsilon\ddot{\varphi} = -k\varphi + M(\Omega - \omega, \varphi).$$

The latter is reduced to a system including an equation with a small parameter $\varepsilon > 0$ at the derivative

$$\begin{cases} \varepsilon\dot{\omega} = -k\varphi + M(\Omega - \omega, \varphi), \\ \dot{\varphi} = \omega. \end{cases} \tag{1}$$

Here φ – is the brake pad rotation angle (relative to the position at which the spring torque is zero), ε – is the brake pad moment of inertia, $k > 0$ – is the system elasticity coefficient, Ω – is the shaft angular velocity, and we will assume that $\Omega = const$. Let $M(\Omega - \omega, \varphi)$ – be a function expressing the dependence of the dry friction torque on the relative velocity $\Omega - \omega$ and angle φ. It follows from the technical data that

$$M(\Omega - \omega, \varphi) = \begin{cases} k\varphi, & |k\varphi| \le M_0(\Omega - \omega), \\ M_0(\Omega - \omega), & |k\varphi| > M_0(\Omega - \omega). \end{cases} \tag{2}$$

We assume that $M_0(\theta) = (\theta - m_0)^2 + M_1$, where $M_1 = \min M_0(\theta)$, $\theta = \Omega - \omega$. Besides, let $\Omega < m_0$.

Let the initial point on the phase plane (φ, ω) have coordinates (φ_0, ω_0), wherein $k\varphi_0 < M_0(\Omega - \omega_0)$. Then the solution is described by a system where the first inequality from (2) is satisfied

$$\begin{cases} \varepsilon\dot{\omega} = 0, \\ \dot{\varphi} = \omega, \end{cases} \quad \omega(0) = \omega_0, \varphi(0) = \varphi_0, \tag{3}$$

and has the following form $\omega = \omega_0$, $\varphi = \omega_0 t + \varphi_0$. Therefore, this solution on the phase plane will correspond to a line segment extending in the direction of increasing angle φ, and from the point where $k\varphi(T) = M_0(\Omega - \omega_0)$, the solution will be functions defined by the following system when the second inequality from (2) takes effect:

$$\begin{cases} \varepsilon\dot{\omega} = -k\varphi + (\Omega - \omega - m_0)^2 + M_1, \\ \dot{\varphi} = \omega, \end{cases} \quad \omega(T) = \omega_0, \varphi(T) = \omega_0 T + \varphi_0. \tag{4}$$

It is known from [4] that at the breakdown point, where the transition to another stable solution occurs,

$$M'_{0\omega} = -2(\Omega - \omega - m_0) = 0,$$

and therefore, at this point $\omega_c = \Omega - m_0$, wherein

$$M'_{0\omega} \begin{cases} < 0, & \omega < \Omega - m_0, \\ > 0, & \omega > \Omega - m_0, \end{cases} \tag{5}$$

from which it follows that the stability region of the root of the system degenerate equation will be determined by the following inequality

$$\Omega - \omega > m_0.$$

At the transition point to another stable branch at $t = T$, the following relations are satisfied

$$\begin{aligned} &\omega = \omega_0, \\ &|k(\omega_0 T + \varphi_0)| = M_0(\Omega - \omega_0) = (\Omega - \omega_0 - m_0)^2 + M_1 = M_0. \end{aligned} \tag{6}$$

2. Constructing the Asymptotics of the Solution

The degenerate system corresponding to (1) under the second condition from (2) has the following form

$$\begin{aligned} &0 = -k\varphi + (\Omega - \omega - m_0)^2 + M_1, \\ &\dot{\varphi} = \omega, \\ &\varphi(T) = \omega_0 T + \varphi_0, \end{aligned}$$

Let us denote its solution by the pair of functions $\bar{\omega}_0(t)$, $\bar{\varphi}_0(t)$. We solve the first (final) equation for ω to obtain

$$(\Omega - \bar{\omega}_0 - m_0)^2 = k\bar{\varphi}_0 - M_1 \geq 0,$$

$$\bar{\omega}_0 = \Omega - m_0 \pm \sqrt{k\bar{\varphi}_0 - M_1}.$$

Taking into account inequalities (4), we classify the branch of the solution corresponding to the "+" sign as unstable, and the branch corresponding to the "–" sign as stable. The phase plane picture reflects the situation where, due to equality (6) corresponding to the moment of time $t = T$, the solution is located on the unstable branch, and therefore "repulses" from it beginning to "abstract" towards the stable branch. Based on the algorithm for constructing the asymptotics of the Tikhonov system solution [5], we select it as the zero term of the first component of the regular part of the asymptotics. For the second component of the regular part, we have the following problem

$$\begin{aligned} &\dot{\bar{\varphi}}_0 = \Omega - m_0 - \sqrt{k\bar{\varphi}_0 - M_1}, \\ &\bar{\varphi}_0(T) = \omega_0 T + \varphi_0. \end{aligned}$$

We solve the equation for the unknown function $\bar{\varphi}_0(t)$, and obtain an integral in the form

$$(\Omega - m_0 - \sqrt{k\bar{\varphi}_0 - M_1})^{2(m_0-\Omega)k^{-1}} \exp\left\{2k^{-1}\left(\Omega - m_0 - \sqrt{k\bar{\varphi}_0 - M_1}\right)\right\} = C\exp\{t - T\}.$$

The constant C is found from the initial condition, namely,

$$\begin{aligned} C = &\left(\Omega - m_0 - \sqrt{k(\omega_0 T + \varphi_0) - M_1}\right)^{2(m_0-\Omega)k^{-1}} \times \\ &\times \exp\left\{2k^{-1}\left(\Omega - m_0 - \sqrt{k(\omega_0 T + \varphi_0) - M_1}\right)\right\}. \end{aligned}$$

Then the first component of the regular part of the asymptotics $\bar{\omega}_0(t)$ will be set by the equality

$$\bar{\omega}_0(t) = \Omega - m_0 - \sqrt{k\bar{\varphi}_0(t) - M_1} = \omega_c - \sqrt{k\bar{\varphi}_0(t) - M_1} \tag{7}$$

after substituting the function we have just defined $\bar{\varphi}_0(t)$.

Thus, we found the zero terms of the regular part of the asymptotics for the considered problem. However, while its second component $\bar{\varphi}_0(t)$ satisfies the initial condition as a solution to the Cauchy problem, this is not the case for the first component $\bar{\omega}_0(t)$. It is a solution to the final equation and, generally speaking, $\bar{\omega}_0(T) \neq \omega_0$. To eliminate this discrepancy in the solution associated with this circumstance, we introduce the function $\Pi\omega_0(\tau)$, called the boundary function, which depends on the stretched variable $\tau = \dfrac{t - T}{\varepsilon}$.

Strictly speaking, we should also introduce the boundary function $\Pi\varphi_0(\tau)$. Thus, we seek an asymptotics in the form of a series in terms of powers of ε

$$\begin{aligned} \omega(t,\varepsilon) &= \bar{\omega}(t,\varepsilon) + \Pi\omega(\tau,\varepsilon) = \sum_{i=0}^{\infty} \varepsilon^i(\bar{\omega}_i(t) + \Pi_i\omega(\tau)), \\ \varphi(t,\varepsilon) &= \bar{\varphi}(t,\varepsilon) + \Pi\varphi(\tau,\varepsilon) = \sum_{i=0}^{\infty} \varepsilon^i(\bar{\varphi}_i(t) + \Pi_i\varphi(\tau)), \end{aligned} \tag{8}$$

where the functions $\bar{\omega}_i$ and $\bar{\varphi}_i$, depending on the argument t, define the regular part of the asymptotics, and the boundary functions $\Pi_i\omega$ and $\Pi_i\varphi$ depend on the stretched variable τ.

$$\varepsilon\frac{d\bar{\omega}}{dt} + \frac{d\Pi\omega}{d\tau} = -k\left(\bar{\varphi}(t) + \Pi\varphi(\tau)\right) + \left(\Omega - m_0 - \bar{\omega}(t) - \Pi\omega(\tau)\right)^2 + M_1,$$

$$\frac{d\bar{\varphi}}{dt} + \frac{1}{\varepsilon}\frac{d\Pi\varphi}{d\tau} = \bar{\omega}(t) + \Pi\omega(\tau). \tag{9}$$

We add the initial conditions to them at $t = T$

$$\bar{\omega}(T) + \Pi\omega(0) = \omega_0, \quad \bar{\varphi}(T) + \Pi\varphi(0) = \omega_0 T + \varphi_0. \tag{10}$$

We equate the functions with the same powers ε, depending on the same variables, on the left-hand and right-hand sides of the equations. The functions of the regular part in zero order have already been found. The problems for the boundary functions $\Pi_0\omega(\tau)$ and $\Pi_0\varphi(\tau)$ will be as follows

$$\begin{aligned} &\frac{d\Pi_0\omega}{d\tau} = -k\Pi_0\varphi - 2(\Omega - m_0 - \bar{\omega}_0(T))\Pi_0\omega + \Pi_0\omega^2, \\ &\Pi_0\omega(0) = \omega_0 - \bar{\omega}_0(T). \end{aligned}$$

$$\begin{aligned} &\frac{d\Pi_0\varphi}{d\tau} = 0, \\ &\Pi_0\varphi(\infty) = 0. \end{aligned}$$

From the second equation with the initial condition, we find that $\Pi_0\varphi(\tau) \equiv 0$. Substituting this solution into the first equation, taking into account the initial condition, we obtain the following relation for the boundary function $\Pi_0\omega(\tau)$

$$\Pi_0\omega(\tau) = 2(\omega_c - \bar{\omega}_0(T)) \cdot \left[1 - \frac{\omega_0 - 2\omega_c + \bar{\omega}_0(T)}{\omega_0 - \bar{\omega}_0(T)} \exp\{2(\omega_c - \bar{\omega}_0(T))\tau\}\right]^{-1}.$$

It is easy to see that, due to formula (7), written for the argument $t = T$, we have the inequality $\bar{\omega}_0(T) < \omega_c$. Therefore, the following exponential estimate is valid for the boundary function found

$$|\Pi_0\omega(\tau)| < C\exp\{-æ\tau\}, \tag{11}$$

where $æ > 0$, $C > 0$ – are some constants.

We continue constructing the asymptotics. From equations (9) in i-th order, we obtain

$$\dot{\bar{\omega}}_{i-1} = -k\bar{\varphi}_i + 2(\omega_c - \bar{\omega}_0)\bar{\omega}_i + \sum_{j=1}^{i-1} \bar{\omega}_{i-j}\bar{\omega}_j, \quad \dot{\bar{\varphi}}_i = \bar{\omega}_i.$$

We express the function $\bar{\omega}_i(t)$ from the first equality to obtain

$$\bar{\omega}_i = 2^{-1}(\omega_c - \bar{\omega}_0)^{-1}\left[k\bar{\varphi}_i + \dot{\bar{\omega}}_{i-1} - \sum_{j=1}^{i-1}\bar{\omega}_{i-j}\bar{\omega}_j\right].$$

We substitute into the second equation of the system to obtain an equation to find the second component of the regular part of the asymptotics of the following form

$$\dot{\bar{\varphi}}_i = \alpha_i(t)\bar{\varphi}_i + \beta_i(t),$$

where the coefficients $\alpha_i(t)$ and $\beta_i(t)$ are expressed through the functions $\bar{\omega}_j$ with indices $j = 0, 1, ..., i-1$. We solve this first-order linear inhomogeneous equation with the initial condition following from (10)

$$\bar{\varphi}_i(T) = 0.$$

after which $\bar{\omega}_i(t)$ also defined. Thus, if the functions of the regular part have been constructed in the previous steps, we can proceed to finding the functions of the next indices. Next, we construct the boundary functions in the i-th order. We can easily see from (9) and (10) that the problem for $\Pi_i\varphi(\tau)$

$$\frac{d\Pi_i\varphi}{d\tau} = \Pi_{i-1}\omega(\tau),$$

$$\Pi_i\varphi(\infty) = 0.$$

Notably, the condition at infinity (standard for boundary functions [5]) should guarantee their decrease at $\varepsilon \to 0$. From this, we arrive at the solution

$$\Pi_i\varphi(\tau) = \int_{\infty}^{\tau}\Pi\omega_{i-1}(s)ds.$$

Now let us explain how the boundary functions $\Pi_i\omega(\tau)$ are constructed. When we substitute the series by the small parameter from the boundary part of the asymptotics and change the variable in the first equation of (9), we should take into account the rule for transforming the right-hand side of the inhomogeneous equations (see [5]). In the case of a quadratic right-hand side, as in our example, we obtain

$$\frac{d\Pi_0\omega}{d\tau} + \varepsilon\frac{d\Pi_1\omega}{d\tau} + \ldots + \varepsilon^i\frac{d\Pi_i\omega}{d\tau} + \ldots = -k\bar{\varphi}_0(T+\varepsilon\tau) - \ldots - k\varepsilon^i\bar{\varphi}_i(T+\varepsilon\tau) +$$
$$+ (\omega_c - \bar{\omega}_0(T+\varepsilon\tau) - \varepsilon\bar{\omega}_1(T+\varepsilon\tau) - \ldots - \varepsilon^i\bar{\omega}_i(T+\varepsilon\tau) - \ldots -$$
$$- \Pi_0\omega(\tau) - \varepsilon\Pi_1\omega(\tau) - \ldots - \varepsilon^i\Pi_i\omega(\tau) - \ldots)^2 + M_1 + k\bar{\varphi}_0(T+\varepsilon\tau) + \ldots +$$
$$+k\varepsilon^i\bar{\varphi}_i(T+\varepsilon\tau) - - (\omega_c - \bar{\omega}_0(T+\varepsilon\tau) - \varepsilon\bar{\omega}_1(T+\varepsilon\tau) - \ \ldots - \varepsilon^i\bar{\omega}_i(T+\varepsilon\tau) - \ldots)^2 - M_1$$

with the initial condition

$$\Pi\omega_i(0) = -\bar{\omega}_i(T).$$

Expanding the functions of the regular part $\bar{\omega}_i$ with the center of expansion $t = T$ into series in terms of powers of $\varepsilon\tau$ and equating the terms with the same powers ε on the right-hand and left-hand sides of the equation, we obtain problems for determining the boundary functions $\Pi_i\omega(\tau)$:

$$\frac{d\Pi\omega_i}{d\tau} = 2\left(-\omega_c + \bar{\omega}_0(T)\right)\Pi_i\omega(\tau) + 2\left(\dot{\bar{\omega}}_0(T)\tau + \bar{\omega}_1(T)\right)\Pi_{i-1}\omega +$$

$$+2\left(\frac{\ddot{\bar{\omega}}_0(T)}{2!}\tau^2 + \dot{\bar{\omega}}_1(T)\tau + \bar{\omega}_2(T)\right)\Pi_{i-2}\omega + \ldots +$$

$$+2\left(\frac{\bar{\omega}_0^{(i)}(T)}{i!}\tau^i + \frac{\bar{\omega}_1^{(i-1)}(T)}{(i-1)!}\tau^{i-1} + \ldots + \dot{\bar{\omega}}_{i-1}(T)\tau + \bar{\omega}_i(T)\right)\Pi_0\omega + \sum_{j=0}^{i}\Pi_{i-j}\omega\Pi_j\omega,$$

$$\Pi_i\omega(0) = -\bar{\omega}_i(T).$$

The resulting initial value problem is formulated for an inhomogeneous equation linear with respect to the desired function and solved using standard methods. Notably, due to known estimates for the linear equation, an exponential estimate of the form (11) will we valid. Thus, we defined the boundary functions $\Pi_i\omega(\tau)$ after which we can eaility find $\Pi_i\varphi(\tau)$ using the saforementioned method. Obviously, an estimate of the form (9) will also be valid for them.

3. Result Statement

The construction of the asymptotic terms is completed. It remains to make a statement about the degree of approximation of the constructed series to the solution of our problem. It follows from [5] that an asymptotic expansion of the solution $x(t,\varepsilon) == (\omega(t,\varepsilon), \varphi(t,\varepsilon))$ to problem (4) is obtained at $\varepsilon \to 0$ on the closed interval $[T, T_1]$, i.e., the following estimate holds:

$$\max_{[T,T_1]} |x(t,\varepsilon) - X_n(t,\varepsilon)| = O(\varepsilon^{n+1}),$$

where $X_n(t,\varepsilon)$ are partial sums of the series (8). This result can be proven similarly to [5]. Thus, the phase plane (φ, ω) has a transition to the stable branch (6) of the solution of the degenerate system. Then, moving along this branch, the system reaches a breakdown point and again finds itself on the original line described by (3). Consequently, relaxation oscillations of the studied model can be traced on the phase plane, as described in [3, 4].

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ОБ ОДНОЙ ЗАДАЧЕ ИЗ ТЕОРИИ РЕЛАКСАЦИОННЫХ КОЛЕБАНИЙ

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