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PROBLEM OF HARD AND OPTIMAL CONTROL OF SOLUTIONS TO THE INITIAL-FINAL PROBLEM FOR NONSTATIONARY SOBOLEV TYPE EQUATION

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The main aim of this work is solving the problem of hard and optimal control of solution to the initial-finial problem for nonstationary Sobolev type equation. We construct a solution to the initial-final problem for the nonstationary equation and show that a unique optimal control of solutions to this problem exists.

Apart from the introduction and bibliography, the article consists of three sections. The first section provides the essentials of the theory of relatively p-bounded operators. In the second section we construct a strong solution to the initial-final problem for nonstationary Sobolev-type equations. The third section contains our proof that there exists a unique optimal control of solutions to the initial-final problem.

Keywords: optimal control; initial-final problem; Sobolev-type equations; relatively bounded operator.

Introduction

Suppose that $\mathfrak{X}$, $\mathfrak{Y}$, and $\mathfrak{U}$ are Hilbert spaces, and then take bounded linear operators $L \in \mathcal{L}(\mathfrak{X};\mathfrak{Y})$ and $B \in \mathcal{L}(\mathfrak{U};\mathfrak{Y})$, assuming that the kernel of L is non-trivial. Take also a closed linear operator $M \in \mathcal{C}l(\mathfrak{X};\mathfrak{Y})$ whose domain is dense in $\mathfrak{X}$.

Consider the Sobolev-type equation [1–4]

$$L\dot{x}(t) = a(t)Mx(t) + Bu(t) \tag{1}$$

with a control vector function $u : [0,\tau] \to \mathfrak{U}$ and a scalar function $a : [0,\tau] \to \mathbb{R}_+$, to be specified later, characterizing the change in time of the parameters of (1). The operators L and M generate the analytic resolving group for the homogeneous stationary equation (1), which means that $a(t) \equiv 1$.

We consider an optimal control problem for (1). Namely, we aim to find a pair $(\hat{x},\hat{u}) \in \mathfrak{X} \times \mathfrak{U}_{ad}$ with

$$J(\hat{x},\hat{u}) = \inf_{(x,u)\in\mathfrak{X}\times\mathfrak{U}_{ad}} J(x,u). \tag{2}$$

Here $\mathfrak{U}_{ad}$ is a closed convex set of admissible controls in the Hilbert space $\mathfrak{U}$ of controls, all pairs (x,u) satisfy the initial-final problem [5] for (1), and $J(x,u)$ is a certain penalty functional in special form.

Previously the authors studied the optimal control problem for solutions to non-stationary Sobolev-type equations (1) with the Showalter–Sidorov condition. In this paper we study the optimal control of solutions to the initial–final problem [5], which is a generalized Showalter–Sidorov problem for (1).

1. Relatively Spectrally Bounded Operators

Recall the standard notation of the theory of relatively p-bounded operators [3].

Starting with two Hilbert spaces $\mathfrak{X}$ and $\mathfrak{Y}$, take a bounded linear operator $L \in \mathcal{L}(\mathfrak{X};\mathfrak{Y})$ with non-trivial kernel and a closed linear operator $M \in \mathcal{C}l(\mathfrak{X};\mathfrak{Y})$ whose domain is dense in $\mathfrak{X}$. Consider the stationary equation

$$L\dot{x}(t) = Mx(t) + f(t), \tag{3}$$

called a *Sobolev-type equation* [3].

The sets

$$\rho^L(M) = \{\mu \in \mathbb{C} : (\mu L - M)^{-1} \in \mathcal{L}(\mathfrak{Y};\mathfrak{X})\} \qquad \text{and} \qquad \sigma^L(M) = \mathbb{C} \setminus \rho^L(M)$$

are called the *L-resolvent set* and the *L-spectrum* of M respectively.

The operator-valued functions $(\mu L - M)^{-1}$, $R^L_\mu(M) = (\mu L - M)^{-1}L$, and $L^L_\mu(M) = L(\mu L - M)^{-1}$ are respectively called the *resolvent*, *right resolvent*, and *left resolvent* of M *with respect to* L (or briefly the *L-resolvent*, *right L-resolvent*, and *left L-resolvent* of M).

Lemma 1. *Let operators $L \in \mathcal{L}(\mathfrak{X};\mathfrak{Y})$ and $M \in \mathcal{C}l(\mathfrak{X};\mathfrak{Y})$. Then the L-resolvent, right and left L-resolvents of M are analytic on $\rho^L(M)$.*

Definition 1. *An operator M is called **spectrally bounded with respect to** an operator L (or briefly (L,σ)-**bounded**) whenever*

$$\exists r_0 > 0 \quad \forall \mu \in \mathbb{C} \quad (|\mu| > r_0) \ \Rightarrow \ (\mu \in \rho^L(M)).$$

Put $\gamma = \{\mu \in \mathbb{C} : |\mu| = r > r_0\}$. The integrals of F. Riesz type

$$P = \frac{1}{2\pi i}\int_\gamma R^L_\mu(M)\,d\mu, \qquad Q = \frac{1}{2\pi i}\int_\gamma L^L_\mu(M)\,d\mu$$

exist by Lemma 1 for every (L,σ)-bounded operator M. The operators $P \in \mathcal{L}(\mathfrak{X})$ and $Q \in \mathcal{L}(\mathfrak{Y})$ are projections [3]. Put $\mathfrak{X}^0 = \ker P$, $\mathfrak{Y}^0 = \ker Q$, $\mathfrak{X}^1 = \mathrm{im}P$ and $\mathfrak{Y}^1 = \mathrm{im}Q$. Denote the restriction of L (M) to $\mathfrak{X}^k$ by L_k (M_k) for $k = 0, 1$.

Theorem 1. *The following claims hold for every (L,σ)-bounded operator M:*

(i) *the operators $L_k, M_k : \mathfrak{X}^k \to \mathfrak{Y}^k$ for $k = 0, 1$;*

(ii) *the operators $M_0 \in \mathcal{L}(\mathfrak{X}^0;\mathfrak{Y}^0)$ and $M_1 \in \mathcal{C}l(\mathfrak{X}^1;\mathfrak{Y}^1)$;*

(iii) *there exists operators $L_1^{-1} \in \mathcal{L}(\mathfrak{Y}^1;\mathfrak{X}^1)$ and $M_0^{-1} \in \mathcal{L}(\mathfrak{Y}^0;\mathfrak{X}^0)$;*

(iv) *there exist analytic resolving operator groups $\{X^t \in \mathcal{L}(\mathfrak{X}) : t \in \mathbb{R}\}$ for the homogeneous equation* (3) *and $\{Y^t \in \mathcal{L}(\mathfrak{Y}) : t \in \mathbb{R}\}$ for the equation*

$$R^L_\beta(M)\dot{y}(t) = M(\beta L - M)^{-1}y(t),$$

where $\beta \in \rho^L(M)$, which are of the form

$$X^t = e^{tL_1^{-1}M_1}P = \frac{1}{2\pi i}\int_\gamma R^L_\mu(M)e^{\mu t}d\mu \quad \text{and} \quad Y^t = e^{tM_1L_1^{-1}}Q = \frac{1}{2\pi i}\int_\gamma L^L_\mu(M)e^{\mu t}d\mu$$

respectively.

Theorem 1 implies the existence of the operators $H = M_0^{-1}L_0 \in \mathcal{L}(\mathfrak{X}^0)$ and $S = L_1^{-1}M_1 \in \mathcal{L}(\mathfrak{X}^1)$.

An (L,σ)-bounded operator M is called $(L,0)$-*bounded* whenever the point ∞ is a *removable singularity* of the L-resolvent of M, that is, $H \equiv \mathbb{O}$; (L,p)-*bounded* whenever the point ∞ is an *order* $p \in \mathbb{N}$ *pole* of the L-resolvent of M, that is, $H^p \neq \mathbb{O}$ and $H^{p+1} \equiv \mathbb{O}$; (L,∞)-*bounded* whenever the point ∞ is an *essential singularity* of the L-resolvent of M, that is, $H^q \neq \mathbb{O}$ for all $q \in \mathbb{N}$.

2. Solvability of the Initial-Final Problem

Take two Hilbert spaces $\mathfrak{X}$ and $\mathfrak{Y}$. For two operators $L \in \mathcal{L}(\mathfrak{X};\mathfrak{Y})$ and $M \in \mathcal{C}l(\mathfrak{X};\mathfrak{Y})$, where M is (L,p)-bounded for $p \in \{0\} \cup \mathbb{N}$.

Let the relative L-spectrum of operator M can be represent as

$$\sigma^L(M) = \sigma_1^L(M) \cup \sigma_2^L(M), \quad \text{where} \quad \sigma_1^L(M) \cap \sigma_2^L(M) = \emptyset.$$

Define the operators $P_j \in \mathcal{L}(\mathfrak{X})$ and $Q_j \in \mathcal{L}(\mathfrak{Y})$ as

$$P_j = \frac{1}{2\pi i}\int\limits_{\gamma_j} R_\mu^L(M)d\mu, \quad Q_j = \frac{1}{2\pi i}\int\limits_{\gamma_j} L_\mu^L(M)d\mu, \quad j = 1,2.$$

These operators are projectors by the relatively spectral theorem [8], and moreover, the results of [8]. Put $\mathfrak{X}_j^1 = \mathrm{im}P_j$ and $\mathfrak{Y}_j^1 = \mathrm{im}Q_j$ for $j = 1,2$.

Consider the initial-final problem

$$P_1(x(0) - x_0) = 0, \quad P_2(x(\tau) - x_\tau) = 0 \tag{4}$$

for (3). Applying to (3) the projections $\mathbb{I} - Q$ and Q_j for $j = 1,2$ we obtain the equivalent system

$$\begin{cases} H\dot{x}^0 = x^0 + M_0^{-1}f^0, \\ \dot{x}_1^1 = S_1 x_0^1 + L_{11}^{-1}f_1^1, \\ \dot{x}_2^1 = S_2 x_\tau^1 + L_{12}^{-1}f_2^1 \end{cases} \tag{5}$$

where $H = M_0^{-1}L_0 \in \mathcal{L}(\mathfrak{X}^0)$ is a degree $p \in \{0\} \cup \mathbb{N}$ nilpotent operator, the operators $S_j = P_j S \in \mathcal{L}(\mathfrak{X}_j^1)$ and $L_{1j}^{-1} = P_j L_1^{-1} \in \mathcal{L}(\mathfrak{Y}_j^1;\mathfrak{X}_j^1)$, $j = 1,2$.

Put $\mathbb{N}_0 \equiv \{0\} \cup \mathbb{N}$ and construct the space

$$H^{p+1}(\mathfrak{Y}) = \{\xi \in L_2(0,\tau;\mathfrak{Y}) : \xi^{(p+1)} \in L_2(0,\tau;\mathfrak{Y}),\ p \in \mathbb{N}_0\}$$

which is a Hilbert space with the inner product

$$[\xi,\eta] = \sum_{q=0}^{p+1}\int\limits_0^\tau \left\langle \xi^{(q)}, \eta^{(q)} \right\rangle_{\mathfrak{Y}} dt.$$

Definition 2. *A vector-valued function $x \in H^1(\mathfrak{X})$ is called a* ***strong solution*** *to the initial-final problem (3), (4) whenever it satisfies (3) and the terms of* (4) *almost everywhere.*

Theorem 2. *Let operator M (L,p)-bounded, $p \in \{0\} \cup \mathbb{N}$, then for any vector-functions $x_0, x_\tau \in \mathfrak{X}$, vector-function $u \in H^{p+1}(\mathfrak{U})$ and scalar function $a \in C^{p+1}([0,\tau);\mathbb{R}_+)$ separated from zero there exists a unique solution $x \in H^1(\mathfrak{X})$ for problem* (3), (4)*:*

$$\begin{aligned} x(t) = -\sum_{q=0}^{p} H^q M_0^{-1}(I-Q)\left(\frac{1}{a(t)}\frac{d}{dt}\right)^k \frac{Bu(t)}{a(t)} + \\ +X_1^{A(t)}x_0 + \int_0^t X_1^{A(t)-A(s)}L_1^{-1}Q_1Bu(s)ds - \\ -X_2^{A(\tau)-A(t)}x_\tau - \int_t^\tau X_2^{A(t)-A(s)}L_1^{-1}Q_2Bu(s)ds \end{aligned} \tag{6}$$

with the replacement $A(t) = \int_0^t a(\zeta)d\zeta$

3. Optimal Control of the Problem

For a Hilbert space $\mathfrak{X}$ consider the equation

$$L\dot{x}(t) = a(t)Mx(t) + Bu(t) \tag{7}$$

with operators $L \in \mathcal{L}(\mathfrak{X};\mathfrak{Y})$, $M \in \mathcal{C}l(\mathfrak{X};\mathfrak{Y})$, and $B \in \mathcal{L}(\mathfrak{U};\mathfrak{Y})$, a scalar function $a : [0,\tau) \to \mathbb{R}_+$, as well as vector functions $u : [0,\tau) \to \mathfrak{U}$ to be specified later.

Take a Hilbert space $\mathfrak{Z}$ and an operator $C \in \mathcal{L}(\mathfrak{X};\mathfrak{Z})$. Consider the *penalty functional*

$$J(u) = \alpha \sum_{q=0}^{1} \int_0^\tau \|z^{(q)} - z_d^{(q)}\|_{\mathfrak{Z}}^2 dt + (1-\alpha)\sum_{q=0}^{k}\int_0^\tau \left\langle N_q u^{(q)}, u^{(q)}\right\rangle_{\mathfrak{U}} dt, \qquad z = Cx, \tag{8}$$

where $0 \le k \le p+1$. The operators $N_q \in \mathcal{L}(\mathfrak{U})$ for $q = 0, 1, \ldots, p+1$ are self-adjoint and positive definite, while $z_d = z_d(t,s)$ is an observation from some *space of observations* $\mathfrak{Z}$. Note that if $x \in H^1(\mathfrak{X})$ then $z \in H^1(\mathfrak{Z})$. By analogy with $H^{p+1}(\mathfrak{Y})$, define the space $H^{p+1}(\mathfrak{U})$, which is a Hilbert space because so is $\mathfrak{U}$. We distinguish a convex and closed subset $H^{p+1}_{ad}(\mathfrak{U})$ of the space $H^{p+1}(\mathfrak{U})$, called the *set of admissible controls*.

Note that $\alpha \in (0,1]$ and $(1-\alpha)$ are weights goals of optimal control. Its describe achievement the targets observed value without abrupt changes (the first term in (8)) and minimize the resources expended to control (the second term in (8)). If $\alpha = 1$ in the functional (8) second term vanishes and we get a hard control problem, that is, when optimization of achieving the goal is not interested in the cost of expended resources [8].

Definition 3. *A vector function $v \in H^{p+1}_{ad}(\mathfrak{U})$ is called an* ***optimal control*** *of solutions to problem (4), (7) whenever*

$$J(v) = \min_{(x(u),u)\in\mathfrak{X}\times H^{p+1}_{ad}(\mathfrak{U})} J(u), \tag{9}$$

where the pairs $(x(u),u) \in \mathfrak{X} \times H^{p+1}_{ad}(\mathfrak{U})$ satisfy (4), (7).

By Theorem 2, a unique solution $x \in H^1(\mathfrak{X})$ to problem (4), (7) exists for all vectors $x_0, x_\tau \in \mathfrak{X}$, $u \in H^{p+1}(\mathfrak{U})$ and a function $a \in C^{p+1}([0,\tau);\mathbb{R}_+)$ separated from zero.

We now fix $x_0, x_\tau \in \mathfrak{X}$ for and consider (6) as a mapping $D : u \to x(u)$.

Lemma 2. *Given Hilbert spaces $\mathfrak{X}$, $\mathfrak{Y}$, and $\mathfrak{U}$, take an (L,p)-bounded operator M with $p \in \mathbb{N}_0$, a function $a \in C^{p+1}(\overline{\mathbb{R}}_+;\mathbb{R}_+)$ separated from zero, and fix vectors $x_0, x_\tau \in \mathfrak{X}$. Then the mapping $D : H^{p+1}(\mathfrak{U}) \to H^1(\mathfrak{X})$ defined by (6) is continuous.*

Proof.

Since $B \in \mathcal{L}(H^{p+1}(\mathfrak{U}); H^{p+1}(\mathfrak{Y}))$ and (6) is the solution to (7), this lemma holds by the properties of the operator group X^t and the continuity of $a(t)$ for $t \in \overline{\mathbb{R}}_+$, by analogy with the proof of Theorem 2.

□

Theorem 3. *Take an (L,p)-bounded operator M with $p \in \mathbb{N}_0$ and a function $a \in C^{p+1}([0,\tau);\mathbb{R}_+)$ separated from zero. Then for arbitrary vectors $x_0, x_\tau \in \mathfrak{X}$ and $z_d \in \mathfrak{Z}$, there exists a unique solution $v \in H^{p+1}_{ad}(\mathfrak{U})$ to the optimal control problem* (4), (7)–(9) *with $\alpha \in (0,1)$.*

Proof.

Using the mapping D of Lemma 2, we see that the functional (8) becomes

$$J(u) = \|Cx(t;u) - z_d\|^2_{H^1(\mathfrak{Z})} + [\eta, u],$$

where $\eta^{(k)}(t) = N_k u^{(k)}$ for $k = 0, \ldots, p+1$. Therefore,

$$J(u) = \pi(u,u) - 2\theta(u) + \|z_d - Cx(t;0)\|^2_{H^1(\mathfrak{Z})},$$

where $\pi(u,u) = \|C(x(t;u) - x(t;0))\|^2_{H^1(\mathfrak{Z})} + [\eta, u]$ is a coercive continuous bilinear form on $H^{p+1}(\mathfrak{U})$, and

$$\theta(u) = \langle z_d - Cx(t;0), C(x(t;u) - x(t;0))\rangle_{H^1(\mathfrak{Z})}$$

is a continuous linear form on $H^{p+1}(\mathfrak{U})$. Thus, the theorem is valid by analogy with [6].

□

Corollary 1. . *Take an (L,p)-bounded operator M with $p \in \mathbb{N}_0$ and a function $a \in C^{p+1}([0,\tau);\mathbb{R}_+)$ separated from zero. Then for all vectors $x_0, x_\tau \in \mathfrak{X}$ and $z_d \in \mathfrak{Z}$, there exists a unique solution $v \in H^{p+1}_{ad}(\mathfrak{U})$ to the hard control problem* (4), (7)–(9) *with $\alpha = 1$.*

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