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SOME FORMULAE FOR CALCULATION OF THE MEAN DERIVATIVES

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The paper continues the study of theoretical issues related to finding mean derivatives of stochastic diffusion processes. A theorem on the existence of the backward mean derivative for L_1-random processes in $\mathbb{R}^n$ with continuous sample trajectories is proved using an equality based on the properties of conditional expectation. For Ito stochastic processes of diffusion type in $\mathbb{R}^n$, a relation connecting the backward mean derivative with vector fields is proved.

Keywords: mean derivative; second order stochastic differential equations; stochastic algebraic-differential equations.

Introduction

The notion of mean derivatives was introduced by E. Nelson (see [1–3]) for the needs of his Stochastic Mechanics, a version of quantum mechanics. The equation of motion in the Stochastic Mechanics was the so called Newton–Nelson equation, a second order stochastic equation with mean derivatives where a special second order mean derivative was in use.

Note that in [4] a new mean derivative, called quadratic, in addition to Nelson notions of forward, backward and symmetrical derivatives. Our definition is a slight modification of Nelson ideas. (Nelson defined quadratic derivative as a unit operator multiplied by some constant). The use of quadratic mean deriovative together one of the Nelson's mean derivative allows one recover the process from its mean derivative and it makes possible to obtain solutions to many equations including mean derivatives occurring in mathematical physics.

In this paper we present some methods and formulae for calculation the mean derivatives of stochastic processes. The proofs of some of them were absent in the literature.

1. General Definitions

The newest introduction in the ewtheory of mean derivatives can bi found in [5]. Here we describe only the notions tht are necessary for description of the main reesults. Everywhere below we deal with processes, equations, etc., given on a certain finite time interval $[0, T]$.

Consider a stochastic process $\xi(t)$ in $\mathbb{R}^n$, $t \in [0, T]$, given on a certain probability space $(\Omega, \mathcal{F}, \mathsf{P})$ and such that $\xi(t)$ is L_1-random variable for all t. Every stochastic process $\xi(t)$ in $\mathbb{R}^n$, $t \in [0, l]$, determines three families of σ-subalgebras of σ-algebra $\mathcal{F}$:
(i) the "past" $\mathcal{P}_t^\xi$ generated by pre-images of Borel sets in $\mathbb{R}^n$ by all mappings $\xi(s) : \Omega \to \mathbb{R}^n$ for $0 \le s \le t$;

(ii) the "future" $\mathcal{F}_t^\xi$ generated by pre-images of Borel sets in $\mathbb{R}^n$ by all mappings $\xi(s) : \Omega \to \mathbb{R}^n$ for $t \leq s \leq l$;
(iii) the "present" ("now") $\mathcal{N}_t^\xi$ generated by pre-images of Borel sets in $\mathbb{R}^n$ by the mapping $\xi(t)$.
All families are supposed to be complete, i.e., containing all sets of probability 0.

For convenience we denote the conditional expectation of $\xi(t)$ with respect to $\mathcal{N}_t^\xi$ by $E_t^\xi(\cdot)$. Ordinary ("unconditional") expectation is denoted by E. Strictly speaking, almost surely (a.s.) the sample paths of $\xi(t)$ are not differentiable for almost all t. Thus its "classical" derivatives exist only in the sense of generalized functions. To avoid using the generalized functions, following Nelson (see, e.g., [1–3]) we give

Definition 1. *(i) Forward mean derivative $D\xi(t)$ of $\xi(t)$ at time $t \in [0, T)$ is an L_1-random variable of the form*

$$D\xi(t) = \lim_{\Delta t \to +0} E_t^\xi \left(\frac{\xi(t + \Delta t) - \xi(t)}{\Delta t} \right) \tag{1}$$

where the limit is supposed to exists in $L_1(\Omega, \mathcal{F}, \mathsf{P})$ and $\Delta t \to +0$ means that Δt tends to 0 and $\Delta t > 0$.

(ii) Backward mean derivative $D_*\xi(t)$ of $\xi(t)$ at $t \in (0, T]$ is an L_1-random variable

$$D_*\xi(t) = \lim_{\Delta t \to +0} E_t^\xi \left(\frac{\xi(t) - \xi(t - \Delta t)}{\Delta t} \right) \tag{2}$$

where the conditions and the notation are the same as in (i).

(iii) The derivative $D_S = \frac{1}{2}(D + D_*)$ is called symmetric mean derivative. The vector $v^\xi(t) = v^\xi(t, \xi(t)) = D_S\xi(t)$ is called current velocity of $\xi(t)$

Please note that the current velocities are natural analogues of physical velocities od deterministic processes.

From the properties of conditional expectation (see [6]) it follows that $D\xi(t)$ and $D_*\xi(t)$ can be represented as compositions of $\xi(t)$ and Borel measurable vector fields (regressions)

$$Y^0(t, x) = \lim_{\Delta t \to +0} E \left(\frac{\xi(t + \Delta t) - \xi(t)}{\Delta t} | \xi(t) = x \right)$$

$$Y_*^0(t, x) = \lim_{\Delta t \to +0} E \left(\frac{\xi(t) - \xi(t - \Delta t)}{\Delta t} | \xi(t) = x \right) \tag{3}$$

on $\mathbb{R}^n$. This means that $D\xi(t) = Y^0(t, \xi(t))$ and $D_*\xi(t) = Y_*^0(t, \xi(t))$.

Definition 2. *For an L_1-stochastic process $\xi(t)$, $t \in [0, T]$, its quadratic mean derivative $D_2\xi(t)$ is defined by the formula*

$$D_2\xi(t) = \lim_{\triangle t \to +0} E_t^\xi \left(\frac{(\xi(t + \triangle t) - \xi(t))(\xi(t + \triangle t) - \xi(t))^*}{\triangle t} \right), \tag{4}$$

where $(\xi(t + \triangle t) - \xi(t))$ is a column vector and $(\xi(t + \triangle t) - \xi(t))^*$ is its conjugate, i.e., the row vector, the limit is supposed to exist in $L_1(\Omega, \mathcal{F}, \mathsf{P})$.

One can easily derive that for an Ito process $\xi(t) = \int_0^t a(s)ds + \int_0^t A(s)dw(s)$ its quadratic mean derivative takes the form $D_2\xi(t) = AA^*$.

The mean derivatives defined above particular cases of object defined as follows. Let $x(t)$ and $y(t)$ be L_1 stochastic processes in $\mathbb{R}^n$, given on $(\Omega, \mathcal{F}, \mathsf{P})$. Introduce the forward y mean dreivative of $x(t)$ by the formula

$$D^y x(t) = \lim_{\Delta t \to +0} E_t^y \left(\frac{x(t+\Delta t) - x(t)}{\Delta t} \right) \tag{5}$$

and the backward y mean derivative of $x(t)$ by the formula

$$D_*^y x(t) = \lim_{\Delta t \to +0} E_t^y \left(\frac{x(t) - x(t-\Delta t)}{\Delta t} \right) \tag{6}$$

where, of course, thed limist must exist in $L_1(\Omega, \mathcal{F}, \mathsf{P})$.

2. Main Results

Theorem 1. *Let $g(t)$ and $h(t)$ be L_1-random processes with continuous sample trajectories in $\mathbb{R}^n$, defined for $t \in [0, T]$ on the same probability space. Consider the process $E_t^h g(t)$. Let $Dh(t)$ and $D_* h(t)$ exist. Then:*

(i) $D^h g(t)$ exists if and only if $D^h E_t^h g(t)$ exists, and $D^h E_t^h g(t) = D^h g(t)$;

(ii) $D_*^h g(t)$ exists if and only if $D_*^h E_t^h g(t)$ exists, and $D_*^h E_t^h g(t) = D_*^h g(t)$.

Proof. We choose an arbitrary smooth real-valued function $f : \mathbb{R}^n \to \mathbb{R}$ with compact support. Using equality

$$\begin{aligned} \left(E_{t+\Delta t}^h g(t+\Delta t)\right) f\left(h(t+\Delta t)\right) - \left(E_t^h g(t)\right) f\left(h(t)\right) = \\ \{E_{t+\Delta t}^h g(t+\Delta t) - E_t^h g(t)\} f(h(t+\Delta t)) + \\ E_t^h g(t)\{f(h(t+\Delta t)) - f(h(t))\}, \end{aligned} \tag{7}$$

We we obtain

$$E D_*^h \{\left(E_t^h g(t)\right) f\left(h(t)\right)\},$$

using (2) from the definition of the backward mean derivative,

$$\lim_{\Delta t \to +0} E \left(E_t^h \left(\frac{E_t^h g(t) f\left(h(t)\right) - E_{t-\Delta t}^h g(t-\Delta t) f\left(h(t-\Delta t)\right)}{\Delta t} \right) \right),$$

using the equality (7),

$$\begin{aligned} E \lim_{\Delta t \to +0} E_t^h \frac{E_t^h g(t) - E_{t-\Delta t}^h g(t-\Delta t)}{\Delta t} f\left(h(t)\right) + \\ E \lim_{\Delta t \to +0} E_t^h E_{t-\Delta t}^h g(t-\Delta t) \frac{f\left(h(t)\right) - f\left(h(t-\Delta t)\right)}{\Delta t} \end{aligned}$$

using the definitions of the mean derivative on the right side (1) and on the left (2), we obtain

$$E D_*^h E_t^h g(t) f\left(h(t)\right) - E E_t^h g(t-\Delta t) D^h f\left(h(t-\Delta t)\right),$$

The existence of the second term on the right-hand side follows from the conditions of the lemma. Thus, the limit exists if and only if $D_*^h E_t^h g(t)$ exists. On the other hand, by (iii) the property of the conditional expectation:

$$ED_*^h g(t) f\left(h(t)\right) - Eg(t-\Delta t) D^h f\left(h(t-\Delta t)\right),$$

Similarly, if and only if $D_*^h g(t)$ exists. Obviously,

$$\begin{aligned} ED_*^h E_t^h g(t) f\left(h(t)\right) = ED_*^h g(t) f\left(h(t)\right) \\ EE_t^h g(t-\Delta t) D^h f\left(h(t-\Delta t)\right) = Eg(t-\Delta t) D^h f\left(h(t-\Delta t)\right) \end{aligned}$$

□

Lemma 1.

(i) If $x(t)$ is a martingale with respect to $\mathcal{P}_t^y$, then $D^y x(t) = 0$.

(ii) If $x(t)$ is an inverse martingale with respect to $\mathcal{F}_t^y$, then $D_*^y x(t) = 0$.

The proof of the lemma is based on a technical application of the properties of the conditional expectation and is omitted here.

Theorem 2. *For the Ito process in $\mathbb{R}^n$, the following formulas hold:*

$$\begin{aligned} DZ &= \frac{\partial}{\partial t} Z + (Y^0 \cdot \nabla) Z + \frac{1}{2} tr\ Z''(A, A), \\ D_* Z &= \frac{\partial}{\partial t} Z + (Y_*^0 \cdot \nabla) Z - \frac{1}{2} tr\ Z''(A, A), \end{aligned} \tag{8}$$

where $\nabla = \left(\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}\right)$, tr is the trace, the dot denotes the scalar product in $\mathbb{R}^n$, and the vector fields $Y^0\left(t, \xi(t)\right)$ and $Y_*^0\left(t, \xi(t)\right)$ are introduced in (3)

Proof. The vector field $Z(t, x)$ can be viewed as a continuously differentiable mapping $Z : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$. We expand it in a Taylor series in a neighborhood of the point x_0, t_0.

$$Z(t, x) = Z(t_0, x_0) + \frac{\partial Z}{\partial t} \partial t + Z' \partial x + \frac{1}{2} \frac{\partial^2 Z}{\partial t^2} (\partial t)^2 + \frac{1}{2} Z''(\partial x, \partial x) + \ldots \tag{9}$$

where the primes denote the derivatives of Z with respect to x at the point x_0. We substitute into it the increment of the Ito process $\partial \xi(s) = a(s)\partial s + A(s)\partial w(s)$. We get

$$\begin{aligned} Z(s, \xi(s)) &= Z(t_0, x_0) + \frac{\partial Z}{\partial s} \partial s + Z'\left(a(s)\partial s + A(s)\partial w(s)\right) + \frac{1}{2} \frac{\partial^2 Z}{\partial s^2} (\partial s)^2 + \\ &+ \frac{1}{2} Z''(a(s)\partial s + A(s)\partial w(s), a(s)\partial s + A(s)\partial w(s)) + \ldots = \\ &= Z(t_0, x_0) + \frac{\partial Z}{\partial s} \partial s + Z' a(s) \partial s + Z' A(s) \partial w(s) + \frac{1}{2} \frac{\partial^2 Z}{\partial s^2} (\partial s)^2 + \\ &+ \frac{1}{2} \left(Z''(a(s), a(s))(\partial s)^2 + Z''(a(s)\partial s, A(s)\partial w(s)) + \right. \\ &\left. + Z''(A(s)\partial w(s), a(s)\partial s) + Z''(A(s)\partial w(s), A(s)\partial w(s))\right) + \ldots \end{aligned} \tag{10}$$

Integrating this formula, we find that if $Z(t,x)$ is a continuously differentiable (once with respect to t and twice with respect to x) mapping, then for process $Z(\xi(t))$ the classical Ito formula holds:

$$Z(\xi(s)) = Z(\xi_0) + \int_0^t \left[\frac{\partial Z}{\partial t} + Z'(a(s)) + \frac{1}{2} trZ''(A(s), A(s))\right] ds + \\ + \int_0^t Z'(A(s))\, dw(s), \tag{11}$$

Where

$$trZ''(A(s), A(s)) = \sum_{i=1}^{n} Z''(A(s)e_i, A(s)e_i), \tag{12}$$

$e_1, ..., e_k$ is an arbitrary orthonormal basis in R^k. Indeed, by formulas (i) and (ii) of Theorem 2, exactly one second-order integral

$$\frac{1}{2}\int_0^t Z''(A(s)\, dw(s), A(s)\, dw(s)) = \int_0^t \frac{1}{2} trZ''(A(s), A(s))\, ds \tag{13}$$

is not equal to zero. Since the Ito integral is a martingale with respect to $\mathcal{P}_t^{\xi}$ (see Theorem 1 and Lemma 1), the last term of expression 11 is zero. Also note that $E_t^{\xi}(Z'(\beta)) = ((Y^0 \cdot \nabla)Z)(\xi(t))$. Thus, we obtain the first expression from (8).

Similarly, we can apply the inverse Ito formula to $Z(t, \xi(t))$. Then repeating the steps above:

$$Y_*^0(t, \xi(t)) = D_*\xi(t) = E_t^{\xi}(\beta(t)) - 2u^{\xi}(t, \xi(t)), \\ -2u^{\xi}(t, \xi(t)) = \lim_{\partial t \to +0} E_t^{\xi}\left(\frac{w(t) - w(t - \partial t)}{\partial t}\right). \tag{14}$$

Next,

$$E_t^{\xi}(Z'(\beta(t))) + \lim_{\partial t \to +0} E_t^{\xi}\left(\frac{\int_{t-\partial t}^{t} Z'\, d_* w}{\partial t}\right) = \left((Y_*^0 \cdot \nabla)Z\right)(\xi(t)) \tag{15}$$

Thus, we arrive at the second expression from (8)

□

References

1. Nelson E. Derivation of the Schrödinger Equation from Newtonian Mechanics. *Phys. Reviews*, 1966, vol. 150, no. 4, pp. 1079–1085.
2. Nelson E. *Dynamical Theory of Brownian Motion.* Princeton, Princeton University Press, 1967.
3. Nelson E. *Quantum Fluctuations*, Princeton, Princeton University Press, 1985.
4. Azarina S.V., Gliklikh Yu.E. Differential Inclusins with Mean Derivatives. *Dynamic Systems and Applications*, 2007, vol. 16, no. 1. P. 49–71.
5. Gliklikh Yu.E. Stochastic Equations and Inclusions with Mean Derivatives and Their Applications. *Journal of Mathematical Sciences*, 2024, vol. 282, no. 2, pp. 111–253.

6. Parthasarathy K.R. *Introduction to Probability and Measure.* New York, Springer-Verlag, 1978.

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НЕКОТОРЫЕ ФОРМУЛЫ ДЛЯ ВЫЧИСЛЕНИЯ ПРОИЗВОДНЫХ В СРЕДНЕМ

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В статье продолжено исследование теоретических вопросов, связанных с нахождением производных в среднем стохастических процессов диффузионного типа. Доказана теорема существования производной в среднем слева для L_1-случайных процессов в $\mathbb{R}^n$ с непрерывными выборочными траекториями, используя равенство, основанное на свойствах условного математического ожидания. Для диффузионных стохастических процессов Ито в $\mathbb{R}^n$ доказано соотношение, связывающее производную в среднем слева с векторными полями.

Ключевые слова: производные в среднем; стохастические дифференциальные уравнения второго порядка; стохастические алгебро-дифференциальные уравнения.

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