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START CONTROL OF POSITIVE SOLUTIONS TO SOBOLEV TYPE EQUATIONS

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Using degenerate holomorphic groups of operators generated by linear continuous operators L and M, a positive solution to a linear homogeneous Sobolev type equation is obtained; necessary and sufficient conditions for the positivity of these groups are found. Start control of positive solutions to the Showalter–Sidorov problem is investigated.

Keywords: start control problem; positive solutions; initial problem; abstract homogeneous Sobolev type equation.

Introduction

Currently, mathematical modeling and computational experiment are integral parts of general approaches to the study of various applied problems. This includes problems posed by the currently relevant State Program of the Chelyabinsk Region "Environmental Protection of the Chelyabinsk Region". Clean air, water, and soil are vital components of our environment. Their proper maintenance is a factor contributing to the socio-economic development of an industrial region [1, 2]. A significant number of works are devoted to the study of pollution caused by industrial activity. To forecast possible pollution of the aforementioned natural resources, as well as to create purification systems, mathematical models of fluid filtration in soil (in fractured-porous media) and other models with elements of hydrodynamic similarity theory are used, upon which a significant number of mathematical methods used in the practice of hydro- and aerodynamic research are built. The application of positive solutions is also relevant in the theory of optimal dynamic measurements [3].

These models rely on non-negative and realistic aerodynamic, kinetic, physical, geometric parameters and indicators, and their ratios. Solutions to problems obtained using these models must also be positive [4, 5] to ensure adequacy of characteristics, forecast accuracy, more stable and efficient computations, as well as their optimal implementation. Here are examples of some of these indicators: velocities and directions of flows of the substance under study, average velocity of substance movement, geometric dimensions of the study area, geometric parameters of filters, their drag forces; forces causing a pressure gradient in a medium; particle trajectories in a substance and their kinematic characteristics; concentration of impurities in the atmosphere; source power, etc.

1. Axiomatics and structure of the positive approach

A vector space $\mathcal{B}$ is called a *Riesz space* [4] if an order relation $\geq$ (satisfying the axioms of reflexivity, transitivity, and antisymmetry) is defined on it, which is consistent with the vector structure, i.e.,

$$(x \geq y) \Rightarrow (x+z \geq y+z) \quad \text{for all} \quad x,y,z \in \mathcal{B}, \quad \text{and}$$

$$(x \geq y) \Rightarrow (\alpha x \geq \alpha y) \quad \text{for all} \quad x,y \in \mathcal{B} \quad \text{and} \quad \alpha \in \overline{\mathbf{R}_+}, \quad \text{where} \quad \overline{\mathbf{R}_+} = \{0\} \cup \mathbf{R}_+.$$

In turn, a Riesz space $\mathcal{B}$ is called a *functional Riesz space* if one can define elements $x_+ = \max\{x, 0\}$ and $x_- = \min\{-x, 0\}$ for any $x \in \mathcal{B}$ such that $x_+, x_- \in \mathcal{B}$. Further, a functional Riesz space is called a *normed functional Riesz space* if a norm $||\cdot||_{\mathcal{B}}$ is defined on it such that

$$(|x| \geq |y|) \Rightarrow (||x||_{\mathcal{B}} \geq ||y||_{\mathcal{B}}) \quad \text{for all} \quad x, y \in \mathcal{B}. \tag{1}$$

Here $|x| = x_+ + x_-$. A complete normed functional Riesz space will be called a *Banach lattice.*

Now let $\mathcal{B}$ be a Banach space. A convex set $\mathcal{C} \subset \mathcal{B}$ is called a *cone* if $\mathcal{C} + \mathcal{C} = \mathcal{C}$ and $\alpha\mathcal{C} \subset \mathcal{C}$ for all $\alpha \in \overline{\mathbf{R}_+}$. A cone $\mathcal{C}$ is called *proper* if $\mathcal{C} \cap (-\mathcal{C}) = \{0\}$, and *generating* if $\mathcal{B} = \mathcal{C} - \mathcal{C}$, i.e., for all vectors $x \in \mathcal{B}$ there exist vectors $y, z \in \mathcal{C}$ such that $x = y - z$. If in a Banach space $\mathcal{B}$ there exists a proper generating cone $\mathcal{C}$, then an order relation $\geq$ can be defined on $\mathcal{B}$ such that $(x \geq y) \Leftrightarrow (x - y \in \mathcal{C})$. The relation $\geq$, due to the properties of the cone $\mathcal{C}$, is consistent with the vector structure of the space $\mathcal{B}$, and if for each vector $x \in \mathcal{B}$ one can define a vector $|x| \in \mathcal{B}$ such that (1) holds, then the space $\mathcal{B}$ becomes a Banach lattice. On the other hand, if $\mathcal{B}$ is a Banach lattice, then the set $\mathcal{B}_+ = \{x \in \mathcal{B} : x \geq 0\}$ will be a proper generating cone.

So, let $\mathcal{B} = (\mathcal{B}, \mathcal{C})$ be a Banach lattice. Here $\mathcal{C}$ is a proper generating cone, and note that, generally speaking, $\mathcal{C}$ may not coincide with the *canonical cone* $\mathcal{B}_+$. An operator $A \in \mathcal{L}(\mathcal{B})$ is called *positive* if $Ax \geq 0$ for any $x \in \mathcal{C}$. A group of operators $V^\bullet = \{V^t : t \in \mathbf{R}\}$ acting on the space $\mathcal{B}$ is called *positive* if $V^t x \geq 0$ for any $x \in \mathcal{C}$ and $t \in \mathbf{R}$. If $V^\bullet$ is a degenerate group, then its identity V^0 is a projector, which splits the space $\mathcal{B}$ into the direct sum $\mathcal{B} = \mathcal{B}^0 \oplus \mathcal{B}^1$, where $\mathcal{B}^0 = \ker V^0$ and $\mathcal{B}^1 = \mathrm{im} V^0$. Since $V^t = V^0 V^t V^0$, then $\mathcal{B}^0 = \ker V^t$, and $\mathcal{B}^1 = \mathrm{im} V^t$. Hence, we can define $\ker V^\bullet = \mathcal{B}^0$ and $\mathrm{im} V^\bullet = \mathcal{B}^1$. If a degenerate group $V^\bullet$ is also positive, then $\mathcal{B}^1$ is a Banach lattice with the proper generating cone $\mathcal{C}^1 = \{x \in \mathcal{C} : V^0 x = x\} = \mathcal{B}^1 \cap \mathcal{C}$. If it turns out that the space $\mathcal{B}^0$ is also a Banach lattice with a generating cone $\mathcal{C}^0$ and an order relation $\succeq$, then the cone $\mathcal{C}^* = \mathcal{C}^0 \oplus \mathcal{C}^1$ can generate a new Banach lattice structure on the space $\mathcal{B}$ with the order relation $\circledcirc$, i.e.,

$$(x \circledcirc y) \leftrightarrow (x_0 \succeq y_0) \wedge (x_1 \geq y_1).$$

In what follows, we consider the Banach space $\mathcal{X}(\mathcal{Y})$ as a Banach lattice $\mathcal{X} = (\mathcal{X}, \mathcal{C}_\mathcal{X})$ $(\mathcal{Y} = (\mathcal{Y}, \mathcal{C}_\mathcal{Y}))$, where $\mathcal{C}_\mathcal{X}$ $(\mathcal{C}_\mathcal{Y})$ are proper generating cones.

2. Positive solutions to a linear Sobolev type equation

Let $\mathcal{X} = (\mathcal{X}, \mathcal{C}_\mathcal{X})$ and $\mathcal{Y} = (\mathcal{Y}, \mathcal{C}_\mathcal{Y})$ be Banach lattices, where $\mathcal{C}_\mathcal{X}$ and $\mathcal{C}_\mathcal{Y}$ are proper generating cones [4]. Let the operators $L \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ and $M \in \mathcal{C}l(\mathcal{X}, \mathcal{Y})$, and let the operator M be positively (L, p)-bounded [4]. Consider the linear homogeneous Sobolev type equation

$$L\dot{x} = Mx. \tag{2}$$

A vector-function $x \in C^1(\mathbf{R}, \mathcal{X})$ satisfying this equation is called a *solution of equation (2)*. A solution $x = x(t)$ is called a *solution of the Cauchy problem* if, for some $x_0 \in \mathcal{X}$, it satisfies the condition

$$x(0) = x_0. \tag{3}$$

A solution $x = x(t)$ is called a *solution of the Showalter–Sidorov problem* if it satisfies the condition

$$P(x(0) - x_0) = 0. \tag{4}$$

As is easy to verify, the vector-function $x(t) = X^t x_0$, where $\{X^t : t \in \mathbf{R}\}$ is a degenerate holomorphic group of operators of the form

$$X^t = \frac{1}{2\pi i} \int_\gamma R^L_\mu(M) e^{\mu t} d\mu, \quad t \in \mathbf{R}, \tag{5}$$

$$Y^t = \frac{1}{2\pi i} \int_\gamma L^L_\mu(M) e^{\mu t} d\mu, \quad t \in \mathbf{R}, \tag{6}$$

is a solution of equation (2) for any $x_0 \in \mathcal{X}$, and it is also a solution of problem (4) for any $x_0 \in \mathcal{X}$. The question arises about the existence and uniqueness of a solution to problem (2), (3) and the question about the uniqueness of a solution to problem (2), (4). To solve these questions, recall that a set $\mathcal{B} \subset \mathcal{X}$ is called the phase space of equation (2) if any its solution $x(t) \in \mathcal{B}$ for each $t \in \mathbf{R}$; and for any $x_0 \in \mathcal{B}$ there exists a unique solution $x \in C^1(\mathbf{R}, \mathcal{X})$ of problem (3) for equation (2). The following holds

Theorem 1. *[4] Let the operator M be (L, p)-bounded, $p \in \{0\} \cup \mathbf{N}$. Then the following statements are equivalent.*

(i) $(\mu R^L_\mu(M))^{p+1}$ $((\mu L^L_\mu(M))^{p+1})$ is positive for all sufficiently large $\mu \in \mathbf{R}_+$.

(ii) the degenerate holomorphic group $X^\bullet$ $(Y^\bullet)$ is positive.

Theorem 2. *[4] Let the operator M be (L, p)- positively bounded, $p \in \{0\} \cup \mathbf{N}$. Then*

(i) the phase space of equation (2) is the subspace $\mathcal{X}^1$;

(ii) for any $x_0 \in \mathcal{X}$ there exists a unique solution $x = x(t)$ of problem (2), (4), which, moreover, has the form $x(t) = X^t x_0$.

Theorem 2 gives a complete answer to both questions posed. However, regarding the uniqueness of the solution to problem (2), (4), the following should be said. By statement (i) of this theorem, any solution of equation (2) lies in the space $\mathcal{X}^1$ pointwise, i.e., $x(t) \in \mathcal{X}^1$ for all $t \in \mathbf{R}$. This means that if we represent an arbitrary initial vector x_0 as $x_0 = x_0^0 + x_0^1$, where $x_0^k \in \mathcal{X}^k, k = 0, 1$ (according to G.A. Sviridyuk's splitting theorem [6]), then the solution $x(t) = X^t x_0$ of problem (2), (4) will also be the unique solution of this problem with the initial condition $v_0 = v_0^0 + x_0^1$, where the vector $v_0^0 \in \mathcal{X}^0$ is arbitrary. This circumstance must be taken into account to correctly understand what follows.

Let now $\mathcal{X} = (\mathcal{X}, \mathcal{C}_\mathcal{X})$ be a Banach lattice and the group $X^\bullet = \{X^t : t \in \mathbf{R}\}$ be positive. As follows from the reasoning in Section 1, the phase space of equation (2) will also be a Banach lattice, i.e., $\mathcal{X}^1 = (\mathcal{X}^1, \mathcal{C}^1_\mathcal{X})$, where the proper generating cone $\mathcal{C}^1_\mathcal{X} = \mathcal{X}^1 \cap \mathcal{C}_\mathcal{X} \neq \{0\}$. Therefore, the following holds

Corollary 1. *[4] Let the conditions of Theorem 1 be satisfied, $\mathcal{X}$ be a Banach lattice, and the degenerate holomorphic group $X^\bullet$ be positive. Then for any $x_0 \in \mathcal{C}^1_\mathcal{X}$ there exists* *a unique positive solution $x = x(t)$ of problem (2), (3), which, moreover, has the form $x(t) = X^t x_0$.*

Let us proceed to consider positive solutions of problem (2), (4). Under the conditions of Corollary 1, the solution $x(t) = X^t x_0$ of problem (2), (4) will be positive for any initial vector $x_0 \in \mathcal{X}$ such that $Px_0 \in \mathcal{C}^1_{\mathcal{X}}$.

Corollary 2. *[4] Let the conditions of Corollary 1 be satisfied. Then for any $x_0 \in \mathcal{X}$ such that $Px_0 \in \mathcal{C}^1_{\mathcal{X}}$, there exists a unique positive solution $x = x(t)$ of problem (2), (4), which, moreover, has the form $x(t) = X^t x_0$.*

3. Start control

When searching for start control for positive solutions of a Sobolev type equation, it is necessary to find such control that ensures the initial condition of solution positivity. Previously, issues of start control for studying Sobolev type equations were investigated, including in the context of elasticity [7] and filtration.

Let $L \in \mathcal{L}(\mathcal{X};\mathcal{Y})$ and $M \in \mathcal{C}l(X;Y)$, with $\ker L \neq \{0\}$. M is called positively bounded if there exists a number $\mu \in \mathcal{C}$ such that $det(\mu L - M)^{-1} \neq 0$, and there exists $p \in \{0\} \cup N$ such that for $p = 0$, the L-resolvent $(\mu L - M)^{-1}$ of the operator M has a removable singular point at ∞; otherwise, p is equal to the order of the pole at ∞. The terminology, definitions, results, and axiomatics are described in more detail in [4, 5]. Let us introduce a certain closed convex set U_{ad} in the space of controls U.

The start control problem consists in finding, among the set of admissible pairs $u \times x(u) \in U_{ad} \times X$ satisfying the system of equations

$$L\dot{x}(t) = Mx(t), \tag{7}$$

the Showalter–Sidorov initial condition

$$[R^l_\mu(M)]^{p+1}(x(0) - u) = 0, \tag{8}$$

such $v \times x(v)$ that

$$J(v) = \min_{u \in U_{ad}} J(u), \quad J(u) = \sum_{q=0}^{1} \int_0^\tau ||x^q(u) - x_0(t)||^2 dt, \tag{9}$$

where $x(u)$ is a strong solution of (7),(8), $R^l_\mu(M) = (\mu L - M)^{-1}L$ is the right L-resolvent of M, $\tau \in R_+$, $x(t)$ and $x_0(t)$ are the actual and planned observed values of the function, respectively. The form of the functional (9) means having only one control goal – achieving planned indicators; such a problem statement is called rigid start control. The following holds

Theorem 3. *Let the operator M be (L,p)-bounded, $p \in \{0\} \cup \mathbf{N}$, $x_0 \in \mathcal{C}_{\mathcal{X}}$. Then there exists a unique rigid start control of positive solutions of (7), (8).*

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СТАРТОВОЕ УПРАВЛЕНИЕ ПОЗИТИВНЫМИ РЕШЕНИЯМИ УРАВНЕНИЙ СОБОЛЕВСКОГО ТИПА

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При помощи вырожденных голоморфных групп операторов, порожденных линейными и непрерывными операторами L и M, получено позитивное решение линейного однородного уравнения соболевского типа, найдены необходимые и достаточные условия позитивности этих групп. Исследовано стартовое управление позитивными решениями задачи Шоуолтера–Сидорова.

Ключевые слова: задача стартового управления; позитивные решения; начальная задача; абстрактное однородное уравнение соболевского типа.

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