

SOME ESTIMATES OF REGULAR SOLUTIONS TO THE INITIAL–BOUNDARY VALUE PROBLEM FOR THE QUASIHYDRODYNAMIC SYSTEM IN THE SHALLOW WATER APPROXIMATION

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Many important natural processes involving the motion of a fluid with a free surface in a gravitational field and characterized by a negligibly small vertical scale compared to the horizontal scale, for example, modeling tidal waves, flood flows, and tsunamis, are described by the shallow water equations. The article is devoted to studying the question of obtaining a priori estimates in Sobolev spaces for solutions to an initial-boundary value problem for a quasi-hydrodynamic system of equations in the shallow water approximation. These estimates to some extent characterize the behavior of a solution as $t \rightarrow \infty$. We assume that a solution to the problem exists globally in time and is regular, i.e., all generalized derivatives occurring into equations exist and summable to some power. The proofs of a priori estimates rely on the embedding theorem, interpolation properties of Sobolev spaces and, in particular, interpolation inequalities of various types.

Keywords: initial-boundary value problem; quasihydrodynamic system; shallow water approximation; existence; a priori estimate.

Introduction

We consider the following system of quasihydrodynamic equations in the shallow water approximation (see paragraph 3.1 in [1]):

$$\begin{aligned} \frac{\partial h}{\partial t} + \operatorname{div}(h\vec{u}) &= \operatorname{div}(h\vec{w}), \quad \vec{w} = \tau((\vec{u}, \nabla)\vec{u} + g\nabla h), \\ \frac{\partial(h\vec{u})}{\partial t} + \operatorname{div}(h\vec{u} \otimes \vec{u}) + g\nabla \left(\frac{h^2}{2} \right) &= 2\operatorname{div}(\nu h \hat{\sigma}(\vec{u})) + \operatorname{div}(h\vec{w} \otimes \vec{u} + h\vec{u} \otimes \vec{w}), \\ (t, x) &\in Q = (0, T) \times G, \quad G \subset \mathbb{R}^2. \end{aligned} \quad (1)$$

– where the strain rate tensor $\hat{\sigma}$ is of the form:

$$\hat{\sigma}(\vec{u}) = \hat{\sigma} = \frac{1}{2}[(\nabla \otimes \vec{u}) + (\nabla \otimes \vec{u})^T], \quad \sigma_{ij} = \frac{1}{2}(u_{ix_j} + u_{jx_i}),$$

G is a bounded domain with boundary $\Gamma \in C^2$, the fluid kinematic viscosity coefficient ν and the characteristic relaxation time τ are positive constants. Let $S = (0, T) \times \Gamma$. The vector $\vec{u} = (\vec{u}_1(t, x_1, x_2), \vec{u}_2(t, x_1, x_2))$ is the height-averaged flow velocity. The function $h = h(t, x_1, x_2)$ is interpreted as the vertical distance from the flat bottom of the water body, located in the $x_1 \otimes x_2$ plane, to the free surface of the fluid. The system includes

Galileo's constant $g = 9.8 \text{ (m/c}^2\text{)}$, equal to the modulus of the free fall acceleration in the Earth gravitational field. The symbols div and ∇ stand for the divergence and gradient, respectively.

The system (1) is supplemented with initial-boundary conditions

$$\vec{u}|_S = 0, \quad (\vec{w} \cdot \vec{\nu})|_S = 0, \quad \vec{u}|_{t=0} = \vec{u}_0, \quad h|_{t=0} = h_0, \quad (2)$$

where $\vec{\nu}$ is the vector of the outward unit normal to Γ . Note that the second condition in (2) implies that $\frac{\partial h}{\partial n}|_S = 0$.

Many important natural processes involving the motion of a fluid with a free surface in a gravitational field and characterized by a negligibly small vertical scale compared to the horizontal scale, for example, modeling tidal waves, flood flows, and tsunamis, are described by the shallow water equations [2–4]. The system of quasihydrodynamic equations in shallow water theory (1) is an analogue of the Saint-Venant equation [1], which is a system of nonlinear hyperbolic partial differential equations that approximate the complete system of Euler equations [5,6], and the analogue of which in gas dynamics is the quasigasdynamic system of equations derived by Elizarova T.G. and Chetverushkin B.N. [7,8]. The derivation of the Saint-Venant equations and their detailed analysis are presented in [9–11]. It should be noted that a large part of the literature is devoted to the study of the issues of existence and uniqueness of solutions of the Saint-Venant equations [12–16]. In the article [12], the authors investigate the uniqueness of a classical solution to the system of one-dimensional Saint-Venant equations. The local and global existence of classical solutions for dissipative shallow water equations are considered in [13] and [14], respectively. In [15] the Cauchy problem for one-dimensional Saint-Venant equations is studied using a topological approach based on the theory of the sum of two fixed point operators in Banach spaces. The results of the study of the questions of existence and uniqueness of regular solutions in Sobolev classes of initial-boundary value problems for a quasihydrodynamic system of equations in the shallow water approximation in the two-dimensional case are presented in [16]. In the works of Lyapidevskii V.Y., Chesnokov A.A. and others, models are analyzed that reflect the influence of inhomogeneities of the velocity and density fields on the structure of waves in stratified and multiphase flows described by hyperbolic systems of differential and integrodifferential equations in the shallow water approximation [17,18]. It should be noted that the development and elaboration of models of layered fluid motion were significantly influenced by the work of Ovsyannikov L.V., devoted to the study of models of two-layer shallow water [19]. Some results of numerical solution of problems in the shallow water approximation are presented in [20–22]. We can refer also to the articles by Teshukov V.M. [23,24]. Despite the fact that the equations in the shallow water approximation have been studied in considerable depth, it appears that there are no theoretical results addressing the issues of generalized solvability of initial-boundary value problems for regularized Saint-Venant equations. The authors aim to fill this gap in this work. We will consider the system of quasihydrodynamic equations in the shallow water approximation for the two-dimensional case (see paragraph 3.1 in [1]) and obtain a priori estimates that, to some extent, characterize the behavior of a solution in the Sobolev classes of the initial-boundary value problem as $t \rightarrow \infty$.

1. Preliminaries

First, we present some definitions. As usually, the expression (u, v) stands for the inner product in the space $L_2(G)$, i. e., $(u, v) = \int_G u(x)v(x) dx$. We use the same notation (unless otherwise specified) in the case of the space $L_2(G)$ of vector-functions $\vec{u} = (u_1, u_2)$. Sometimes we also use the notation $\vec{u} \cdot \vec{v}$ which denotes the inner product in $\mathbb{R}^2$. Thus, for vectors $\vec{u}, \vec{v}$ we have $(\vec{u}, \vec{v}) = \int_G \vec{u} \cdot \vec{v} dx$. Let E be a Banach space. By $L_p(G; E)$ (G is a domain in $\mathbb{R}^n$), we mean the space of measurable functions defined on G with values in E and the finite norm $\|u(x)\|_E \|_{L_p(G)}$. The notations for Sobolev spaces $W_p^s(G)$, $W_p^s(Q)$, etc. are standard (see [25]). We also use the anisotropic Sobolev spaces $W_p^{s,r}(Q) = L_p(G; W_p^s(0, T)) \cap L_p(0, T; W_p^r(G))$. Definitions of Hölder spaces $C^{\alpha, \beta}(\overline{Q})$, $C^\alpha(\overline{Q})$, etc. can be found, for example, in [26]. We consider all spaces and coefficients of the equation (1) to be real. To simplify the notation, we denote the spaces of vector-functions and scalar functions by the same symbol and the membership $\vec{v} \in W_p^s(G)$ of the vector-function v means that each coordinate of this vector-function belongs to $W_p^s(G)$. By the norm of a vector, we mean the sum of the norms of coordinates, unless otherwise stated.

The definitions validate the equalities

$$\operatorname{div}(h(\vec{u} - \vec{w}) \otimes \vec{u}) = h(\vec{u} - \vec{w}, \nabla) \vec{u} + h \operatorname{div}(\vec{u} - \vec{w}) \vec{u} + \nabla h \cdot (\vec{u} - \vec{w}) \vec{u}, \quad (3)$$

$$\operatorname{div}(h \vec{u} \otimes \vec{w}) = h(\vec{u}, \nabla) \vec{w} + h \operatorname{div} \vec{u} \vec{w} + \nabla h \cdot \vec{u} \vec{w}. \quad (4)$$

Next, we use the representations

$$\operatorname{div}(\hat{\sigma}(\vec{u})) = \Delta \vec{u} + \nabla(\operatorname{div} \vec{u}), \quad (5)$$

$$\operatorname{div}(h \hat{\sigma}(\vec{u})) = h \operatorname{div} \hat{\sigma}(\vec{u}) + \vec{H}, \quad \vec{H} = \left(\sum_{i=1}^2 h_{x_i} \sigma_{i1}, \sum_{i=1}^2 h_{x_i} \sigma_{i2} \right)^T. \quad (6)$$

We assume below that $h_0 \in W_p^{2-2/p}(G)$ ($p > 4$). In this case the embedding theorems [25] yield $h_0 \in C^{2-4/p}(\overline{G})$, put $m_0 = \inf_{x \in G} h_0(x)$, $M_0 = \sup_{x \in G} h_0(x)$.

The proof of existence and uniqueness of a regular solutions to the problem (1), (2) is exhibited in [16]:

Theorem 1. *Let $\vec{u}_0 \in W_p^{2-2/p}(G)$, $\vec{u}_0|_\Gamma = 0$, $h_0 \in W_p^{2-2/p}(G)$ ($p > 4$), $h_0(x) > 0$ in $\overline{Q}$, $\frac{\partial h_0}{\partial n}|_\Gamma = 0$. Then there is a constant $q_0 > 0$ independent of the data $\vec{u}_0, h_0$ such that if*

$$\|\vec{u}_0\|_{L_\infty(G)} \left(M_0 + \frac{M_0}{m_0} + \frac{1}{m_0} \right) + \frac{M_0}{m_0} \|\vec{u}_0\|_{L_\infty(G)}^2 \leq q_0$$

in G , where the constants M_0, m_0 are such that $M_0 \geq h_0(x) \geq m_0 > 0$ for almost all $x \in G$, then on some interval $[0, \gamma_0]$ there exists a solution to the problem (1), (2) such that $\vec{u}, h \in W_p^{1,2}(Q^{\gamma_0})$ and $h(t, x) > 0$ in $\overline{Q^{\gamma_0}}$, where $Q^{\gamma_0} = (0, \gamma_0) \times G$. If $(\vec{u}_1, h_1), (\vec{u}_2, h_2)$ are two solutions of the problem (1), (2) and there is a constant $\delta_0 > 0$ such that $h_i(t, x) \geq \delta_0$ ($i = 1, 2$), then $\vec{u}_1 = \vec{u}_2$, $h_1 = h_2$.

2. Main Results

Theorem 2. *Let the data of the problem $\vec{u}_0, h_0$ satisfy the conditions of Theorem 1, the solution of the problem (1), (2) exists globally in time and is regular, i.e. $\vec{u}, h \in W_p^{1,2}(Q^T)$ for each $T > 0$. If $h \geq 0$ in $Q = (0, \infty) \times G$, then the following equality holds:*

$$\frac{\partial}{\partial t} \int_G \left(\frac{|\vec{u}|^2}{2} h + \frac{g h^2}{2} \right) dx + \frac{1}{\tau} \int_G h |\vec{w}|^2 dx + \int_G \sum_{i,j=1}^2 \sigma_{ij}^2 h dx = 0, t \in (0, \infty),$$

$$\sigma_{ij} = \frac{1}{2} (u_{1x_j} + u_{jx_i})$$

and, respectively, the estimate

$$\max_{t \geq 0} \int_G (|\vec{u}|^2 h + g h^2) dx \leq \int_G (|\vec{u}_0|^2 h_0 + g h_0^2) dx = M.$$

If there are constants $m_0 < M_0$ such that $M_0 \geq h(t, x) \geq m_0 > 0$ almost everywhere, then the following estimate is valid:

$$\begin{aligned} \|\vec{u}\|_{L_2(0, \infty; W_2^1(G))} + \|g \nabla h + (\vec{u}, \nabla) \vec{u}\|_{L_2(Q)} &\leq C_1(M), \\ \|\vec{u}\|_{L_\infty(0, \infty; L_2(G))} + \|h\|_{L_\infty(0, \infty; L_2(G))} &\leq C_2(M), \\ \|(\vec{u}, \nabla) \vec{u}\|_{L_{q_0}(0, \infty; L_{p_0}(G))} &\leq C_3(M), \quad p_0 \in [1, 2), \quad q_0 = 2p_0/(3p_0 - 2), \\ \|\nabla h\|_{L_2(0, \infty; L_1(G))} &\leq C_4(M), \end{aligned} \tag{7}$$

where the constants C_i depend only on the value M , the Lebesgue measure $\mu(G)$ of the domain G , τ and the norms of the initial data.

Proof. We multiply the equations in (1) by $h, \vec{u}$, respectively, and integrate the result over G . From the first equation of the system (1) we have:

$$(gh_t, h) - (h\vec{u}, \nabla h)g + g(h\vec{w}, \nabla h) = 0. \tag{8}$$

Next, we infer

$$(h\vec{u}_t, \vec{u}) + \left(\frac{h_t \vec{u}}{2}, \vec{u} \right) = \frac{\partial}{\partial t} \int_G \frac{|\vec{u}|^2}{2} h dx. \tag{9}$$

From the second equation of the system it follows that

$$\begin{aligned} \frac{\partial}{\partial t} \int_G \frac{|\vec{u}|^2}{2} h dx - \left(h(\vec{w} - \vec{u}), \frac{\nabla |\vec{u}|^2}{2} \right) + \\ + (h(\vec{u}, \nabla) \vec{u}, \vec{w}) + g \left(\vec{u}, \frac{\nabla h^2}{2} \right) - (h(\vec{u} - \vec{w}, \nabla) \vec{u}, \vec{u}) + \int_G \sum \sigma_{ij}^2 h dx = 0. \end{aligned} \tag{10}$$

Denote

$$\left(h(\vec{w} - \vec{u}), \frac{\nabla |\vec{u}|^2}{2} \right) = -(h(\vec{u} - \vec{w}, \nabla) \vec{u}, \vec{u}), \quad (11)$$

and rewrite (10) as follows:

$$\frac{\partial}{\partial t} \int_G \frac{|\vec{u}|^2}{2} h dx + (h(\vec{u}, \nabla) \vec{u}, \vec{w}) + g \left(\vec{u}, \frac{\nabla h^2}{2} \right) + \int_G \sum \sigma_{ij}^2 h dx = 0. \quad (12)$$

Adding (12) to (8), multiplied by g , we obtain that

$$\begin{aligned} \frac{\partial}{\partial t} \int_G \left(\frac{|\vec{u}|^2}{2} h + \frac{g h^2}{2} \right) dx + (h(\vec{u}, \nabla) \vec{u}, \vec{w}) + g \left(\vec{u}, \frac{\nabla h^2}{2} \right) - (h \vec{u}, \nabla h) g + (h \vec{w}, \nabla h) g + \\ + \int_G \sum \sigma_{ij}^2 h dx. \end{aligned} \quad (13)$$

Thus, we arrive at the equality

$$\frac{\partial}{\partial t} \int_G \left(\frac{|\vec{u}|^2}{2} h + \frac{g h^2}{2} \right) dx + \frac{1}{\tau} \int_G h |\vec{w}|^2 dx + \int_G \sum \sigma_{ij}^2 h dx = 0. \quad (14)$$

If there exist constants $m_0 < M_0$ such that $M_0 \geq h(t, x) \geq m_0 > 0$ almost everywhere, then, using Proposition 1.2 §1.1.3 in [27], we obtain the inequality

$$\frac{\partial}{\partial t} \int_G \left(\frac{|\vec{u}|^2}{2} h + \frac{g h^2}{2} \right) dx + \int_G \left(\frac{h}{\tau} |\vec{w}|^2 + m_0 c_0 (|\nabla \vec{u}|^2 + |\vec{u}|^2) \right) \nu dx \leq 0. \quad (15)$$

Integrating this inequality with respect to t , we obtain the estimates

$$\begin{aligned} \max_t \int_G |\vec{u}|^2 h + g h^2 dx + \int_Q \left(\frac{h}{\tau} |\vec{w}|^2 + m_0 c_0 (|\nabla \vec{u}|^2 + |\vec{u}|^2) \right) dx dt \leq \\ \leq 2 \int_G (|\vec{u}_0|^2 h_0 + g h_0^2) dx = M. \end{aligned} \quad (16)$$

As a result, we can conclude that

$$\|\vec{u}\|_{L_2(0, \infty; W_2^1(G))} + \|g \nabla h + (\vec{u}, \nabla) \vec{u}\|_{L_2(Q)} \leq C_1(M), \quad (17)$$

$$\|\vec{u}\|_{L_\infty(0, \infty; L_2(G))} + \|h\|_{L_\infty(0, \infty; L_2(G))} \leq C_1(M), \quad (18)$$

where $C_1(M)$ is some constant depending on M, τ . Proceed with the remaining estimates. The Hölder inequality yields

$$\|(\vec{u}, \nabla) \vec{u}\|_{L_{p_0}(G)} \leq \|\nabla \vec{u}\|_{L_2(G)} \cdot \|\vec{u}\|_{L_{p_0 q}(G)}, \quad (19)$$

where $q = \frac{2}{2-p_0}$. Assume that $p_0 \in [1, 2)$. If $p_0 > 1$, then we use the embedding theorem [25]: $W_2^s(G) \subset L_r(G)$, $r = 2p_0/(2-p_0)$ with $s = 2(p_0-1)/p_0$. The inequality $s < 1$ is equivalent to the inequality $p_0 < 2$. From (19), it follows the estimate

$$\|(\vec{u}, \nabla) \vec{u}\|_{L_{p_0}(G)} \leq C_1 \|\nabla \vec{u}\|_{L_2(G)} \cdot \|\vec{u}\|_{W_2^s(G)}. \quad (20)$$

Now we estimate the last factor using the interpolation inequality:

$$\|\vec{u}\|_{W_2^s(G)} \leq C \|\vec{u}\|_{W_2^1(G)}^s \cdot \|\vec{u}\|_{L_2(G)}^{1-s}. \quad (21)$$

The inequality (20) implies that

$$\|(\vec{u}, \nabla)\vec{u}\|_{L_{p_0}(G)} \leq C \|\vec{u}\|_{W_2^1(G)}^{1+s} \cdot \|\vec{u}\|_{L_2(G)}^{1-s}. \quad (22)$$

In the case of $p_0 = 1$ we have an inequality (i.e. we do not use the embedding theorem)

$$\|(\vec{u}, \nabla)\vec{u}\|_{L_1(G)} \leq C \|\nabla\vec{u}\|_{L_2(G)} \cdot \|\vec{u}\|_{L_2(G)}. \quad (23)$$

The estimates (18), (22), (23) validate the inequality

$$\|(\vec{u}, \nabla)\vec{u}\|_{L_{q_0}(0,\infty;L_{p_0}(G))} \leq C_2 \left(\int_0^\infty \|\vec{u}\|_{W_2^1(G)}^{q_0(1+s)} dt \right)^{1/q_0}, \quad (24)$$

where we choose $q_0(1+s) = 2$, i.e. $q_0 = 2p_0/(3p_0 - 2)$, $p_0 \in [1, 2)$. Thus, we have

$$\|(\vec{u}, \nabla)\vec{u}\|_{L_{q_0}(0,\infty;L_{p_0}(G))} \leq C_3(M), \quad p_0 \in [1, 2), \quad q_0 = 2p_0/(3p_0 - 2). \quad (25)$$

In view of (17), we derive the inequality

$$\|w\|_{L_2(G)} \geq (\mu(G))^{1/2} \|w\|_{L_1(G)} \geq c_0 \|\nabla h\|_{L_1(G)} - c_1 \|(u, \nabla)u\|_{L_1(G)},$$

where the triangle inequality and Hölder's inequality are involved, c_i are positive constants. From here we obtain the inequality

$$\|\nabla h\|_{L_1(G)} \leq c_2 \|w\|_{L_2(G)} + c_3 \|(u, \nabla)u\|_{L_1(G)},$$

which implies the estimate (see (25), (17))

$$\|\nabla h\|_{L_2(0,\infty;L_1(G))} \leq c_2 \|w\|_{L_2(0,\infty;L_2(G))} + c_3 \|(u, \nabla)u\|_{L_2(0,\infty;L_1(G))} \leq C_3(M). \quad (26)$$

Conclusions

The questions of obtaining a priori estimates in Sobolev spaces for solutions to an initial-boundary value problem for a quasi-hydrodynamic system of equations in the shallow water approximation are examined. These estimates to some extent characterize the behavior of a solution as $t \rightarrow \infty$. It is assumed that a solution to the problem exists globally in time and is regular, i.e., all generalized derivatives occurring into equations exist and summable to some power. The proofs of a priori estimates rely on the embedding theorem, interpolation properties of Sobolev spaces and, in particular, interpolation inequalities of various types.

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НЕКОТОРЫЕ ОЦЕНКИ РЕГУЛЯРНЫХ РЕШЕНИЙ НАЧАЛЬНО-КРАЕВОЙ ЗАДАЧИ ДЛЯ КВАЗИГИДРОДИНАМИЧЕСКОЙ СИСТЕМЫ В ПРИБЛИЖЕНИИ МЕЛКОЙ ВОДЫ

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Многие естественные процессы, связанные с описанием движения жидкости со свободной поверхностью в гравитационном поле и характеризующиеся ничтожно малым вертикальным масштабом по сравнению с горизонтальным, например, моделирование приливных волн, наводнений, цунами описываются уравнениями мелкой воды. Статья посвящена получению априорных оценок в пространствах Соболева для решений одной начально-краевой задачи для квазигидродинамической системы уравнений в приближении мелкой воды. Оценки в некоторой степени характеризуют поведение решений при $t \rightarrow \infty$. Мы предполагаем, что решение задачи существует глобально по времени и является регулярным, т.е., все обобщенные производные, входящие в уравнение, существуют и суммируемы с некоторой степенью. Доказательства опираются на теоремы вложения, интерполяционные свойства пространств Соболева, в частности, на интерполяционные неравенства различных типов.

Ключевые слова: начально-краевая задача; квазигидродинамическая система; приближение мелкой воды; существование; априорные оценки.

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