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ANALYSIS OF CRITICAL STATES OF INHOMOGENEOUS JOINTS WITH AN INDIRECT CONTACT BOUNDARY

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Mathematical models of critical states of discrete heterogeneous joints under conditions of plane deformation are considered. The boundary between sections of different strengths is assumed to be a two-link polyline or contains such a fragment. The purpose of the work is to study the features of the characteristic fields (sliding lines) of such models and to find the stress state at the inner boundary. A characteristic variant of such a field is given; possible locations of stress rupture lines (lines of violation of smoothness of characteristics) and zones of plastic flow in a durable part of the joint are indicated. It has been established how the value of the coefficient of heterogeneity affects the appearance of a plastic section in a durable part of the joint.

Keywords: discretely heterogeneous joints; critical states; stress state; conjugation problem; quasi-linear hyperbolic equations; characteristics.

Introduction

The work is devoted to the study of critical states that occur in the plastic zone during the loading of heterogeneous joints. In both model and applied situations, the internal boundary (the boundary between sections of different strengths) was often assumed to be rectilinear [1–8]. In fact, it can have a complex form [9–11]. The paper considers discretely heterogeneous joints with an internal boundary containing a two-link polyline. A two-link polyline is a model example that reveals the main features of the stress state in the vicinity of the contact boundary. The work is a continuation, addition and development of articles [12] and [13]. In [12], the basic concepts used in the work are formulated: the object of study is discrete heterogeneous compounds; the subject of study is critical states of structures and samples (states of pre - destruction); features and characteristics of the joint deformation process – plane deformation, loss of stability of the plastic deformation process, contact hardening, complete and incomplete implementation of contact hardening; concepts related to the theoretical methodology of investigation of the studied processes are the stress state of an object, a system of quasi–linear partial differential equations of hyperbolic type, fields of characteristics, in particular, uniform and fan-centered fields of characteristics, Riemann invariants along characteristics, and the conjugation problem for stresses at the inner boundary. The strength of the joint and the heterogeneous structure is determined by the magnitude of the external load, leading to a critical condition, which is exhaustively characterized by a field of characteristics. The aim of the work is to analyze the critical state of an heterogeneous joint with an inclined inner boundary, in particular, when the inner boundary contains a two-link polyline. A new situation has been studied when the angle of inclination of the boundary is negative, which was not considered in [12]. The description of the characteristic fields is given. The conditions under which a more durable section enters into plastic deformation are being studied. The normal and tangential stresses at the contact boundaries at different angles of inclination are calculated.

1. Preliminary Information

The stress state of the plastic section in dimensionless unknowns is determined by a system of equations [11, pp. 13–14, etc.]:

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0, \quad \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} = 0,$$

$$(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2 = 4k^2.$$

and the boundary conditions on the free boundary

$$\tau_{x,y}(1, y) = 0; \quad \sigma_x(1, y) = 0,$$

where $k = k^+$ on the more durable section, and $k = k^-$ on the less durable one. This system is of the hyperbolic type. Its Riemann invariants (Genka integrals) along the characteristics (sliding lines) are known: $\sigma \pm 2k\gamma = c$, where c is a constant along the characteristic, γ is the angle of inclination of the characteristics to Ox axis (Fig. 1), the plus sign in the above formula is for a family of characteristics with an acute angle γ. The equilibrium equations in dimensionless unknowns at the contact boundary (see Fig. 1) have the form:

$$\sigma_{y'}^+ = \sigma_{y'}^-, \quad \tau_{x'y'}^+ = \tau_{x'y'}^-. \tag{1}$$

Here $(x'; y')$ are Cartesian coordinates with the Ox' axis directed along the contact boundary. Let $k^\pm$ be plastic constants corresponding to parts of joints of different strength, $K = k^+/k^-$. In dimensionless unknowns, the system of equations (1) can be written:

$$K\sigma_{y'}^+ = \sigma_{y'}^-, \quad K\tau_{x'y'}^+ = \tau_{x'y'}^-. \tag{2}$$

The stress conjugation problem (internal boundary problem) is formulated as follows: find the stresses τ_{xy} and σ_y on the contact surface from the boundary conditions on the free boundary and the conjugation conditions (2). It was shown in [12, 13] that if the plastic regions are adjacent to the inner boundary (for example, $AFPF_1D_1A$ section in Fig. 1), then

$$\sigma_{y'}^- = 1 + 2\omega^- + \cos(2\omega^- + 2\alpha), \quad \sigma_{y'}^+ = 1 - 2\omega^+ + \cos(2\omega^+ + 2\alpha), \tag{3}$$

$$\tau_{x'y'}^- = \sin(2\omega^- + 2\alpha), \quad \tau_{x'y'}^+ = \sin(2\omega^+ + 2\alpha) \tag{4}$$

Using the last expressions, we can write the system (2) in the form [12, 13]:

$$1 + 2\omega^- + \cos(2\omega^- + 2\alpha) = K(1 - 2\omega^+ + \cos(2\omega^+ + 2\alpha)), \tag{5}$$

$$\sin(2\omega^- + 2\alpha) = K\sin(2\omega^+ + 2\alpha). \tag{6}$$

Incomplete implementation of contact hardening in the vicinity of a free boundary. FF_1, AH_1 segments are stress rupture lines in the stronger part of the joint. An important feature of heterogeneous connections are stress rupture lines. They exist either because of the peculiarities of the sample shape [14, pp. 164–168] or because of the heterogeneity [15].

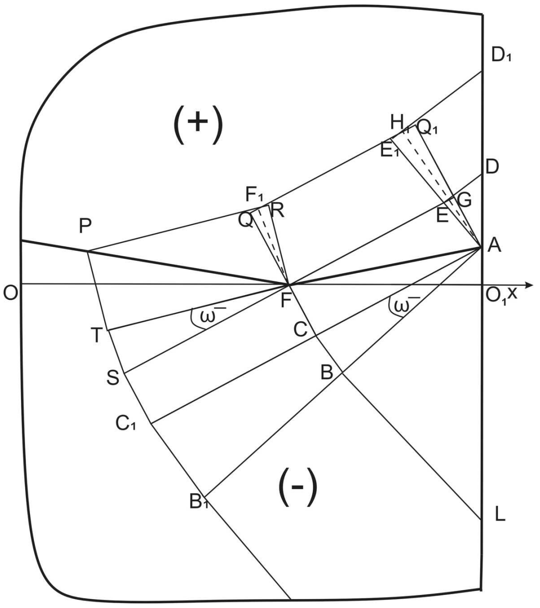

Fig. 1. The case when the angles between the links of the contact boundary and a straight line OO_1 orthogonal to the free boundary have different signs: $\angle AFO_1 > 0, \quad \angle PFO < 0$

2. The Main Results

Fig. 1 shows a fragment of the field of characteristics (sliding lines) of an inhomogeneous connection with a boundary containing a two-link line, in which a link not adjacent to the free boundary forms a negative angle with an axis orthogonal to the free surface. The stress rupture lines are indicated by dotted lines. In the critical state shown in Fig. 1, contact hardening is not fully realized, because $\angle\omega^- < \angle FAB$ in the neighborhood and in the neighborhood $\angle\omega^- + \alpha < \pi/4$. Partial implementation occurs when [12]. In this case, a stress rupture and a violation of the smoothness of the characteristics occurs (among other things) along a line FF_1 that does not pass through point A, i.e. the intersection of the outer and inner boundaries (a case that has not been reported before). When studying the stress state at the boundary PFA , the segments PF and FA should be considered separately. They have different angles of inclination. The magnitude of the angle depends, as can be seen from (5) and (6), on α and K. Therefore, the full or incomplete implementation of contact hardening is determined by these two parameters. Fig. 2 shows a line in the coordinate plane (K, α) separating the zones of full implementation of contact hardening (the base metal does not deform plastically), and incomplete. In this case, the base material passes into a plastic state in the vicinity of the contact boundary. Using the least squares method, the dependence of Fig. 2 is found in an analytical form:

$$\alpha = \frac{1,41}{K} - 0.63 - 0.03K. \tag{7}$$

Numerical solutions of the system (5), (6) for different conditions were obtained in [13] and, using another iterative procedure, were refined by the authors in [12]. This solution allows us to calculate (in this paper) the values of dimensionless normal and tangential stresses using formulas (3), (4). Here are shown (Figs. 3) some results for different angles α.

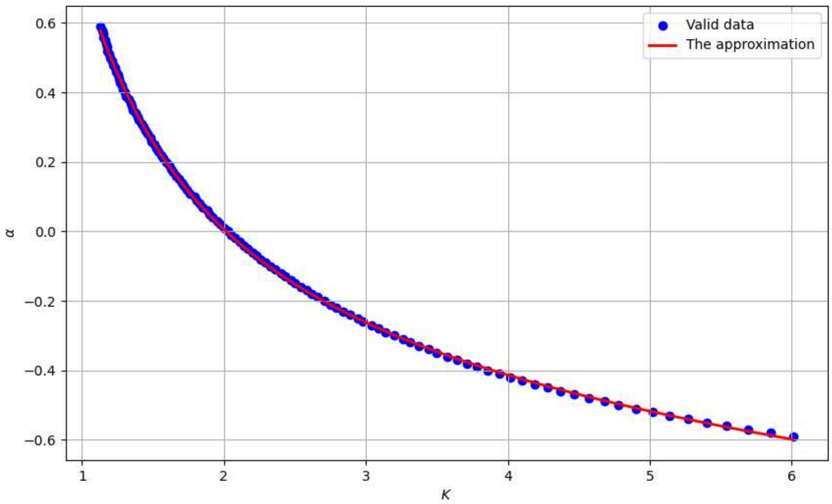

Fig. 2. The zones of involvement of the more durable layer material in plastic deformation, depending on K and alpha. There is no plastic deformation above the curve

The iterative algorithm based on formulas (5) and (6) works only in the area of incomplete involvement of the less durable part of the joint in plastic deformation (from the bottom of the curve in Fig. 2), that is, by virtue of (7), provided

$$\alpha < \frac{1,41}{K} - 0.63 - 0.03K.$$

This explains the decrease in the area of values of K in Fig. 3, c) and d).

3. Conclusion

As noted in [12], if the boundary between the less durable and more durable parts of the joint contains a two-link polyline, the zone of involvement of the material of the more durable section in plastic deformation may not be adjacent to the free boundary, and the stress rupture lines may be located in the less durable part of the joint. The paper describes a specific situation in which critical states of discrete inhomogeneous compounds with these features are realized. In a previously unexplored case, the angles of inclination of the characteristic fields and the normal and tangential stresses at the contact boundary are found. The results obtained can be used to find the critical external load on the joints of the type under study.

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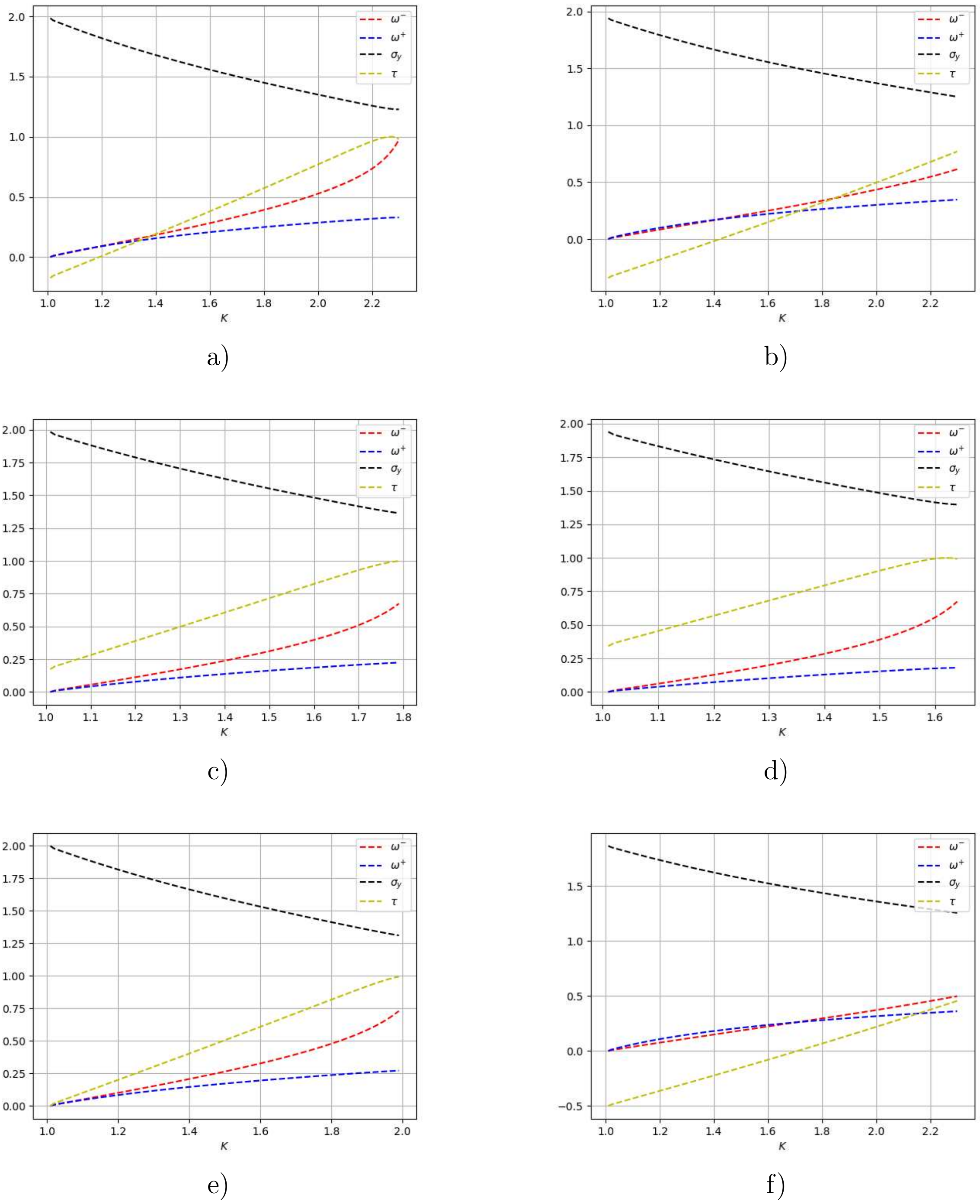

Fig. 3. The dependence $\omega^-, \omega^+, \tau_{xy}, \sigma_y$ and stresses on the contact surface on the coefficient K: a) $\alpha = -\frac{5\pi}{180}$, b) $\alpha = -\frac{10\pi}{180}$, c) $\alpha = \frac{5\pi}{180}$, d) $\alpha = \frac{10\pi}{180}$, e) $\alpha = 0$, f) $\alpha = -\frac{15\pi}{180}$.

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АНАЛИЗ КРИТИЧЕСКИХ СОСТОЯНИЯХ НЕОДНОРОДНЫХ СОЕДИНЕНИЙ С НЕПРЯМОЛИНЕЙНОЙ КОНТАКТНОЙ ГРАНИЦЕЙ

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Рассматриваются математические модели критических состояний дискретно-неоднородных соединений в условиях плоской деформации. Граница между участками разной прочности предполагается двухзвенной ломаной или содержит такой фрагмент. Цель работы: исследование особенностей полей характеристик (линий скольжения) таких моделей и нахождение напряженного состояния на внутренней границе. Приведен характерный вариант такого поля; указаны возможные места расположения линий разрыва напряжений (линий нарушения гладкости характеристик) и зоны пластического течения в более прочной части соединения. Установлено, как величина коэффициента неоднородности влияет на возникновение пластического участка в более прочной части соединения.

Ключевые слова: дискретно неоднородные соединения; критические состояния; напряженное состояние; задача сопряжения; квазилинейные уравнения гиперболического типа; характеристики.

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