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CALCULATION OF EIGENVALUES OF COUETTE SPECTRAL PROBLEM BY METHOD OF REGULARIZED TRACES

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One can find the eigenvalues of non-selfadjoint operators only by numerical methods. The use of these methods is associated with large computational difficulties. Therefore, the development of a new method for calculating of eigenvalues of non-self-adjoint operators has great theoretical and practical interest. Non iterative method for finding of eigenvalues of perturbed self-adjoint operators is developed on the basis of the theory of regularized traces. This method is called the method of regularized traces. The linear formulas for computing of the eigenvalues of the discrete operators, which are semi-bounded from below, were found. Using them, one can compute the eigenvalues of perturbed self-adjoint operator with any their number. Note that for this computation it does not matter whether the eigenvalues with less number are known or not. Numerical calculations of eigenvalues for the spectral problems, which are generated by the equations of mathematical physics, show that for large numbers of eigenvalues the proposed formulas give more exact result than the Galerkin method. In addition, the obtained formulas allow to compute the eigenvalues of perturbed self-adjoint operator with very large number, such that the use of the Galerkin method becomes difficult. The algorithm of application of the method of regularized traces for finding of eigenvalues of the Couette spectral problem of hydrodynamic stability theory is constructed. This problem studies the stability of the flow of a tough liquid between two rotating axisymmetric cylinders to small perturbations of the basic flow. A feature of this problem is the fact that the differential operator is a matrix one. Numerical experiments have shown the high computational efficiency of the proposed algorithm of computing of the eigenvalues of the studied spectral problem.

The algorithm of application of the method of regularized traces for spectral problems, which are generated by the matrix discrete operators limited from below, is constructed in the paper.

Keywords: eigenvalues and eigenfunctions of operators; corrections of the perturbation theory; discrete operators; self-adjoint operators; the theory of hydrodynamic stability.

Introduction

This article is a continuation of works [1–15], which are associated with the development of non iterative numerical method for computing of eigenvalues of perturbed self-adjoint operators by the method of regularized traces (RT). The basic ideas of the method of regularized traces are the following. To solve the problem of finding of the eigenvalues of operator $T+P$

$$\Big(T+P\Big)u=\mu u, \tag{1}$$

where T – a discrete semi-bounded from below operator, P – limited operator defined in a separable Hilbert space H. Suppose that the eigenvalues $\{\beta_n\}_{n=1}^{\infty}$ of the operator T are known and are numbered in non-decreasing order of their values with regard to algebraic multiplicity, and orthonormal eigenfunctions $\{v_n\}_{n=1}^{\infty}$, corresponding to these eigenvalues. Denote the multiplicity of the eigenvalue β_n by ν_n, and the number of all unequal to each other eigenvalues β_n, which lie inside the circle T_{n_0} of radius $\rho_{n_0} = \dfrac{|\beta_{n_0+1} + \beta_{n_0}|}{2}$ with center in the beginning of coordinate complex plane by n_0. Let $\{\mu_n\}_{n=1}^{\infty}$ be eigenvalues of the operator $T + P$, which are numbered in non-decreasing order of their real parts. If inequalities $q_n = \dfrac{2\|P\|}{|\beta_{n+\nu_n} - \beta_n|} < 1$ are true for all $n \geq n_0$, then the first $m_0 = \sum\limits_{n=1}^{n_0} \nu_n$ of the eigenvalues of $\{\mu_n\}_{n=1}^{m_0}$ of the operator $T + P$ are solutions of the system of m_0 nonlinear equations of the form

$$\sum_{k=1}^{m_0} \mu_k^p = \sum_{k=1}^{m_0} \beta_k^p + \sum_{k=1}^{\infty} \alpha_k^{(p)}(m_0), \;\; p = \overline{1, m_0}. \tag{2}$$

Here $\alpha_k^{(p)}(m_0) = \dfrac{(-1)^k p}{2\pi k i} Sp \int\limits_{T_{n_0}} \beta^{p-1} \Big[P R_\beta(T) \Big]^k d\beta$ are k-tide corrections to the perturbation theory of the operator $T + P$ of integer order p, $R_\beta(T)$ is the resolvent of the operator T.

Numerical reiterating method of regularized traces (MS) is based on the system of equations (2). This method allows to find the eigenvalues of perturbed self-adjoint operators in the case when the self-adjoint operators have eigenvalues of an arbitrary multiplicity.

If eigenfunctions $\{v_n\}_{n=1}^{\infty}$ of the operator T are basis in H, then eigenvalues $\{\mu_n\}_{n=1}^{m_0}$ of the operator $T + P$ are calculated by the formulas [15]:

$$\mu_n = \beta_n + (P v_n, v_n) + \delta(n), \;\; n = \overline{1, m_0}, \tag{3}$$

where $\delta(n)$ satisfy to the estimates $|\delta(n)| \leq (2n-1)\rho_n \dfrac{q^2}{1-q}$.

An algorithm of use of the PC method for finding of the eigenvalues of the spectral Couette problem of the hydrodynamic theory of stability is developed in the paper.

1. Spectral Couette Problem

Consider the problem of hydrodynamic stability of the axisymmetric flow of a viscous incompressible fluid between two rotating concentric cylinders. Let us to introduce a cylindrical coordinate system with axis OZ along axis of cylinders. In the case of rotational symmetry, equation of the first approximation for small perturbations relative with regard to the amplitudes of the stream function $\widehat{\psi}(r)$ and transversal velocity component $\widehat{v}(r)$ are of the form [16]

$$\begin{cases} \dfrac{1}{r} T^2 \widehat{\psi} + 2\lambda R v \widehat{v} = \mu T \widehat{\psi}, \\ \dfrac{1}{r} T \widehat{v} - 2\lambda R A \widehat{\psi} = \mu \widehat{v}. \end{cases} \qquad r_1 < r < r_2, \tag{4}$$

Here $A = \dfrac{\dfrac{\Omega_2}{\Omega_1}\Big(\dfrac{R_2}{R_1}\Big)^2 - 1}{\Big(\dfrac{R_2}{R_1}\Big)^2 - 1}$, $B = -\dfrac{\Big(\dfrac{\Omega_2}{\Omega_1} - 1\Big)\Big(\dfrac{R_2}{R_1}\Big)^2}{\Big(\dfrac{R_2}{R_1}\Big)^2 - 1}$, R_1 and R_2 – radiuses of inner and outer cylinders, Ω_1 and Ω_2 are their angular velocities of rotation, R – the dimensionless

Reynolds number, λ – wave number, $\mu = R\sigma$, σ – spectral parameter of the Couette problem, $T\widehat{v} = -r\dfrac{d^2\widehat{v}}{dr^2} - \dfrac{d\widehat{v}}{dr} + (\dfrac{1}{r} + \lambda^2 r)\widehat{v}$, $v = Ar + \dfrac{B}{r}$, $r_1 = 1$, $r_2 = R_2/R_1$. The boundary conditions for the differential equations system (4) are following:

$$\widehat{v}\Big|_{r=r_1,r_2} = \widehat{\psi}\Big|_{r=r_1,r_2} = \frac{d\widehat{\psi}}{dr}\Big|_{r=r_1,r_2} = 0. \tag{5}$$

If the real part of σ_r of complex numbers $\sigma = \sigma_r + \sigma_i i$ is positive, then due to linear theory, the perturbation is unstable. If $\sigma_r < 0$, then perturbation decays [16].

In separable Hilbert space $L_2^r[r_1, r_2]$ with weight r we introduce the matrix operator

$$\mathbf{G} = \begin{pmatrix} \dfrac{1}{r}T^2 - \mu T & 2\lambda R v \\ -2\lambda R A & \dfrac{1}{r}T - \mu \end{pmatrix}.$$

The domain of definition of $\mathbf{D_G}$ of the matrix operator $\mathbf{G}$ consists of all of the matrix-columns $\mathbf{\Phi}$ of the form $\begin{pmatrix} \widehat{\psi} \\ \widehat{v} \end{pmatrix}$, the elements of which are functions of the class

$$\widehat{\psi}(r) \in C^4(r_1, r_2)\bigcap C^1[r_1, r_2],\ \frac{1}{r}T^2\widehat{\psi} \in L_2^r[r_1, r_2],$$

$$\widehat{v}(r) \in C^2(r_1, r_2)\bigcap C^1[r_1, r_2],\ \frac{1}{r}T\widehat{v} \in L_2^r[r_1, r_2],$$

satisfying the boundary conditions (5), i.e.

$$\mathbf{D_G} = \begin{pmatrix} \Big\{\widehat{\psi} \mid \widehat{\psi} \in C^4(r_1, r_2)\bigcap C^1[r_1, r_2],\ \dfrac{1}{r}T^2\widehat{\psi} \in L_2^r[r_1, r_2], \\ \widehat{\psi}(r)\Big|_{r=r_1,r_2} = \dfrac{d\widehat{\psi}(r)}{dr}\Big|_{r=r_1,r_2} = 0\Big\} \\ \Big\{\widehat{v} \mid \widehat{v} \in C^2(r_1, r_2)\bigcap C^1[r_1, r_2], \dfrac{1}{r}T\widehat{v} \in L_2^r[r_1, r_2], \\ \widehat{v}(r)\Big|_{r=r_1,r_2} = 0\Big\} \end{pmatrix}.$$

Then the system of equations (4) can be written in the form

$$\mathbf{GU} = \mathbf{0}, \quad \mathbf{U} \in \mathbf{D_G}. \tag{6}$$

One can not to apply the method of regularized traces directly to find the eigenvalues of the spectral problem (6). It is so because a matrix differential operator $\mathbf{G}$ cannot be represented as a sum of matrices of self-adjoint and bounded operators. To overcome this difficulty, we build an auxiliary spectral problem, such that the set of its eigenvalues coincides with the set of eigenvalues of the Couette problem (6).

To this end we consider the differential operator T_1. Suppose that

$$T_1 f = -r\frac{d^2 f}{dr^2} - \frac{df}{dr} + \Big(\frac{1}{r} + \lambda^2 r\Big) f$$

with domain of definition

$$D_{T_1} = \Big\{ f \mid f \in C^4(r_1, r_2) \bigcap C^1[r_1, r_2],\ \frac{1}{r} T_1^2 f \in L_2^r[r_1, r_2],\ f(r_1) = f(r_2) = 0 \Big\},$$

and inhomogeneous boundary problem

$$\begin{array}{cc} T_1 f = w(r), & r_1 < r < r_2, \\ f(r_1) = 0, & f(r_2) = 0. \end{array} \tag{7}$$

Theorem 1. *The solution of the boundary problem (7) in the domain D_{T_1} is unique and is expressed by the formula*

$$f(r) = \frac{G_1(\lambda r, \lambda r_1)}{G_1(\lambda r_2, \lambda r_1)} \int_{r_1}^{r_2} \xi, G_1(\lambda r_2, \lambda \xi) w(\xi) d\xi - \int_{r_1}^{r} \xi, G_1(\lambda r, \lambda \xi) w(\xi) d\xi, \qquad r_1 \le r \le r_2,$$

where $G_1(\xi, \eta) = I_1(\xi) K_1(\eta) - I_1(\eta) K_1(\xi)$, $I_1(r)$ – Bessel function of the first order of imaginary argument, $K_1(r)$ – the function of Makdonalda of the first order.

Remark 1. The operator T_1 has an inverse operator T_1^{-1} in D_{T_1}, since the boundary problem (7) has a unique solution in D_{T_1}.

$$T_1^{-1} f = \frac{G_1(\lambda r, \lambda r_1)}{G_1(\lambda r_2, \lambda r_1)} \int_{r_1}^{r_2} \xi, G_1(\lambda r_2, \lambda \xi) f(\xi) d\xi - \int_{r_1}^{r} \xi, G_1(\lambda r, \lambda \xi) f(\xi) d\xi.$$

Lemma 1. *The operator T_1^{-1} on the set D_{T_1} is a "right inverse" of the operator T.*

Substitute

$$\mathbf{U} = \begin{pmatrix} T_1^{-1} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \widehat{f} \\ \widehat{v} \end{pmatrix} = \mathbf{QF}.$$

Then

$$\mathbf{GU} = \begin{pmatrix} \frac{1}{r} T^2 - \mu T & 2\lambda R v \\ -2\lambda R A & \frac{1}{r} T - \mu \end{pmatrix} \begin{pmatrix} T_1^{-1} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \widehat{f} \\ \widehat{v} \end{pmatrix} =$$

$$= \begin{pmatrix} \frac{1}{r} T - \mu & 2\lambda R v \\ -2\lambda R A T_1^{-1} & \frac{1}{r} T - \mu \end{pmatrix} \begin{pmatrix} \widehat{f} \\ \widehat{v} \end{pmatrix} =$$

$$= \left[\begin{pmatrix} \frac{1}{r} T & 0 \\ 0 & \frac{1}{r} T \end{pmatrix} + 2\lambda R \begin{pmatrix} 0 & v \\ -A T_1^{-1} & 0 \end{pmatrix} - \mu \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right] \begin{pmatrix} \widehat{f} \\ \widehat{v} \end{pmatrix} =$$

$$= \Big(\frac{1}{r} \mathbf{T} + \mathbf{P} - \mu \mathbf{E} \Big) \mathbf{F}.$$

Here

$$\mathbf{T} = \begin{pmatrix} T & 0 \\ 0 & T \end{pmatrix}, \ \mathbf{P} = \begin{pmatrix} 0 & 2\lambda R v \\ -2\lambda R A T_1^{-1} & 0 \end{pmatrix},$$

$$\mathbf{Q} = \begin{pmatrix} T_1^{-1} & 0 \\ 0 & 1 \end{pmatrix}, \ \mathbf{E} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

From (6), we obtain

$$\Big(\frac{1}{r}\mathbf{T} + \mathbf{P}\Big)\mathbf{F} = \mu \mathbf{F}, \ \mathbf{F} \in \mathbf{D_T}, \tag{8}$$

where

$$\mathbf{F} = \begin{pmatrix} \widehat{f} \\ \widehat{v} \end{pmatrix}, \ \mathbf{U} = \begin{pmatrix} \widehat{\psi} \\ \widehat{v} \end{pmatrix},$$

$$\mathbf{D_T} = \left(\begin{array}{c} \Big\{\widehat{f} \mid \widehat{f} \in C^2(r_1, r_2) \bigcap C^1[r_1, r_2], \ \frac{1}{r} T\widehat{f} \in L_2^r[r_1, r_2], \\ \frac{d\widehat{f}(r)}{dr}\Big|_{r=r_1,r_2} = 0\Big\} \\ \Big\{\widehat{v} \mid \widehat{v} \in C^2(r_1, r_2) \bigcap C^1[r_1, r_2], \frac{1}{r} T\widehat{v} \in L_2^r[r_1, r_2], \\ \widehat{v}(r)\Big|_{r=r_1,r_2} = 0\Big\} \end{array} \right).$$

The sets of eigenvalues of the spectral problems (6) and (8) coincide, and their eigenfunctions $\mathbf{F}$ and $\mathbf{U}$ are related by the equality

$$\mathbf{U} = \mathbf{QF}.$$

Theorem 2. *Spectral problem*

$$\frac{1}{r}\mathbf{TF}_0 = \beta \mathbf{F}_0, \ \mathbf{F}_0 \in \mathbf{D_T}, \ \mathbf{F}_0 = \begin{pmatrix} \widehat{f}_0 \\ \widehat{v}_0 \end{pmatrix} \tag{9}$$

has set of eigenvalues $\{\beta_n\}_{n=1}^{\infty}$*:*

$$\beta_n \in A_g \bigcup A_q,$$

$$A_g = \{\lambda^2 + g_n^2\}_{n=1}^{\infty}, \ A_q = \{\lambda^2 + q_n^2\}_{n=1}^{\infty}$$

and many eigenfunctions:

$$\mathbf{F}_{0_n} = \left\{ \begin{pmatrix} \chi_{A_g}(\beta_n)\widehat{f}_{0_n}(r) \\ \chi_{A_q}(\beta_n)\widehat{v}_{0_n}(r) \end{pmatrix} \right\}_{n=1}^{\infty}, \ r_1 \le r \le r_2. \tag{10}$$

Here

$$\widehat{f}_{0_n}(r) = C_{2n}\Big[S_1(g_n)Y_1(g_n r) - S_2(g_n)J_1(g_n r)\Big],$$

$$S_1(g) = -\frac{1}{\lambda^2 + g^2}\Big[\lambda r_2 G_2(\beta r_2, \lambda r_2)J_1(gr_2) - g r_1 G_1(\lambda r_2, \lambda r_1)J_0(gr_1) - \beta r_1 G_2(\lambda r_1, \lambda r_2)J_1(gr_1)\Big],$$

$$S_2(g) = -\frac{1}{{}^2 + g^2}\Big[\lambda r_2 G_2(\lambda r_2, \lambda r_2)Y_1(gr_2) - g r_1 G_1(\lambda r_2, \lambda r_1)Y_0(gr_1) - \lambda r_1 G_2(\lambda r_1, \beta r_2)Y_1(gr_1)\Big],$$

g_n *– roots of the equation:*

$$\begin{gathered} \Big[r_2 G_1(\lambda r_2, \lambda r_1) g Y_0(gr_2) - r_2\lambda G_2(\lambda r_2, \lambda r_1)Y_1(gr_2) + r_1\lambda G_2(\lambda r_1, \lambda r_1)Y_1(gr_1)\Big]\times \\ \times\Big[r_2\lambda G_2(\lambda r_2, \lambda r_2)J_1(gr_2) - r_1 G_1(\lambda r_2, \lambda r_1) g J_0(gr_1) - r_1\lambda G_2(\lambda r_1, \lambda r_2)J_1(gr_1)\Big]- \\ -\Big[r_2\lambda G_2(\lambda r_2, \lambda r_2)Y_1(gr_2) - r_1 G_1(\lambda r_2, r_1) g Y_0(gr_1) - r_1\lambda G_2(r_1, \lambda r_2)Y_1(gr_1)\Big]\times \\ \times\Big[r_2 G_1(\lambda r_2, \lambda r_1) g J_0(gr_2) - r_2\lambda G_2(\lambda r_2, \lambda r_1)J_1(gr_2) + r_1\lambda G_2(\lambda r_1, \lambda r_1)J_1(gr_1)\Big] = 0, \end{gathered}$$

$\widehat{v}_{0_n}(r) = C_{4n}\Big[J_1(q_n r_1)Y_1(q_n r) - Y_1(q_n r_1)J_1(q_n r)\Big]$, q_n *– roots of the equation:*

$$J_1(qr_1)Y_1(qr_2) - J_1(qr_2)Y_1(qr_1) = 0,$$

χ_{A_η} *u* χ_{A_ζ} *– characteristic functions of sets* A_η *and* A_ζ *respectively,* $G_1(\xi,\eta) = I_1(\xi)K_1(\eta) - I_1(\eta)K_1(\xi)$, $G_2(\xi,\eta) = I_1(\xi)K_1(\eta) + I_1(\eta)K_1(\xi)$.

Theorem 3. *A set of eigenvalues* $\{\beta_n\}_{n=1}^{\infty}$ *of the problem (9) has no finite limit points, and all eigenvalues are real, non-negative and simple.*

Theorem 4. *Eigenfunctions (10) of the spectral problem (9) corresponding to different eigenvalues are orthogonal in* $L_2^r[r_1, r_2]$.

Theorem 5. *An operator* $\mathbf{P}$ *is restricted in* $L_2^r[r_1, r_2]$ *and*

$$||\mathbf{P}|| \le 2\lambda R \max_{r_1 \le r \le r_2}\Big(|v|, \frac{|A|}{\lambda^2}\Big).$$

Theorem 6. *An operator* $\dfrac{\mathbf{1}}{\mathbf{r}}\mathbf{T}$ *with the domain of definition* $\mathbf{D_T}$ *is a discrete semi-bounded from below in* $L_2^r[r_1, r_2]$.

Scalar product $\Big(\mathbf{PF}_{0_n}, \mathbf{F}_{0_m}\Big)_r$ for any $n, m \in N$ is computed by the formulas:

$$V_{nm} = \Big(\mathbf{PF}_{0_n}, \mathbf{F}_{0_m}\Big)_r = \int_{r_1}^{r_2} r\Big[\mathbf{PF}_{0_n}\Big]^T \mathbf{F}_{0_m} dr =$$

$$= 2\lambda R\Big[\chi_{A_g}(\beta_n)\chi_{A_q}(\beta_m)\int_{r_1}^{r_2} rv(r)\widehat{v}_{0_n}(r)\widehat{f}_{0_m}(r)dr - \qquad (11)$$

$$-\chi_{A_q}(\beta_n)\chi_{A_g}(\beta_m)\int_{r_1}^{r_2} rv(r)\widehat{v}_{0_m}(r)T_1^{-1}\Big(\widehat{f}_{0_n}(r)\Big)dr\Big].$$

To compute the eigenvalues of the spectral problem (8) and, therefore, the Couette problem (4), (5) by the method of regularized traces, we propose the following algorithm:

1. Using theorem 1, to find the eigenvalues $\{\beta_k\}_{k=1}^{k_0}$ of spectrum problem (9). To enumerate founded eigenvalues of β_k in descending order of value.

2. To find the orthogonal system of eigenfunctions $\{\mathbf{F}_0\}_{k=1}^{k_0}$ of the boundary problem (9) and to normalize it.

3. To find n_0 from inequalities

$$\frac{2||\mathbf{P}||}{|\beta_{n+1} + \beta_n|} < 1$$

such that they are true for all $n \le n_0 \le k_0$.

4. To calculate the scalar product of V_{nn} by the formulas (11) for all $n \le n_0$.

5. To calculate approximate eigenvalues $\{\widetilde{\mu}_n\}_{n=1}^{n_0}$ of the boundary problem (9) using equations (2) for all $n \le n_0$.

6. Using $\mu = R\sigma$, to find approximate eigenvalues $\{\widetilde{\sigma}_n\}_{n=1}^{n_0}$ of the spectral problem (4), (5).

The results of the calculation of the first eigenvalues $\widetilde{\sigma}_n$ of the Couette problem (4), (5), which are obtained by the method of RS, were compared with calculation of them by the method of Galerkin. In all cases, the results coincide with good veracity.

Table shows the first four eigenvalues of the Couette problem for the case $\lambda = 10$, $\dfrac{R_2}{R_1} = 3$, $\dfrac{\Omega_2}{\Omega_1} = 0,5$.

Table

The results of the calculation of approximate eigenvalues $\widetilde{\sigma}_j$ of the spectral problem (4), (5) by method of RS

j	$\widetilde{\sigma}_j$	$\widetilde{\sigma}_j$
	$R = 10000$	$R = 100000$
1	$-0,010707 - 1,446012i$	$-0,001071 - 1,446017i$
2	$-0,010707 + 1,446012i$	$-0,010707 + 1,446017i$
3	$-0,011396 - 1,344948i$	$-0,001139 - 1,344955i$
4	$-0,011396 + 1,344948i$	$-0,001139 + 1,344955i$

Numerical experiments have shown the high computational efficiency of the constructed algorithm for computing of the eigenvalues of the studied spectral problem.

Conclusion

The application of the method of regularized traces for computing of the eigenvalues of the spectral problem and the Couette problem is developed. To this end the auxiliary boundary problem (9) is constructed. This problem is generated by the matrix perturbed self-adjoint operator, such that its eigenvalues set coincides with the same set of Couette problem. Theorems that allow to justify the legitimacy of application of the RT method for finding of the eigenvalues of the constructed problem are obtained. The algorithm of application of method for the RT spectral problems generated by the discrete matrix operators limited from below is constructed.

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ВЫЧИСЛЕНИЕ СОБСТВЕННЫХ ЗНАЧЕНИЙ СПЕКТРАЛЬНОЙ ЗАДАЧИ КУЭТТА МЕТОДОМ РЕГУЛЯРИЗОВАННЫХ СЛЕДОВ

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Собственные числа несамосопряженных операторов можно найти только численными методами, применение которых связано с большими вычислительными трудностями. Поэтому разработка нового метода вычисления собственных значений несамосопряженных операторов представляет большой теоретический и практический интерес. На основе теории регуляризованных следов разработан неитерационный метод нахождения собственных значений возмущенных самосопряженных операторов, который был назван методом регуляризованных следов. Найдены линейные формулы для вычисления собственных значений дискретных полуограниченных снизу операторов. Используя их, можно вычислять собственные значения возмущенного самосопряженного оператора с любым их номером, независимо от того, известны ли собственные значения с предыдущими номерами или нет. Численные расчеты собственных значений для спектральных задач, порожденные уравнениями математической физики, показывают, что предлагаемые формулы при больших номерах собственных значений дают результат точнее, чем метод Галеркина. Кроме того, по найденным формулам можно вычислять собственные значения возмущенного самосопряженного оператора с очень большими номерами, когда применение метода Галеркина становится затруднительным. Разработан алгоритм применения метода регуляризованных следов для нахождения собственных значений спектральной задачи Куэтта гидродинамической теории устойчивости, которая исследует устойчивость течения вязкой жидкости между двумя вращающимися осесимметричными цилиндрами к малым возмущениям основного течения. Особеностью задачи является тот факт, что дифференциальный оператор является матричным. Проведенные численные эксперименты показали высокую вычислительную эффективность разработанного алгоритма вычисления собственных значений исследуемой спектральной задачи.

В работе построен алгоритм применения метода регуляризованных следов к спектральным задачам, порожденным матричными дискретными ограниченными снизу операторами.

Ключевые слова: собственные значения и собственные функции операторов; поправки теории возмущений; дискретные операторы; самосопряженные операторы; гидродинамическая теории устойчивости.

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