

DIFFERENTIAL OPERATORS ON THE PROJECTIVE PLANE

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A formula for the addition of even spherical harmonics was obtained in [1]. This formula became the starting point for the derivation of the first regularized trace for the Laplace – Beltrami operator with potential on the projective plane. Due to this formula, it is not need to find asymptotic formulas for the associated Legendre polynomials by the three parameters, which was an impossible problem for a long time. The obtained results are the basis for the calculation of the perturbation theory corrections and allow the following application for the formulas for regularized traces of elliptic differential operators. In this paper we consider the problem of summation of divergent series class. We propose a method for calculating of the corrections of perturbation theory for a differential operator with potential on the real projective plane. The method is applicable, in particular, to take sum of series with factorial growth of its members.

Keywords: operator Laplace – Bochner; projective plane; Hilbert space; Legendre polynomial; Lipschitz condition.

1. Addition Theorem

Consider spherical function of order n [9]:

$$P_n(\cos \alpha) = 2 \sum_{l=0}^n \frac{(n-l)!}{\delta_l(n+l)!} P_n^{(l)}(\cos \theta') P_n^{(l)}(\cos \theta) \cos l(\varphi - \varphi'),$$

where α is an angle between the radius vectors in spherical coordinates (θ, φ) and (θ', φ') ;

$$\cos \alpha = \sin \theta \sin \theta' \cos(\varphi - \varphi') + \cos \theta \cos \theta';$$

$\delta_l = 2$ at $l = 0$ and $\delta_l = 1$ at $l > 0$;

$$P_n^{(l)}(x) = (1-x^2)^{l/2} \frac{d^l P_n(x)}{dx^l} = \frac{(1-x^2)^{l/2}}{2^n n!} \frac{d^{n+l}}{dx^{n+l}} [(x^2-1)^n],$$

Legendre polynomial

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} [(x^2-1)^n], \quad -1 < x < 1.$$

Using the addition theorem for Legendre polynomials [11, 13], we write even spherical functions. We assign the negative superscript to the functions which contain $\cos(l\varphi)$, and the positive superscript to $\sin(l\varphi)$.

$$\begin{aligned} Y_n^{(-l)}(\theta, \varphi) &= P_n^{(l)}(\cos \theta) \cos(l\varphi), & n \equiv 0 \pmod{2}, \\ Y_n^{(l)}(\theta, \varphi) &= P_n^{(l)}(\cos \theta) \sin(l\varphi), & n \equiv 1 \pmod{2}, \quad l \leq n. \end{aligned}$$

Theorem 1. *In*

$$v_{nl}(\theta, \varphi) = \sqrt{\frac{2(n-l)!}{\delta_l(n+l)!}} \cos(l\varphi) P_n^{(l)}(\cos \theta),$$

$$v'_{nl}(\theta', \varphi') = \sqrt{\frac{2(n-l)!}{\delta_l(n+l)!}} \cos(l\varphi') P_n^{(l)}(\cos \theta'), \quad l \leq n$$

are spherical harmonics, then for the even spherical functions we can use the following formulas

$$\sum_{l \equiv 1 \pmod{2}}^n v_{nl} v'_{nl} = L_1(P_n(\cos \alpha)),$$

$$\sum_{l \equiv 0 \pmod{2}}^n v_{nl} v'_{nl} = L_2(P_n(\cos \alpha)),$$

where

$$L_1 \equiv \sum_{m=0}^n \left| \sin \frac{\pi m}{2} \right| \prod_{k=0, k \neq m}^n \frac{-\frac{d^2}{d\varphi^2} - k^2}{m^2 - k^2},$$

$$L_2 \equiv \sum_{m=0}^n \left| \cos \frac{\pi m}{2} \right| \prod_{k=0, k \neq m}^n \frac{-\frac{d^2}{d\varphi^2} - k^2}{m^2 - k^2}.$$

Proof. In order to prove the theorem, we consider the following relations [12, 14]

$$\begin{aligned} L_1(\cos(l\varphi)) &= 0, & l &\equiv 0 \pmod{2}, \\ L_1(\cos(l\varphi)) &= \cos(l\varphi), & l &\equiv 1 \pmod{2}, \\ L_1(\sin(l\varphi)) &= 0, & l &\equiv 0 \pmod{2}, \\ L_1(\sin(l\varphi)) &= \sin(l\varphi), & l &\equiv 1 \pmod{2}, \\ \\ L_2(\cos(l\varphi)) &= 0, & l &\equiv 1 \pmod{2}, \\ L_2(\cos(l\varphi)) &= \cos(l\varphi), & l &\equiv 0 \pmod{2}, \\ L_2(\sin(l\varphi)) &= 0, & l &\equiv 1 \pmod{2}, \\ L_2(\sin(l\varphi)) &= \sin(l\varphi), & l &\equiv 0 \pmod{2}, \end{aligned}$$

which follow from the sequence of parities

$$\begin{aligned} L_1(\cos(l\varphi)) &= \sum_{m=0}^n \left| \sin \frac{\pi m}{2} \right| \prod_{k=0, k \neq m}^n \frac{l^2 - k^2}{m^2 - k^2} \cos(l\varphi) = \\ &= \sum_{m \equiv 1 \pmod{2}}^n \prod_{k=0, k \neq m}^n \frac{l^2 - k^2}{m^2 - k^2} \cos(l\varphi) = \\ &= \begin{cases} 0, & l \equiv 0 \pmod{2}, \\ \cos(l\varphi) \prod_{k=0}^n \frac{l^2 - k^2}{l^2 - k^2}, & l \equiv 1 \pmod{2} \end{cases} = \begin{cases} 0, & l \equiv 0 \pmod{2}, \\ \cos(l\varphi), & l \equiv 1 \pmod{2}. \end{cases} \\ \\ L_1(\sin(l\varphi)) &= \sum_{m=0}^n \left| \sin \frac{\pi m}{2} \right| \prod_{k=0, k \neq m}^n \frac{l^2 - k^2}{m^2 - k^2} \sin(l\varphi) = \end{aligned}$$

$$\begin{aligned}
 &= \sum_{m \equiv 1 \pmod{2}}^n \prod_{k=0, k \neq m}^n \frac{l^2 - k^2}{m^2 - k^2} \sin(l\varphi) = \\
 &= \begin{cases} 0, & l \equiv 0 \pmod{2}, \\ \sin(l\varphi) \prod_{k=0}^n \frac{l^2 - k^2}{l^2 - k^2}, & l \equiv 1 \pmod{2} \end{cases} = \begin{cases} 0, & l \equiv 0 \pmod{2}, \\ \sin(l\varphi), & l \equiv 1 \pmod{2}. \end{cases} \\
 L_2(\cos(l\varphi)) &= \sum_{m=0}^n \left| \cos \frac{\pi m}{2} \right| \prod_{k=0, k \neq m}^n \frac{l^2 - k^2}{m^2 - k^2} \cos(l\varphi) = \\
 &= \sum_{m \equiv 0 \pmod{2}}^n \prod_{k=0, k \neq m}^n \frac{l^2 - k^2}{m^2 - k^2} \cos(l\varphi) = \\
 &= \begin{cases} 0, & l \equiv 1 \pmod{2}, \\ \cos(l\varphi) \prod_{k=0}^n \frac{l^2 - k^2}{l^2 - k^2}, & l \equiv 0 \pmod{2} \end{cases} = \begin{cases} 0, & l \equiv 1 \pmod{2}, \\ \cos(l\varphi), & l \equiv 0 \pmod{2}, \end{cases} \\
 L_2(\sin l\varphi) &= \sum_{m=0}^n \left| \sin \frac{\pi m}{2} \right| \prod_{k=0, k \neq m}^n \frac{l^2 - k^2}{m^2 - k^2} \sin(l\varphi) = \\
 &= \sum_{m \equiv 0 \pmod{2}}^n \prod_{k=0, k \neq m}^n \frac{l^2 - k^2}{m^2 - k^2} \sin(l\varphi) = \\
 &= \begin{cases} 0, & l \equiv 1 \pmod{2}, \\ \sin(l\varphi) \prod_{k=0}^n \frac{l^2 - k^2}{l^2 - k^2}, & l \equiv 0 \pmod{2} \end{cases} = \begin{cases} 0, & l \equiv 1 \pmod{2}, \\ \sin(l\varphi), & l \equiv 0 \pmod{2}. \end{cases}
 \end{aligned}$$

Then

$$\begin{aligned}
 \sum_{l=0}^n v_{nl} v'_{nl} &= 2 \sum_{l=0}^n \frac{(n-l)!}{\delta_l(n+l)!} P_n^{(l)}(\cos \theta) P_n^{(l)}(\cos \theta') \cos(l\varphi) \cos(l\varphi') = \\
 &= \sum_{l=0}^n \frac{2(n-l)!}{\delta_l(n+l)!} P_n^{(l)}(\cos \theta) P_n^{(l)}(\cos \theta') [\cos l(\varphi + \varphi') + \cos l(\varphi - \varphi')].
 \end{aligned}$$

After replacing variable $\varphi' \rightarrow 2\pi - \varphi'$, we get that [7]

$$\cos l(\varphi + \varphi') = \cos l(\varphi + 2\pi - \varphi') = \cos l(\varphi - \varphi').$$

Therefore

$$\sum_{l=0}^n v_{nl} v'_{nl} = 2 \sum_{l=0}^n \frac{2(n-l)!}{\delta_l(n+l)!} P_n^{(l)}(\cos \theta) P_n^{(l)}(\cos \theta') \cos l(\varphi - \varphi').$$

After using the theorem [2], we get that

$$\sum_{l=0}^n v_{nl} v'_{nl} = P_n(\cos \alpha),$$

it is obvious that

$$L_1 \left(\sum_{l=0}^n v_{nl} v'_{nl} \right) = \sum_{l \equiv 1 \pmod{2}}^n v_{nl} v'_{nl},$$

$$L_2 \left(\sum_{l=0}^n v_{nl} v'_{nl} \right) = \sum_{l \equiv 0 \pmod{2}}^n v_{nl} v'_{nl}.$$

Therefore [6, 8]

$$\sum_{l \equiv 1 \pmod{2}}^n v_{nl} v'_{nl} = L_1(P_n(\cos \alpha)),$$

$$\sum_{l \equiv 0 \pmod{2}}^n v_{nl} v'_{nl} = L_2(P_n(\cos \alpha)).$$

□

2. Analysis of the Perturbation Theory Corrections

Let T be Laplace – Bochner with potential on the projective plane F , acting on a Hilbert space H of functions square-integrable with respect to Haar: $\sin \theta d\theta d\varphi$ (θ, φ are spherical coordinates), $\lambda_n = n(n+1)$ ($n = \overline{0, \infty}$) – the eigenvalues of the operator T , $Vn = 2n+1$ – the multiplicity of λ_n ; Vn, i ($i = \overline{0, 2n}$) – eigenfunctions T forming an orthonormal system of spherical functions. Assume also that $l_n = \{\lambda | \lambda = \lambda_n + n + 1 + i\rho, -\infty < \rho < \infty\}$ – vertical lines in the complex plane. We denote eigenvalues of $T+P$ by $\mu_{n,i}$, combined with the algebraic multiplicity such that

$$|\mu_{n,i} - n(n+1)| \leq \text{const.}$$

It is clear that

$$\sum_{i=0}^{2n} \mu_{n,i} = n(n+1)(2n+1) + \sum_{i=0}^{2n} (Pv_{ni}, v_{ni}) + \alpha_n(p) + \beta_n(p) + O\left(\frac{1}{n^2}\right),$$

where $\alpha_n(p)$ is the second amendment to the amount of perturbation theory $\sum_{i=0}^{2n} \mu_{n,i}$, $\beta_n(p)$ – the third amendment of the perturbation theory. The first amendment is a constant [10]

$$\sum_{i=0}^{2n} (Pv_{ni}, v_{ni}) = \frac{2m+1}{4\pi} \iint_F p(\theta, \varphi) \sin \theta d\varphi d\theta = \text{const.}$$

The second amendment of the perturbation theory has the form

$$\alpha_n(p) = -\frac{1}{2\pi i} Sp \left\{ \left[\int_{l_n} - \int_{l_{n-1}} \right] \lambda [(T - \lambda E)^{-1} P^2] (T - \lambda E)^{-1} d\lambda \right\} = \sum_{k=1, k \neq n}^{\infty} \frac{\alpha_{k,n}}{|\lambda_k - \lambda_n|},$$

$$\alpha_{k,n} = \sum_{i=0}^{2k} \sum_{j=0}^{2n} (Pv_{ki}, v'_{nj})(Pv_{nj}, v'_{ki}).$$

In addition theorem for even spherical harmonics [3]

$$\begin{aligned}
 \alpha_{k,n} &= \sum_{i=0}^{2k} \sum_{j=0}^{2n} (P_{v_{ki}}, v'_{n_j})(P_{v_{nj}}, v'_{ki}) = \\
 &= \frac{(2k+1)(2n+1)}{4\pi^2} \sum_{i=0}^{2k} \sum_{j=0}^{2n} \int_F \int_F \int_F p(\theta, \varphi) v_{ki}(\theta, \varphi) v_{nj}(\theta, \varphi) \sin(\theta) d\theta d\varphi \times \\
 &\quad \times p(\theta', \varphi') v'_{n_j}(\theta', \varphi') v'_{ki}(\theta', \varphi') \sin(\theta') d\theta' d\varphi' = \\
 &= \frac{(2k+1)(2n+1)}{4\pi^2} \int_F \int_F \int_F p(\theta, \varphi) p(\theta', \varphi') \sin(\theta) \sin(\theta') \times \\
 &\quad \times \sum_{i=0}^{2k} v_{ki}(\theta, \varphi) v'_{ki}(\theta', \varphi') \sum_{i=0}^{2n} v_{nj}(\theta, \varphi) v'_{nj}(\theta', \varphi') d\theta d\varphi d\theta' d\varphi' = \\
 &= \frac{(2k+1)(2n+1)}{4\pi^2} \int_F \int_F \int_F p(\theta, \varphi) p(\theta', \varphi') \sin(\theta) \sin(\theta') \times \\
 &\quad \times [L_1(P_{2k}(\cos \alpha)) + L_2(P_{2k}(\cos \alpha))] \times \\
 &\quad \times [L_1(P_{2n}(\cos \alpha)) + L_2(P_{2n}(\cos \alpha))] d\theta d\varphi d\theta' d\varphi' = \\
 &= \frac{(2k+1)(2n+1)}{4\pi^2} \int_F \int_F \int_F p(\theta, \varphi) p(\theta', \varphi') \sin(\theta) \sin(\theta') \times \\
 &\quad \times \sum_{i=1}^2 L_i(P_{2k}(\cos \alpha)) \sum_{i=1}^2 L_i(P_{2n}(\cos \alpha)) d\theta d\varphi d\theta' d\varphi',
 \end{aligned}$$

where

$$\cos \alpha = \cos \theta \cos \theta' + \sin \theta \sin \theta' \cos(\varphi - \varphi').$$

Legendre polynomials P_{2k} , P_{2n} normalized by the condition $P_{2k}(1) = P_{2n}(1) = 1$. Define the function

$$f(\alpha) = \int_F \int_F p(\theta, \varphi) \sin \theta d\theta d\varphi \left[\int_{T(\alpha)} p(\theta', \varphi') \sin \theta' \frac{\phi(\theta, \theta', \alpha) d\theta'}{\sin \alpha} \right],$$

where $T(\alpha)$ is an intersection of the cone with apex at the center of the sphere central angle, $2\alpha (0 \leq \alpha \leq \pi)$ and an axis is defined by spherical coordinates θ, φ , with a sphere in spherical coordinates θ', φ' ;

$$\phi(\theta, \theta', \alpha) = \pm \frac{\sin \alpha}{\sqrt{(-\cos \alpha + \cos(\theta - \theta'))(\cos \alpha - \cos(\theta + \theta'))}};$$

$$L_j = L_j \left(\frac{d}{d\theta} \right), L'_j = L_j \left(\frac{d}{d\theta'} \right), (j = 1, 2).$$

By hypothesis, the function f is Lipschitz. Therefore the function is also Lipschitz:

$$\begin{aligned}
 |f(\alpha) - f(\beta)| &= \int \int_F p(\theta, \varphi) L_j \sin \theta d\theta d\varphi \times \\
 &\times \left[\int \int_{T(\alpha)} p(\theta', \varphi') L'_j \sin \theta' \phi(\theta, \theta', \alpha) d\theta' - \int \int_{T(\alpha)} p(\theta', \varphi') L'_j \sin \theta' \phi(\theta, \theta', \alpha) d\theta' \right] \leq \\
 &\leq \text{const} |\alpha - \beta|.
 \end{aligned}$$

Hence the function f is absolutely continuous. Therefore at each point $[0, \pi]$ it has a finite derivative $f'(\alpha)$, which is an integrable function.

Based on the foregoing, the second correction of perturbation theory has the form ($\varepsilon > 0$ will be chosen later)

$$\begin{aligned}
 \alpha_n(p) &= -\frac{(2n+1)}{4\pi^2} \sum_{k=1, k \neq n}^{\infty} \frac{(2k+1)}{|\lambda_k - \lambda_n|} \int_0^{\pi} f(\alpha) P_{2k}(\cos \alpha) P_{2n}(\cos \alpha) \sin \alpha d\alpha = \\
 &= -\frac{(2n+1)}{4\pi^2} \sum_{k=1, k \neq n}^{\infty} \frac{(2k+1)}{|\lambda_k - \lambda_n|} \left\{ \left[\int_0^{\varepsilon} + \int_{\pi-\varepsilon}^{\pi} \right] f(\alpha) P_{2k}(\cos \alpha) P_{2n}(\cos \alpha) \sin \alpha d\alpha + \right. \\
 &\quad \left. + \int_{\varepsilon}^{\pi-\varepsilon} f(\alpha) P_{2k}(\cos \alpha) P_{2n}(\cos \alpha) \sin \alpha d\alpha \right\}.
 \end{aligned}$$

For Legendre polynomials known [1] asymptotic expansion Stieltjes uniform remainder estimate

$$\begin{aligned}
 P_k(\cos \alpha) &= \frac{\cos\{(2k+1/2)\alpha - \pi/4\}}{(\sin \alpha)^{1/2}} \left[\frac{(2/\pi)^{1/2}}{2k^{1/2}} + \frac{O(1)}{2k^{3/2}} \right] + \\
 &+ \frac{\sin\{(2k+3/2)\alpha - \pi/4\}}{(\sin \alpha)^{3/2}} \frac{O(1)}{2k^{3/2}} + \frac{O(1)}{(\sin \alpha)^{5/2} 2k^{5/2}}.
 \end{aligned}$$

Using this asymptotic expansion, we transform $\alpha_n(p)$ as follows:

$$\begin{aligned}
 \alpha_n(p) &= -\frac{(2n+1)}{4\pi^2} \sum_{k=1, k \neq n}^{\infty} \frac{(2k+1)}{|\lambda_k - \lambda_n|} \left\{ \left[\int_0^{\varepsilon} + \int_{\pi-\varepsilon}^{\pi} \right] f(\alpha) \sin \alpha P_{2k}(\cos \alpha) P_{2n}(\cos \alpha) d\alpha + \right. \\
 &+ \int_{\varepsilon}^{\pi-\varepsilon} f(\alpha) \cos\{(2k+1/2)\alpha - \pi/4\} \cos\{(2n+1/2)\alpha - \pi/4\} \times \\
 &\quad \times \left\{ \frac{2/\pi}{k^{1/2} n^{1/2}} + \frac{O(1)}{k^{3/2} n^{1/2}} + \frac{O(1)}{k^{1/2} n^{3/2}} + \frac{O(1)}{k^{3/2} n^{3/2}} \right\} d\alpha + \\
 &+ \int_{\varepsilon}^{\pi-\varepsilon} \frac{f(\alpha)}{\sin \alpha} \cos\{(2k+1/2)\alpha - \pi/4\} \sin\{(2n+3/2)\alpha - \pi/4\} \times
 \end{aligned}$$

$$\begin{aligned}
 & \times \left\{ \frac{O(1)}{k^{1/2}n^{3/2}} + \frac{O(1)}{k^{3/2}n^{3/2}} \right\} d\alpha + \\
 & + \int_{\varepsilon}^{\pi-\varepsilon} \frac{f(\alpha)}{\sin^2 \alpha} \cos\{(2k+1/2)\alpha - \pi/4\} \frac{O(1)}{k^{1/2}n^{5/2}} d\alpha + \\
 & + \int_{\varepsilon}^{\pi-\varepsilon} \frac{f(\alpha)}{\sin \alpha} \sin\{(2k+3/2)\alpha - \pi/4\} \times \\
 & \times \cos\{(2n+1/2)\alpha - \pi/4\} \frac{O(1)}{k^{3/2}n^{1/2}} d\alpha + \\
 & + \int_{\varepsilon}^{\pi-\varepsilon} \frac{f(\alpha)}{\sin^2 \alpha} \sin\{(2k+3/2)\alpha - \pi/4\} \sin\{(2n+3/2)\alpha - \pi/4\} \frac{O(1)}{k^{3/2}n^{3/2}} d\alpha + \\
 & + \int_{\varepsilon}^{\pi-\varepsilon} \frac{f(\alpha)}{\sin^3 \alpha} \sin\{(2k+3/2)\alpha - \pi/4\} \frac{O(1)}{k^{3/2}n^{5/2}} d\alpha + \\
 & + \int_{\varepsilon}^{\pi-\varepsilon} \frac{f(\alpha)}{\sin^2 \alpha} \cos\{(2k+1/2)\alpha - \pi/4\} \frac{O(1)}{k^{1/2}n^{5/2}} d\alpha + \\
 & + \left. \int_{\varepsilon}^{\pi-\varepsilon} \frac{f(\alpha)}{\sin^3 \alpha} \sin\{(2n+3/2)\alpha - \pi/4\} \frac{O(1)}{k^{5/2}n^{3/2}} d\alpha + \int_{\varepsilon}^{\pi-\varepsilon} \frac{f(\alpha)}{\sin^4 \alpha} \frac{O(1)}{k^{5/2}n^{5/2}} d\alpha \right\}.
 \end{aligned}$$

Finally,

$$\begin{aligned}
 \alpha_n(p) &= O(\varepsilon^2 \ln n) + O\left(\frac{\varepsilon \ln n}{n}\right) + O\left(\frac{\varepsilon}{n^{3/2}}\right) + O\left(\frac{\varepsilon}{n}\right) + \\
 & + O\left(\frac{\varepsilon \ln n}{n^2}\right) + O\left(\frac{\varepsilon \ln n}{n^2}\right) + O\left(\frac{\varepsilon}{n^2}\right) + O\left(\frac{\varepsilon \ln n}{n^2}\right) - \\
 & - \frac{1}{4\pi^2 n} \int_{\varepsilon}^{\pi-\varepsilon} f(\alpha) \operatorname{ctg} \alpha (\pi - \alpha) d\alpha + O\left(\frac{\varepsilon \ln n}{n^2}\right) + O\left(\frac{\ln n}{n^{3/2}}\right) + O\left(\frac{\ln \varepsilon \ln n}{n^{3/2}}\right) + \\
 & + O\left(\frac{\ln n}{n^3 \varepsilon}\right) + O\left(\frac{\ln \varepsilon}{n^{3/2}}\right) + O\left(\frac{1}{\varepsilon n^{3/2}}\right) + O\left(\frac{1}{n^3 \varepsilon^3}\right).
 \end{aligned}$$

Choose $\varepsilon = \frac{1}{n^{3/4}}$ and find

$$\alpha_n(p) = \frac{1}{4\pi^2 n} \int_{\varepsilon}^{\pi-\varepsilon} f(\alpha) \operatorname{ctg} \alpha (\pi - \alpha) d\alpha + O\left(\frac{\ln n}{n^{3/2}}\right).$$

The third amendment of the perturbation theory

$$\beta_n(p) = \frac{1}{6\pi i} Sp \left[\int_{l_n} - \int_{l_{n-1}} \right] [P(T - \lambda E)^{-1}]^3 d\lambda =$$

$$\begin{aligned}
 &= \frac{1}{6\pi i} \left[\int_{l_n} - \int_{l_{n-1}} \right] \sum \frac{(2n+1)(2k+1)(2l+1)}{(\lambda_n - \lambda)(\lambda_k - \lambda)(\lambda_l - \lambda)} d\lambda \times \\
 &\times \left(\frac{1}{4\pi} \right)^3 \iiint_F \iiint_F \iiint_F p(\theta, \varphi) p(\theta', \varphi') p(\theta'', \varphi'') \times \\
 &\quad \times P_n(\cos \alpha) P_n(\cos \beta) P_n(\cos \gamma) \times \\
 &\quad \times \sin \theta \sin \theta' \sin \theta'' d\varphi d\varphi' d\varphi'' d\theta d\theta' d\theta'', \\
 &\quad \cos \alpha = \cos \theta \cos \theta' + \sin \theta \sin \theta' \cos(\varphi - \varphi'), \\
 &\quad \cos \beta = \cos \theta' \cos \theta'' + \sin \theta' \sin \theta'' \cos(\varphi' - \varphi''), \\
 &\quad \cos \gamma = \cos \theta \cos \theta'' + \sin \theta \sin \theta'' \cos(\varphi - \varphi'').
 \end{aligned}$$

Finally,

$$\beta_n(p) = 0.$$

Main theorem.

Theorem 2. *If p -potential Lipschitz two variables, then the eigenvalues $T+P$ the equality*

$$\sum_{i=0}^{2n} \mu_{n,i} - n(n+1)(2n+1) = O\left(\frac{\ln n}{n^{1/2}}\right).$$

Theorem 3. *The first regularized trace for the Laplace – Bochner complex potential Lipschitz has the form*

$$\mu_{0,0} + \lim_{m \rightarrow \infty} \left(\sum_{n=1}^m \left\{ \sum_{i=0}^{2n} \mu_{n,i} - n(n+1)(2n+1) \right\} \right) = \text{const.}$$

And according to k number of absolutely convergent.

Thus, we calculated the regularized trace for the Laplace – Bochner of a non-smooth potential, namely, the potential is Lipschitz.

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Received October 30, 2015

ДИФФЕРЕНЦИАЛЬНЫЕ ОПЕРАТОРЫ НА ПРОЕКТИВНОЙ ПЛОСКОСТИ

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Ранее была получена формула сложения для четных сферических гармоник, которая послужила основой для вывода формулы первого регуляризованного следа оператора Лапласа с потенциалом на проективной плоскости. Благодаря ей удалось обойти нахождение асимптотических формул для присоединенных полиномов Лежандра по трем параметрам, что являлось неразрешимой задачей в течение длительного времени. Полученные результаты стали основой для вычисления поправок теории возмущений с последующим выходом на формулы регуляризованных следов эллиптических дифференциальных операторов. В работе рассматривается задача суммирования класса расходящихся рядов. Предлагается метод вычисления поправок теории возмущений для дифференциального оператора с потенциалом на действительной проективной плоскости. Метод применим, в частности, для суммирования рядов с факториальным ростом членов.

Ключевые слова: оператор Лапласа – Бохнера; проективная плоскость; гильбертово пространство; полином Лежандра; условие Липшица.

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Поступила в редакцию 30 октября 2015.