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PROBABILISTIC PRODUCTION FUNCTION OF MACHINERY: RELIABILITY INTERPRETATION OF THE LEONTIEF TECHNOLOGY

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The question of dividing factors of production into groups is brought up; the definition of the capacity of a group is suggested as a theoretical quantity of output given by only the group's inputs. A possible relation between groups' capacities and probabilistic production functions is described. It is proposed to treat the failure times of machinery as factors of production. Based on the Leontief technology, assuming that the machines' failure times are independently Weibull and Lomax distributed, the probabilistic CES and Cobb-Douglas production functions are respectively obtained for a production line, and the influence of the distributions' parameters on the hazard rate curves of the line's machines is examined. The hypothesis of equality of the Weibull distributions' shape parameters in the derivation of the probabilistic CES production function is justified by the reliability rationale. Based on the Leontief technology, assuming that the machines' failure times have independent generalized Weibull distributions, the probabilistic CES and Cobb-Douglas production functions are obtained in special cases, and the shape of the hazard rate curve is analyzed.

Keywords: Leontief technology; production function; production line; failure time; hazard rate.

Introduction

Suppose an enterprise allocates a range of expendable inputs to manufacture a chosen product. Expendable factors of production may naturally be divided into groups. When it comes to constructing microeconomic-level production functions, physical supplies and time reserves may primarily be distinguished. In particular, physical supplies can be divided into groups, such as raw materials, fuel, parts and subassemblies, etc. Each of the groups can be considered as an aggregated factor; at the same time, it is possible to treat the components of these groups as separate factors of production per se. In both cases the factory can take into account the costs or physical quantities of the inputs.

Suppose a factory employs a production line in order to create a product. A production process must be designed, and it dictates the choice of production machines (e.g., machine tools). These machines are related to fixed factors – they are not consumed as obviously as expendable inputs are. At the same time, during a production process, the reliabilities of the machines are being spent, i.e., they are losing their performance – eventually it causes failures. For microeconomic production functions, we are going to consider an additional group of expendable inputs — the machines' operating times to failure. Note that an operating time to failure describes the actual summary working period (i.e., idle time is not included) of a machine as a random variable. Such a reliability "supply" is being spent every time a work unit is being processed at this machine. Hereafter, operating times to failure will, for the sake of convenience, be called failure times.

Generally, a factory might exploit production machines either according to schedule (when preventive maintenance is used) or until a failure appears (reactive maintenance). We will be concerned with the second approach and consider the failure times of machinery as wholesome inputs. Since they are random variables, it is reasonable to employ probabilistic approaches for constructing production functions.

In the context of probabilistic approaches, inputs' capacities [1] or inputs' unit productivities [2, 3] (see the definitions in the section 2) are treated as random variables. Commitment to a specific microeconomic production function (also called a technology) is postulated for the factory (if its dynamic, i.e., behavior in time, is of interest); however, a group of enterprises can also be assumed to uphold a specific technology (to compare them in static) [1]. In the first case, one can collect the values of the variables at multiple times; in the second case, the values of the variables are collected from multiple enterprises contemporaneously. Traditionally, the Leontief technology is chosen as a microeconomic production principle; the employed inputs are considered non-substitutable. The mentioned random variables are supposed to be independent. We will formulate these thoughts later as an assumption.

Note that sometimes the term "microeconomic production function" is used when a production function is derived from the repairmen problem [4], some of whose parameters are the quantities M and L of machines and repairmen, respectively.

There is division in probabilistic approaches regarding whether an optimization problem is set or not. The approaches which employ optimization to maximize the output (in order to obtain a global production function) customarily deal with unit productivities as random variables [2, 3, 5]. There, a relationship between the unit productivities called a technology menu is utilized. On the contrary, the others [1, 6, 7] use random capacities of inputs and stop at simply stating a microeconomic technology. Here we will uphold the second branch; let us present some results of it if the Leontief technology is assumed.

1) If the inputs' capacities have two-parameter Weibull distributions, then, given the assumption that their shape parameters are equal, the mean (or the median) of the quantity of output can be expressed through the means (or medians, respectively) of the factors' capacities in the form of the CES production function [1, 6]. Such constructs will be called *probabilistic production functions* (note that in the paper [1], probabilistic functions derived from the Leontief technology are referred to as *of Leontief class*).

2) If the inputs' capacities have Pareto [1] or Pareto-like (Lomax, Pareto Type VI) [7] distributions, then, given some restrictions on the distributions' parameters, it is possible to express the mean [1] or the median [7] of the quantity of output in terms of the medians of the factors' capacities in the form of the probabilistic Cobb-Douglas function.

3) Based on the generalized Weibull distribution [8, s. 2.16], it is possible [7] in special cases to obtain both the Cobb-Douglas and CES production functions.

Taking the opportunity, the author would like to point at some flaws in [6, pp. 42-43], where the idea of technology menu is briefly described. Indeed, first, distributions of the unit productivities a_i of the inputs must be considered instead of their per-unit values x_i, i.e., Weibull distributions appear for the parameters a_i; second, the test of second-order sufficient conditions must be taken into account when a global production function is being constructed. The author apologizes to the readers who could have been misled.

1. Goals of the Paper

The Weibull, Lomax, and generalized Weibull distributions are applied in reliability theory – failure times can be described by them. So, it seems interesting to attempt to construct Leontief class probabilistic production functions based on the failure times of a production line's machines and then analyze how restrictions on the distributions' parameters affect the reliability properties of the machines (hazard rate shapes).

Since we assume the Leontief technology, we need to understand why the output of the production line is described by this principle – here we readily recall the series model [9] (also the weakest element model, or the competing risks model).

There is the problem: if we consider a production function of only the failure times of the machines, the contribution of the other inputs is lost. Such a function will show only the *capacity of the group* (an attempt to give a definition to this term is provided in the section 2) of the failure times, i.e., the theoretical potential quantity of output when the other inputs are considered to be "ideal" and therefore neglected. However, it could be possible to obtain production functions of the other groups of inputs, representing their capacities. We should examine how the capacities of the groups relate to the actual quantity of output. If separate probabilistic production functions indicated, for example, the medians of the groups' capacities, then what sense would these measures make? Employing the probabilistic production function of the inputs of only one group seems pointless.

Note that it is a common assumption regarding probabilistic considerations of the CES function that the Weibull distributions of the capacities (or unit factor productivities) of all presented inputs possess the same shape parameter [1, 3, 6]. In [3], the rationale for this hypothesis is given for the R&D domain. However, this assumption turns the mentioned distributions into one-parameter. In this regard, one of the referees in [6] stated that the hypothesis cannot be found in real problems. So, we should examine how the hypothesis may be interpreted when reliability of machines is considered.

2. Formalization

Assume that n expendable factors of production are required to manufacture a product; their quantities are denoted by $X_1, \ldots, X_n$. Let us recall some definitions, omitting rigors.

Definition 1. *The per-unit value x_i of an input X_i is the amount of X_i that is chosen to manufacture one unit of the product; the other inputs are not taken into account.*

Definition 2. *The unit productivity a_i of an input X_i is the reciprocal of the per-unit value x_i of X_i: $a_i \triangleq 1/x_i$.*

Definition 3. *The capacity Q_i of an input X_i is the ratio of the allocated quantity of X_i to its chosen per-unit value x_i. That is, the capacity Q_i shows the theoretical potential quantity of output related to the input X_i.*

$$Q_i \triangleq \frac{X_i}{x_i} = a_i X_i.$$

The quantity of output will be denoted by Q. Generally speaking, it is connected with the inputs' capacities through a (microeconomic) production function $f : \mathbb{R}_+^n \to \mathbb{R}_+$:

$$Q = f\big(Q_1, \ldots, Q_n\big).$$

One of the customary production functions might be deployed (see, for example, [1, 5]), i.e., the special cases of the CES production function as a weighted p-norm of the capacities' vector – the CES function itself, the Cobb-Douglas function, the Leotief principle, and the linear function. The question of input's substitutability is an innate part of the mentioned functions. However, here only the Leontief principle and the linear function contain down-to-earth rules which are readily understandable by a production manager; they are extremes with zero and infinite substitutabilities, respectively. We will focus on the Leontief technology, assuming non-interchangeable nature of industrial components.

Assumption 1. *All expendable inputs $X_1, \ldots, X_n$ required to manufacture the chosen product are non-substitutable, their capacities Q_1, …, Q_n are independent random variables, and an enterprise upholds the Leontief principle to obtain the quantity Q of output:*

$$Q = \min\{Q_1, \ldots, Q_n\}. \tag{1}$$

Since we are going to extract only the reliability of machinery (more precisely, their operating times to failure) from all of the inputs, we need to formalize such treatment.

Let us divide all expendable inputs (and their quantities, respectively) into k nonempty and disjoint *groups* (perhaps of different cardinalities), which will be represented as vectors $\boldsymbol{X}^{(1)} = \big(X_1^{(1)}, \ldots, X_{n_1}^{(1)}\big)$, $\boldsymbol{X}^{(2)} = \big(X_1^{(2)}, \ldots, X_{n_2}^{(2)}\big)$, …, $\boldsymbol{X}^{(k)} = \big(X_1^{(k)}, \ldots, X_{n_k}^{(k)}\big)$, where $n_1 + \ldots + n_k = n$. Each group $\boldsymbol{X}^{(j)}$, $j = 1, \ldots, k$, is associated with the vector $\boldsymbol{Q}^{(j)} = \big(Q_1^{(j)}, \ldots, Q_{n_j}^{(j)}\big)$ which contains the capacities of the inputs of this group.

An analog of Definition 3 should be proposed for a group of inputs – it should indicate the theoretical quantity of output related to the group. To begin with, consider a group of two inputs, say the first two, with the quantities $X_1^{(a)}$, $X_2^{(a)}$ and the per-unit values $x_1^{(a)}$, $x_2^{(a)}$ (let "a" be an index of the group). We readily calculate the separate potential outputs $Q_1^{(a)}$ and $Q_2^{(a)}$; however, what should we do when both the factors are being used? A relation between their individual capacities should be taken into account; at the same time, all the other inputs should be "ideal" in their quantities and it should be forbidden to substitute the other inputs for $X_1^{(a)}$ and $X_2^{(a)}$. The latter is automatically met by Assumption 1.

If we would like to find the capacity of the group $\boldsymbol{X}^{(a)} = \big(X_1, X_2\big)$, all the inputs other than X_1 and X_2 should be taken in such amount that we do not even think about their capacities, that is, $Q_3 \to \infty$, …, $Q_n \to \infty$. Now, it follows from (1) that

$$Q = \min\big\{Q_1^{(a)}, Q_2^{(a)}\big\};$$

let us denote such quantity of output by $Q^{(a)}$ (i.e., Q plus the index of the group) and refer to it as the *capacity $Q^{(a)}$ of the group $\boldsymbol{X}^{(a)}$*:

$$Q^{(a)} = \min\big\{Q_1^{(a)}, Q_2^{(a)}\big\}.$$

It turns out that the capacity $Q^{(a)}$ is defined by the Leontief principle too; obviously, we can use this function as the core of the following definition.

Definition 4. *Given Assumption 1, the capacity $Q^{(j)}$ of a group (or the group capacity of) $\boldsymbol{X}^{(j)}$ equals the potential quantity of output when only the inputs $X_1^{(j)}, \ldots, X_{n_j}^{(j)}$ of this group are employed and the other factors have infinite capacities:*

$$Q^{(j)} \triangleq \min\big\{Q_1^{(j)}, \ldots, Q_{n_j}^{(j)}\big\}. \tag{2}$$

Note that if a group consists of only one input, there are no inconsistencies regarding Definition 4.

Remark 1. We could determine a group capacity in a more general way:

$$Q^{(j)} = f^{(j)}\big(Q_1^{(j)}, \ldots, Q_{n_j}^{(j)}\big),$$

where $f^{(j)}$ – some microeconomic production function pertaining to the chosen group; we will refer to this as a *group production function.*

Remark 2. To express the actual quantity Q of output as a function, we can deal with not only the capacities Q_i, $i = 1, \ldots, n$, of the inputs, but also with the capacities $Q^{(j)}$, $j = 1, \ldots, k$, of all k groups; at the same time, the second option will have another function F:

$$Q = F(Q^{(1)}, \ldots, Q^{(k)}).$$

Let us refer to the function F as a *group-divided production function.*

Remark 3. Obviously, given Assumption 1, the group-divided production function also utilizes the Leontief technology:

$$Q = \min\big\{Q^{(1)}, \ldots, Q^{(k)}\big\}.$$

3. Takeaway from Probabilistic Production Functions

In the current section, we present the key ideas of the approach which does not set an optimization problem to search for a global production function. We have some options. First, all of the inputs might be considered, without any division into groups, – here the inputs' capacities Q_i, $i = 1, \ldots, n$, will be the variables of the chosen microeconomic function f. Second, only the factors of one, say j-th, group might be taken into account, – their capacities will be the variables of the group function $f^{(j)}$.

Given the microeconomic production function f, a relation between the CDF $F_Q(q)$ of the quantity of output Q and the CDF's $F_{Q_i}(q)$, $i = 1, \ldots, n$, of the capacities Q_i of the inputs can be derived, if possible. One can obtain the means, medians, or other measures of these distributions with the CDF's $F_Q(q)$ and $F_{Q_i}(q)$, $i = 1, \ldots, n$. A probabilistic production function is intended to bind together a characteristic (e.g., mean) of the capacity Q and measures of the capacities Q_i, $i = 1, \ldots, n$, of the factors.

Analogously, given the group function $f^{(j)}$, a relation between the CDF $F_{Q^{(j)}}(q)$ of the j-th group capacity $Q^{(j)}$ and the CDF's $F_{Q_i^{(j)}}(q)$, $i = 1, \ldots, n_j$, of the capacities $Q_i^{(j)}$, $i = 1, \ldots, n_j$, of the group's factors can be found. One can calculate the means, medians, or other measures of these distributions ($F_{Q^{(j)}}(q)$ and $F_{Q_i^{(j)}}(q)$, $i = 1, \ldots, n_j$) and then pack these characteristics into the form of a probabilistic group production function.

In order to relate probabilistic group production functions to the actual quantity Q of output, as an option, we may recall Remark 2, more specifically, a group-divided function F. If we applied the just mentioned for the two cases procedure, we could obtain a general probabilistic production function which bound together a characteristic (e.g., mean) of the quantity Q of output and measures of the capacities $Q^{(j)}$, $j = 1, \ldots, k$, of all k groups.

4. Probabilistic Production Functions of Machines' Reliability

Suppose m machines are set in a production line. We focus primarily on automated production lines since work units processing times (see below, τ_i, $i = 1, \ldots, m$) may naturally be considered deterministic in that case.

4.1. Weibull Distribution

Let us represent the failure times T_i, $i = 1 \ldots m$, of the m machines as independent random variables which have Weibull distributions with shape $\beta_i > 0$ and scale $\lambda_i > 0$ parameters. Their CDF's are given by

$$F_{T_i}(t) = 1 - e^{-\lambda_i \cdot t^{\beta_i}}, \quad t \geq 0. \tag{3}$$

Let τ_i be the work unit processing time in the i-th workstation. It is the per-unit value of the failure time of this station. The capacity of the failure time of the i-th machine equals

$$Q_i^{(e)} = T_i / \tau_i. \tag{4}$$

("e" – "equipment" – denotes the group of the reliability inputs). Hence, the CDF's of the reliability capacities are expressed as (denote $\alpha_i = \lambda_i (\tau_i)^{\beta_i}$)

$$F_{Q_i^{(e)}}(q) = 1 - e^{-\alpha_i \cdot q^{\beta_i}}, \quad q \geq 0. \tag{5}$$

Suppose that the production line stops working when a failure of one of the machines occurs; a changeover or repair comes later. So, we need to obtain the potential quantity $Q^{(e)}$ of output collected between the beginning of exploitation and the changeover or overhaul.

The production line represents a series system, and we have supposed that $Q_1^{(e)}, \ldots, Q_m^{(e)}$ are independent; so here we have the weakest element model. Hence we naturally state that the reliability group capacity $Q^{(e)}$ is restricted by the weakest workstation:

$$Q^{(e)} = \min\{Q_1^{(e)}, \ldots, Q_m^{(e)}\}. \tag{6}$$

It is the Leontief technology (recall (2)) for the group. We have just seen how this economic principle can be interpreted in light of reliability.

The CDF of the group capacity $Q^{(e)}$ is expressed by the known formula [9, s. 1.5.1]:

$$F_{Q^{(e)}}(q) = 1 - \prod_{i=1}^{m} \left(1 - F_{Q_i^{(e)}}(q)\right). \tag{7}$$

Substituting (5) into (7) follows

$$F_{Q^{(e)}}(q) = 1 - e^{-\alpha_1 \cdot q^{\beta_1} - \ldots - \alpha_m \cdot q^{\beta_m}}, \quad q \geq 0. \tag{8}$$

The common hypothesis mentioned in Section 1 should be employed – assume that the shape parameters are equal:

$$\beta_1 = \dots = \beta_m \equiv \beta. \tag{9}$$

Now, the CDF (8) takes the form

$$F_{Q^{(e)}}(q) = 1 - e^{-(\alpha_1 + \ldots + \alpha_m) \cdot q^{\beta}}, \quad q \geq 0.$$

Using well-known expressions for the mean $EQ^{(e)}$ of $Q^{(e)}$ and for the means $EQ_i^{(e)}$ of $Q_i^{(e)}$, one can obtain [6] the probabilistic CES (reliability group) production function of the form

$$EQ^{(e)} = \left(\left(EQ_1^{(e)}\right)^{-\beta} + \ldots + \left(EQ_m^{(e)}\right)^{-\beta}\right)^{-\frac{1}{\beta}},$$

or, using (4) and denoting the mean life of the i-th workstation by ET_i,

$$EQ^{(e)} = \left(\tau_1^{\beta}(ET_1)^{-\beta} + \ldots + \tau_m^{\beta}(ET_m)^{-\beta}\right)^{-\frac{1}{\beta}}.$$

Analogously, assuming (9), we employ the median $M_{Q^{(e)}}$ of $Q^{(e)}$ and the medians $M_{Q_i^{(e)}}$ of the reliability capacities $Q_i^{(e)}$ and combine them into the probabilistic CES function

$$M_{Q^{(e)}} = \left(\left(M_{Q_1^{(e)}}\right)^{-\beta} + \ldots + \left(M_{Q_m^{(e)}}\right)^{-\beta}\right)^{-\frac{1}{\beta}}, \tag{10}$$

or, using (4) and denoting the median of the i-th machine failure time by M_{T_i},

$$M_{Q^{(e)}} = \left(\tau_1^{\beta} M_{T_1}^{-\beta} + \ldots + \tau_m^{\beta} M_{T_m}^{-\beta}\right)^{-\frac{1}{\beta}}.$$

In order to extract the meaning of the hypothesis (9), the sense of the parameters β_i and λ_i should be studied, hence we relate to the hazard rate function $h_i(t)$ of the i-th machine. For the Weibull distribution (3), this function has the form [10]

$$h_i(t) = \lambda_i \beta_i \cdot t^{\beta_i - 1},$$

or, employing (9),

$$h_i(t) = \lambda_i \beta \cdot t^{\beta - 1}.$$

Hence the hypothesis (9) reveals the following simple idea:

It is assumed that all of the workstations of the production line have the same shapes of a hazard rate curve, and these shapes are distinguished by scale factors λ_i.

Consider special cases in this regard [9, 10]:

1) $0 < \beta < 1$: the hazard rates of all of the machines decrease with zero asymptotic value at $t \to \infty$. Here the workstations are not aging but have initial flaws, whose influences are being rooted out gradually. The asymptotic behavior of the curves at $t \to 0$ indicates that failures most probably occur at the beginning of exploitation.

2) $\beta = 1$: the hazard rates are constant – $h_i(t) = \lambda_i$; the Weibull distribution turns into the exponential (this is more applicable for modeling of sudden failures).

3) $1 < \beta < 2$: the hazard rates increase and are upper-bounded – almost no initial flaws (due to [9] burn-in or quality control), but the machines are subject to aging and wear-out.

4) $\beta = 2$: the hazard functions are linear – the beginning of extreme wear-out; the Weibull distribution turns into the Rayleigh distribution.

5) $\beta > 2$: the hazard rates increase and are not upper-bounded (extreme wear-out).

4.2. Lomax Distribution

Suppose the failure times T_i, $i = 1 \ldots m$, are independent random variables which have Lomax distributions with shape $\alpha_i > 0$ and scale $\theta_i > 0$ parameters; the CDF's are

$$F_{T_i}(t) = 1 - \big(1 + (t/\theta_i)\big)^{-\alpha_i}, \quad t \geq 0.$$

Recalling (4), we obtain the CDF's of the reliability capacities of the workstations:

$$F_{Q_i^{(e)}}(q) = 1 - \big(1 + \tau_i \cdot (q/\theta_i)\big)^{-\alpha_i}, \quad q \geq 0.$$

If the Leontief function (6) is employed, then it is not hard to show [7] that when the shape parameters α_i are so small that $2^{1/\alpha_i} \gg 1$, $i = 1, \ldots, m$ (roughly $0 < \alpha_i < 0,3$), then the median of the group capacity is approximately related to the medians of the machines' reliability capacities via the probabilistic Cobb-Douglas function of the form

$$M_{Q^{(e)}} \approx 2^{-\frac{n-1}{\alpha_1+\ldots+\alpha_m}} \left(M_{Q_1^{(e)}}\right)^{\frac{\alpha_1}{\alpha_1+\ldots+\alpha_m}} \left(M_{Q_2^{(e)}}\right)^{\frac{\alpha_2}{\alpha_1+\ldots+\alpha_m}} \ldots \left(M_{Q_m^{(e)}}\right)^{\frac{\alpha_m}{\alpha_1+\ldots+\alpha_m}}.$$

The hazard function of the i-th workstation is expressed as [10]

$$h_i(t) = \frac{\alpha_i}{\theta_i + t}.$$

Regardless of the values of α_i and θ_i, all of the machines have decreasing hazard rates with zero asymptotic value as $t \to \infty$. The workstations have gradually fading initial flaws and are not subject to aging; the failures most probably occur at the beginning of exploitation.

4.3. Generalized Weibull Distribution

In [7], the possibility of constructing the probabilistic Cobb-Douglas and CES production functions based on the generalized Weibull distribution [8, s. 2.16] is presented.

Suppose the failure times T_i, $i = 1 \ldots m$, of the machines are independent random variables which have generalized Weibull distributions with shape $\alpha_i > 0$, scale $\theta_i > 0$, and inequality $\beta_i > 0$ parameters [8, s. 2.16, with relabelings]:

$$F_{T_i}(t) = 1 - \big(1 + (1/\alpha_i)(t/\theta_i)^{\beta_i}\big)^{-\alpha_i}, \quad t \geq 0. \tag{11}$$

Using (4), we obtain the CDF's of the machines' reliability capacities:

$$F_{Q_i^{(e)}}(q) = 1 - \left[1 + \frac{1}{\alpha_i}\left(\frac{q}{\theta_i/\tau_i}\right)^{\beta_i}\right]^{-\alpha_i}, \quad q \geq 0.$$

The group capacity $Q^{(e)}$ is a result of the Leontief technology (6); so it can be shown [7] that when the shape parameters α_i are so small that $2^{1/\alpha_i} \gg 1$, $i = 1, \ldots, m$ (roughly $0 < \alpha_i < 0,3$), then the median of the group capacity is approximately related to the medians of the machines' reliability capacities via the Cobb-Douglas function of the form

$$M_{Q^{(e)}} \approx 2^{-\frac{n-1}{\alpha_1\beta_1+\ldots+\alpha_m\beta_m}} \left(M_{Q_1^{(e)}}\right)^{\frac{\alpha_1\beta_1}{\alpha_1\beta_1+\ldots+\alpha_m\beta_m}} \left(M_{Q_2^{(e)}}\right)^{\frac{\alpha_2\beta_2}{\alpha_1\beta_1+\ldots+\alpha_m\beta_m}} \ldots \left(M_{Q_m^{(e)}}\right)^{\frac{\alpha_m\beta_m}{\alpha_1\beta_1+\ldots+\alpha_m\beta_m}},$$

or, using (4) and introducing constants A, $\gamma_1, \ldots, \gamma_m$ (note $\gamma_1 + \ldots + \gamma_m = 1$),

$$M_{Q^{(e)}} \approx A \cdot M_{T_1}^{\gamma_1} M_{T_2}^{\gamma_2} \ldots M_{T_m}^{\gamma_m}.$$

Moreover, when the shape parameters are arbitrarily large, $\alpha_i \to \infty$, $i = 1, \ldots, m$, the distributions (11) turn into Weibull distributions, and the medians bind together by [7]

$$\left(M_{Q^{(e)}}/M_{Q_1^{(e)}}\right)^{\beta_1} + \ldots + \left(M_{Q^{(e)}}/M_{Q_m^{(e)}}\right)^{\beta_m} = 1,$$

hence, using the hypothesis (9), we obtain the probabilistic CES function of the form (10):

$$M_{Q^{(e)}} = \left(\left(M_{Q_1^{(e)}}\right)^{-\beta} + \ldots + \left(M_{Q_m^{(e)}}\right)^{-\beta}\right)^{-\frac{1}{\beta}}.$$

When $\alpha_i \to \infty$, $i = 1, \ldots, m$, the means can also be used to construct the CES function.

Regarding reliability behavior, the hazard rate of the i-th workstation is given by [8]

$$h_i(t) = \frac{\beta_i (t/\theta_i)^{\beta_i - 1}}{\theta_i \left(1 + (1/\alpha_i) \cdot (t/\theta_i)^{\beta_i}\right)}.$$

The shape of this function varies as the values of β_i, α_i change [8]; taking into account the above restrictions on these parameters, we can distinguish the following cases:

1) $0 < \beta_i \leq 1$, $0 < \alpha_i < 0,3$: the i-th hazard function decreases with zero asymptotic value as $t \to \infty$ – only the fading initial flaws influence; the machine is not prone to aging.

2) $\beta_i > 1$, $0 < \alpha_i < 0,3$: the hazard curve has the form of inverted bathtub – the non-aging machine has such flaws whose hazard initially increases and then decreases.

3) $\alpha_i \to \infty$: here is the transition towards the Weibull distribution, whose hazard curve's shape regarding the values of β_i has been analyzed earlier.

Note that the classic bathtub-shaped hazard curve relates to the values $0 < \beta_i < 1$ and $\alpha_i < 0$; however, only $\alpha_i > 0$, $i = 1, \ldots, m$, are considered in the current model.

Conclusion

In order to consider production functions of only the reliability characteristics of a production line's machines, a modification of the theoretical framework has been required – we have introduced a production function of a chosen group of inputs, which indicates the capacity of this group, i.e., the corresponding potential quantity of output.

For production machines combined in a line, it is correct to employ the Leotief function of the capacities of the machines' failure times to evaluate the potential output. If the failure times have Weibull, Lomax, or generalized Weibull distributions, it is possible to obtain probabilistic production functions which relate a measure (e.g., mean) of the capacity of the reliability group to measures of the failure distributions. The parameters of these distributions determine the behavior of the hazard rates, which has also been examined.

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ВЕРОЯТНОСТНАЯ ПРОИЗВОДСТВЕННАЯ ФУНКЦИЯ ПАРКА ОБОРУДОВАНИЯ: ИНТЕРПРЕТАЦИЯ ФУНКЦИИ ЛЕОНТЬЕВА С ТОЧКИ ЗРЕНИЯ НАДЕЖНОСТИ

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Поставлен вопрос о разделении факторов производства на наборы; предложено определение мощности набора факторов как теоретического количества выпускаемой продукции, получаемого при задействовании мощностей только факторов выбранного набора. Рассмотрена возможная взаимосвязь мощностей наборов факторов и вероятностных производственных функций. Предложено рассматривать наработки оборудования до отказа как компоненты производства. Для поточной линии по наработкам машин, независимо распределенных по законам Вейбулла и Ломакса, на основе функции Леонтьева построены вероятностные производственные функции CES и Кобба-Дугласа соответственно и проанализировано влияние параметров распределений на формы кривых опасности отказа машин поточной линии. С позиций теории надежности дано обоснование гипотезе о равенстве параметров формы распределений Вейбулла при построении вероятностной производственной CES-функции. По независимым наработкам до отказа, имеющим обобщенные распределения Вейбулла, на основе принципа Леонтьева построены в частных случаях вероятностные производственные функции CES и Кобба-Дугласа и проанализирована форма кривой опасности отказа.

Ключевые слова: функция Леонтьева; производственная функция; поточная линия; наработка до отказа; функция опасности отказа.

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