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THE DIFFUSION EQUATION IN SPACES OF DIFFERENTIAL K-FORMS DEFINED ON A RIEMANNIAN MANIFOLD WITHOUT BOUNDARY, USING THE EXAMPLE OF A SPHERICAL APPROXIMATION OF THE EARTH'S SURFACE

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This article substantiates a new approach to systems studies of the propagation of temperatures, fluid flows, and electromagnetic waves on manifolds without boundary (this applies to the heat equation, the Navier–Stokes system of equations, and the Maxwell system of equations) using, as a generalization, the invariant form of the diffusion equation in spaces of differential forms of different ranks and pseudodifferential operators on them (one of these is the Laplace–Beltrami operator). The spherical surface of the Earth is given as an example of a manifold without boundary, and the need for such studies is raised. The relationship between the solutions of the heat conduction equations, the Navier–Stokes system of equations, and the Maxwell system of equations is noted, through the spectrum of the Laplace–Beltrami operator, on a manifold without boundary (in our case, on a spherical surface of Earth's radius).

Keywords: differential forms; Riemannian manifold; Laplace–Beltrami operator; pseudodifferential operators.

Introduction

We consider a new approach to the system study of the propagation of temperatures, fluid flows, and electromagnetic waves on a single surface – a spherical approximation of the Earth's surface. This is accomplished by searching for solutions of the heat equation, the Navier–Stokes system, and Maxwell's system simultaneously, using as a generalization the invariant form of the diffusion equation in spaces of differential forms of different ranks and pseudodifferential operators on these forms, the principal one being the Laplace–Beltrami operator. The spherical surface of the Earth is given as an example of a manifold without boundary, but the solutions themselves can also be considered on manifolds without boundary with more complicated global geometry, in particular the torus. The connection between the solutions of the heat equation, the Navier–Stokes system, and Maxwell's system occurs through the spectrum of the Laplace–Beltrami operator on a manifold without boundary, in our example on the spherical surface of the Earth. An example of possible results of such investigations is presented in the form of three graphs: the distribution of temperature (0-forms) over the entire Earth's surface; the distribution of wind speed (the first of two components of the solution – the 1-form of the displacement gradient, while the second component is the 0-form of pressure) over the entire Earth's surface (although a weak analogy between wind in air space and fluid flows is used for illustration, because water does not cover the entire Earth's surface and the Navier–Stokes system must be solved on a surface with boundary). Nevertheless, the analogy in this graph works sufficiently well, because a potential (gradient) flow and pressure are present in the Navier–Stokes system and can be measured for air at every point of the Earth's surface); and the magnetic field flux (2-form) through the surface of the Earth. In all this, the question of solving initial-boundary value problems was not raised, but only the solvability of the equations as a whole. A certain automatic separation of solutions arises due to the different ranks of the differential forms. The general information in the solutions is described by the eigenvalues of the Laplace–Beltrami operator from the invariant form of the differential diffusion equation, and is mainly related to the global geometry of the manifold without boundary. In the example of the Earth's surface, the common part is expressed via the discrete, finite-multiplicity spectrum of the Laplace–Beltrami operator on the sphere.

1. Differential k-Forms

Let E be a two-dimensional smooth compact oriented Riemannian manifold without boundary. Using the theory presented in [1–2], we construct spaces of smooth differential q-forms $H_q = H_q(E)$ for $q = 0, 1, 2$ of the form

$$\omega(t, x_1, x_2) = \sum_{i_1+i_2=q} \chi_{i_1,i_2}(t, x_1, x_2) dx_{i_1} \wedge dx_{i_2}, \tag{1}$$

where the indices i_1, i_2 can take the values 0 or 1.

If we write equations in invariant form in the space of differential forms, we must use the Laplace–Beltrami operator

$$\Delta_{LB} u = d * d * u + *d * du = d\delta + \delta d, \tag{2}$$

where $*$ is the Hodge operator and d is the gradient that raises the rank of the form, while $\delta = *d*$ is the divergence that lowers the rank of the form. It generalizes the usual Laplace operator up to sign.

In these spaces one can introduce the inner product

$$(\chi, \xi)_0 = \int_{\mathbb{E}} \chi \wedge *\xi, \qquad \chi, \xi \in H_q. \tag{3}$$

By virtue of the Hodge–Kodaira theorem, the space of differential forms completed with respect to the norm obtained from the inner product (3) splits into the direct sum of potential, solenoidal, and harmonic differential forms. The operator $\delta = *d*$ is adjoint to d with respect to the inner product (3). In our case, for simplicity, we will consider the restriction to the subspace containing no harmonic ($\Delta_{LB} = 0$) forms.

2. System Analysis Based on the Diffusion Equation

Consider the linear inhomogeneous diffusion equation [3]

$$V_t = D\Delta_{LB} V + f. \tag{4}$$

where $u = u(t, x_1, x_2)$ is the concentration of a substance, depending on time, and defined in a two-dimensional space. The operator Δ_{LB} is given in (2).

2.1. Linear Heat Equation

Now take the space of differential 0-forms (i.e., forms of rank 0 exactly coinciding with the space of functions of three variables). Take the unknown function $V = T$ to be the temperature $T = T(t, \varphi, \psi)$ and the diffusion coefficient equal to the thermal diffusivity $D = a^2 \geq 0$. Then the space of differential 0-forms defined on a 2-dimensional manifold without boundary (here and below in our examples the manifold without boundary will be $E = \mathbb{E}$, the spherical approximation of the Earth's surface with local coordinates φ, ψ (in real life the latitude and longitude of a point on the Earth, respectively) is denoted $H_0 = H_0(\mathbb{E})$). The diffusion equation (4) in the homogeneous case $f \equiv 0$ turns into the linear heat equation [4] on the sphere

$$T_t = a^2(-\Delta_{LB} T). \tag{5}$$

It describes heat (cold) waves on a manifold without boundary from the space H_0. The solution of equation (5) will be a 0-form determining the temperature distribution on the Earth's surface, as in Fig. 1. The color of the points in the figure changes like a rainbow from cold – blue to hot – red and corresponds to the air temperature in the shade in the near-surface layer over the entire Earth's surface (which we approximate by the sphere $\mathbb{E}$).

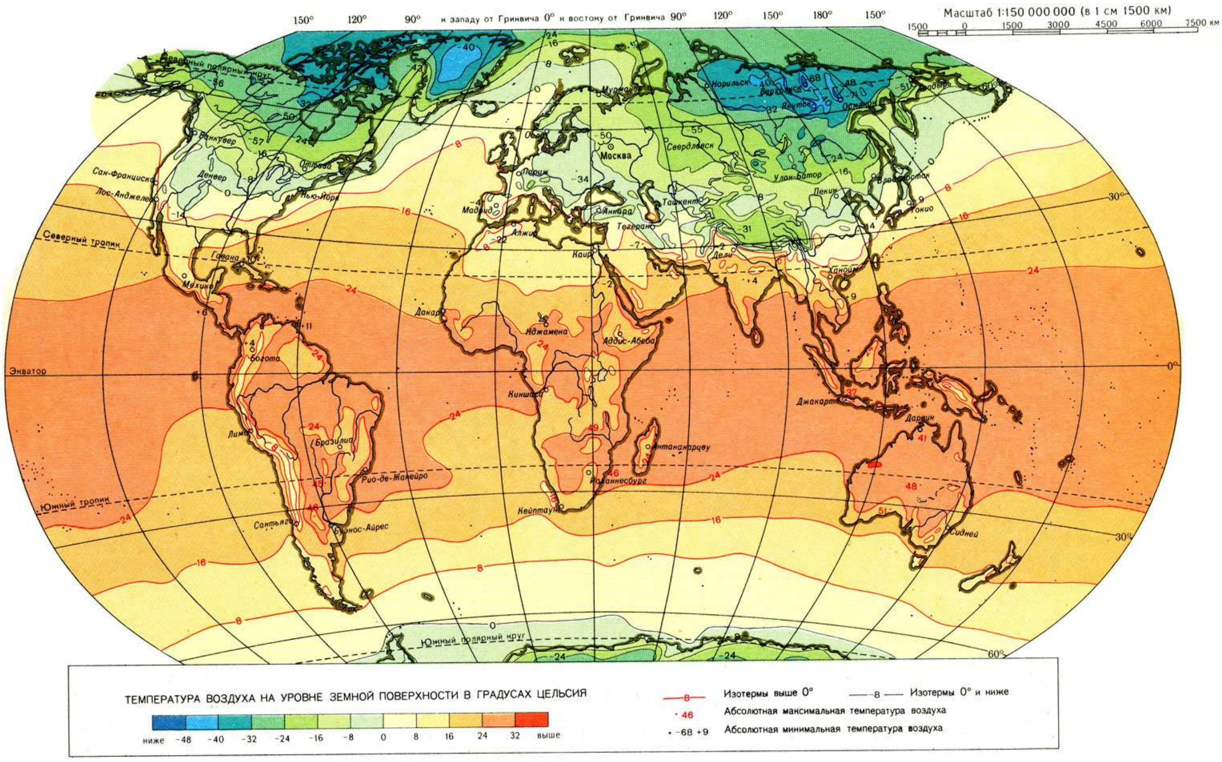

Fig. 1. Graph of the temperature T distribution over the entire Earth's surface $\mathbb{E}$ [8]

2.2. Navier–Stokes System of Equations

Consider a version of the Navier–Stokes system [5] with the convective term dropped, of the form

$$\begin{cases} \dfrac{\partial \overrightarrow{u}}{\partial t} = \nu \Delta \overrightarrow{u} + p, \\ \operatorname{div} \overrightarrow{u} = 0. \end{cases} \tag{6}$$

Here p is the fluid pressure, ν the viscosity coefficient. One of the common methods for solving this system is the method of potentials. It consists in the fact that the resolving vector-function $\overrightarrow{u}$ from equation (6) has the form of a potential, i.e., is the gradient of some function $\overrightarrow{u} = \nabla U$. In the case of differential forms, the divergence operator coincides with δ and the gradient operator ∇ coincides with the differential operator d; this means that the second equation of the system takes the form $\delta dU = 0$. But this is part of the Laplace–Beltrami operator $\Delta_{LB} = d\delta + \delta d$, by which we replace the usual Laplacian in the first equation. Then, using the potential method, the system is replaced by an equation in the space of differential 1-forms u instead of the fluid velocity vector $\overrightarrow{u}$ and the 0-form p coinciding with the pressure function p.

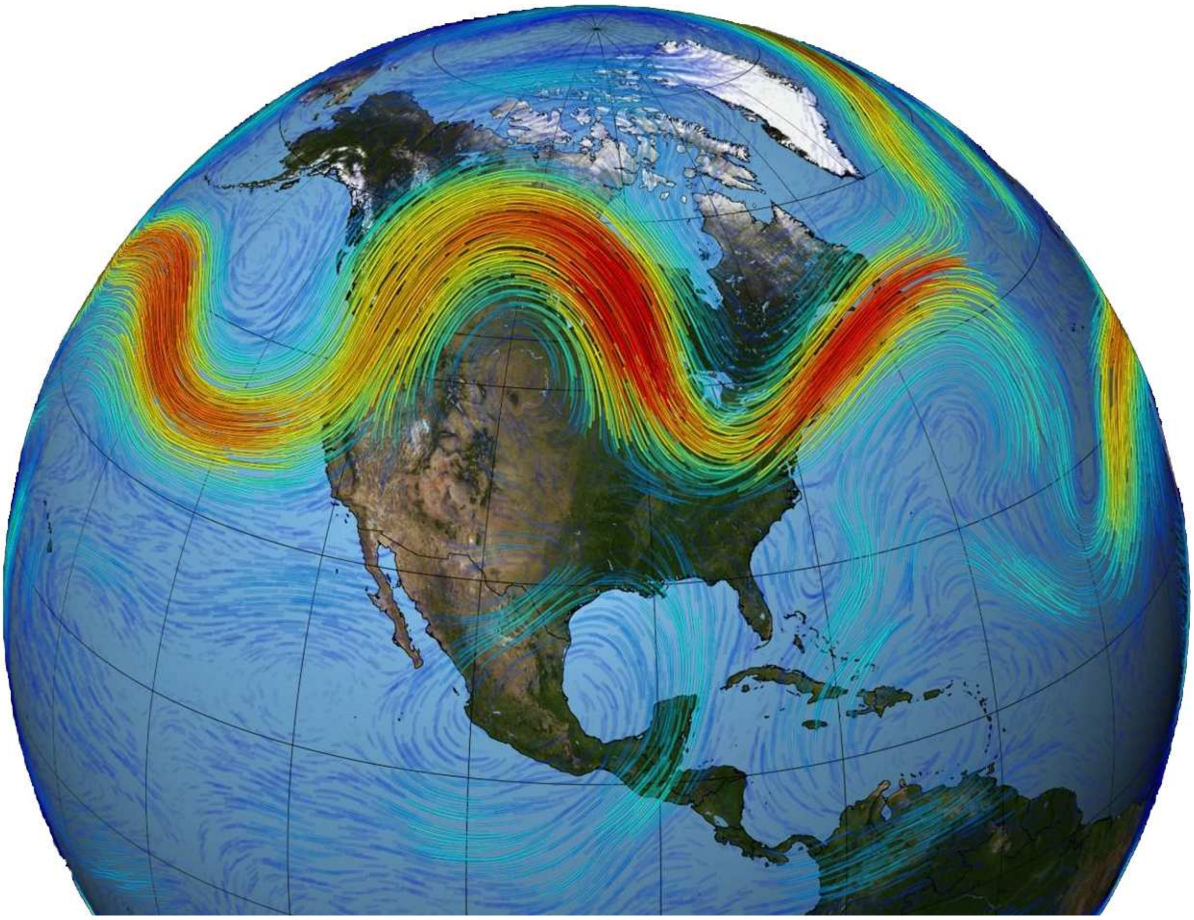

Fig. 2. Graph in which color indicates the gradient of the air flow (wind speed) on the Earth's surface $\mathbb{E}$ [9]

We obtain the equation

$$u_t = \nu\Delta_{LB}u + p \ \Rightarrow \ u_t = \nu(0 + d\delta)u + p, \tag{7}$$

corresponding to the diffusion equation (4) with $V = u$ and $f = p$. It should be noted that in this case the pressure is not given, but is also sought, and the solution of (7) is a pair of differential forms (u, p) of different ranks. For illustration, in the case of the spherical approximation of the Earth's surface $\mathbb{E}$, we apply our imagination, because seas and the liquid they contain form a surface with boundary, while we solve the equation by a method for a surface without boundary. In Fig. 2, the fields of wind speeds are shown (the entire sphere of the air ocean), which in this case is used as an analogy for a viscous fluid modeled by the Navier–Stokes system. In the figure made and processed at NASA, an increase in wind speed corresponds to a transition through the colors of the rainbow from the smallest wind speed shown in blue to the fastest winds in red. That is, if water covered the entire Earth, we could numerically solve the Navier–Stokes system and find the ocean current speeds without the influence of continents.

2.3. Maxwell's System of Equations

We write the first-order differential form

$$A = \tau(t, \varphi, \psi)dt - A_\varphi(\varphi, \psi)d\varphi - A_\psi(\varphi, \psi)d\psi \tag{8}$$

in which only the component τ depends both on time t and on the local coordinates on the sphere (φ, ψ) of fixed radius; the other two terms have coefficients depending only on (φ, ψ). If the current function is the 2-form J, and the electromagnetic field tensor is the 2-form F, then Maxwell's system [6] in invariant form using pseudodifferential operators becomes

$$dF = 0, \qquad \delta F = J. \tag{9}$$

Taking into account the substitution

$$F = \delta A$$

we find that the solution of Maxwell's system F will be a 2-form, and the corresponding homogeneous diffusion equation (9) on the left $(V_t = A(\tau)_t)$, and on the right $V = A(\varphi, \psi)$ for 1-forms of the form (8)

$$A(\tau)_t = \Delta A(\varphi, \psi) + J. \tag{10}$$

Figure 3 shows an image in which colors indicate the intensity of the magnetic field flux through points on the Earth's surface, obtained by the European Space Agency in 2019. The color of the points in the figure changes from points with weak magnetic field flux – blue, to points with strong flux – scarlet. The flux was measured on the sphere $\mathbb{E}$ approximating the Earth's surface.

3. Spectrum of the Laplace–Beltrami Operator and the Connection Between the Solutions of the Equations

If the diffusion equation is solvable, then the coefficients of the differential form of the solution from formula (1) can be represented as a series

$$\chi_{i_1,i_2}(t,\varphi,\psi) = \sum_{k=0}^{\infty}(\lambda_{k1}\varphi_k + \lambda_{k2}\psi_k).$$

where the spectrum of the Laplace–Beltrami operator consists of two parts

$$\{\lambda_k\} = \{\lambda_{k1}\}\bigcup\{\lambda_{k2}\},$$

and φ_k, ψ_k are the eigenfunctions corresponding to the variables φ, ψ. The eigenfunctions for differential q-forms of different ranks q will be different, but the set of eigenvalues of the Laplace–Beltrami operator [7] is the same (ignoring multiplicities).

This means that the amplitude of the terms of the differential forms is determined by the same set of eigenvalues

$$\lambda_k = \frac{k^2}{R^2}$$

of the Laplace–Beltrami operator from the diffusion equation. Moreover, the variable parameter of the global geometry of the sphere, the radius R, is hidden in them.

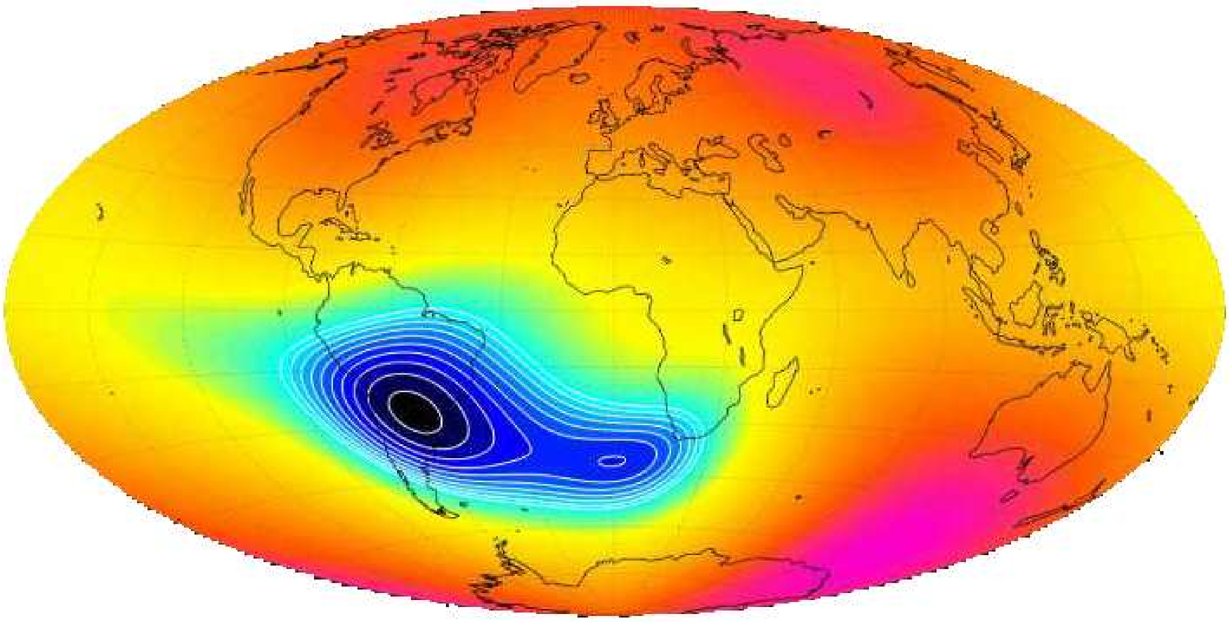

Fig. 3. Graph [10] in which color indicates the intensity of the Earth's magnetic field flux at all points of the Earth's surface, in nanoteslas

Conclusion

Let us summarize the system study of the following three types of waves on a single surface without boundary: heat (cold) propagation waves, fluid (water) waves, and electromagnetic waves passing through this surface. We have found that the amplitude of the coefficients of q-forms of the waves and the parameters of the global geometry of the surface (for a sphere, this is the radius) enter the coefficients of the solution of the invariant equation through the eigenvalues of the Laplace–Beltrami operator on differential forms defined on this Riemannian surface (or more general manifold) without boundary. Corresponding investigations for ellipsoids (in particular the Krasovsky ellipsoid) and for the flat torus are also planned for publication in the near future.

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УРАВНЕНИЕ ДИФФУЗИИ В ПРОСТРАНСТВАХ ДИФФЕРЕНЦИАЛЬНЫХ K-ФОРМ, ЗАДАННЫХ НА РИМАНОВОМ МНОГООБРАЗИИ БЕЗ КРАЯ НА ПРИМЕРЕ СФЕРИЧЕСКОГО ПРИБЛИЖЕНИЯ ЗЕМНОЙ ПОВЕРХНОСТИ

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Статья содержит обоснования нового подхода к системным исследованиям распространения температур, потоков жидкости и электромагнитных волн на многообразиях без края (идет речь о уравнении теплопроводности, системы уравнений Навье – Стокса и системе уравнений Максвелла) с использованием в качестве обобщения инвариантной формы записи уравнения диффузии в пространствах дифференциальных форм разного ранга и псевдодифференциальных операторов на них(один из них оператор Лапласа – Бельтрами). Как пример многообразия без края приводится сферическая поверхность земного шара и ставится вопрос о необходимости проведения подобных исследований и, отмечается, связь между решениями уравнений теплопроводности, системы уравнений Навье – Стокса и системы уравнений Максвелла через спектр оператора Лапласа – Бельтрами на многообразии без края(в нашем случае на сферической поверхности радиуса Земли).

Ключевые слова: дифференциальные формы; риманово многообразие; оператор Лапласа – Бельтрами; псевдодифференциальные операторы.

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