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ON KERNELS AND IMAGES OF RESOLVING ANALYTIC DEGENERATE SEMIGROUPS IN QUASI-SOBOLEV SPACES

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The theory of holomorphic degenerate semigroups of operators was constructed earlier in Banach spaces and Frechet spaces. However, examples of study of mathematical objects show the necessity of their consideration in more general cases. In this article the theory of holomorphic degenerate semigroups of operators is transferred to quasi-Sobolev spaces of sequences which are quasi-normed and even quasi-Banach spaces.

The article besides the introduction and references contains two paragraphs. In the first, resolving analytic degenerate semigroups are constructed. The second paragraph includes the study of kernels and images of analytic semigroups.

Keywords: kernels and images; analytic degenerate semigroup; quasi-Banach spaces; quasi-Sobolev spaces.

Introduction

Let $\mathfrak{U}$ and $\mathfrak{F}$ be quasi-Banach spaces, operators $L \in \mathcal{L}(\mathfrak{U};\mathfrak{F})$ and $M \in \mathcal{C}l(\mathfrak{U};\mathfrak{F})$. Consider an *L-resolvent set* $\rho^L(M) = \{\mu \in \mathbb{C} : (\mu L - M)^{-1} \in \mathcal{L}(\mathfrak{U};\mathfrak{F})\}$ and an *L-spectrum* $\sigma^L(M) = \mathbb{C}\backslash\rho^L(M)$ of an operator M. It is easy to see that, the set $\rho^L(M)$ is always open, therefore the L-spectrum of operator M is always closed. Define $R^L_\alpha(M) = (\alpha L - M)^{-1}L$ and $L^L_\alpha(M) = L(\alpha L - M)^{-1}$ as right and left L-resolvents of operator M.

Consider equations

$$R^L_\alpha(M)\dot{u} = (\alpha L - M)^{-1}Mu \tag{1}$$

and

$$L^L_\alpha(M)\dot{f} = M(\alpha L - M)^{-1}f \tag{2}$$

for $\alpha \in \rho^L(M)$, which are equivalent to a linear Sobolev type equation

$$L\dot{u} = Mu. \tag{3}$$

They can be considered in the framework of equation

$$A\dot{v} = Bv, \tag{4}$$

where operators $A \in \mathfrak{V}, B \in \mathcal{C}l(\mathfrak{V})$, $\mathfrak{V}$ is a quasi-Banach space. Our goal is to construct and study the properties of holomorphic degenerate operator semigroups resolving Sobolev type equation of the form (4).

The mapping $V \in C(\mathbb{R}; L(\mathfrak{V}))$ is called a semigroup of operators if for all $s, t \in \mathbb{R}_+$

$$V^sV^t = V^{s+t}. \tag{5}$$

The semigroup $\{V^t\colon\ t\in\mathbb{R}_+\}$ is called holomorphic if it can be analitically continued to some sector of complex plane containing half axis $\mathbb{R}_+$ preserving property (5). A holomorphic semigroup is called degenerate if its unit $P{=}s{-}\lim_{t\to 0+}V^t$ is a projector in $\mathfrak{V}$.

Firstly holomorphic degenerate semigroups appeared in [1] as resolving semigroups for evolution Sobolev type equation (3) in Banach spaces. Explicit theory of such semigroups can be found in [2]. These results were stread to locally convex spaces [3].

Interest in Sobolev type equations has recently increased significantly [4–6], moreover, there arose a necessity for their consideration in quasi-Banach spaces. The need is dictated not so much by the desire to fill up the theory but by the aspiration to comprehend non-classical models of mathematical physics in quasi-Banach spaces [7, 8]. Our goal is to spread these ideas to one class of evolution Sobolev type equations in quasi-Banach spaces of sequences.

1. Resolving Analytic Degenerate Semigroups

Definition 1. *A solution of (4) will be a function $v(t)\in C^\infty(\mathbb{R}_+;\mathfrak{V})$, satisfying this equation.*

Definition 2. *The semigroup $V^\bullet\in C^\infty(\mathbb{R}_+;\mathcal{L}(\mathfrak{V}))$ is called a resolving operator semigroup of equation (4), if*

(i) $V^sV^t=V^{s+t}\quad\forall s,t\in\mathbb{R}_+$;

(ii) for any $v_0\in\mathfrak{V}$ a function $v(t)=V^tv_0$ will be a solution of this equation.

Remark 1. Availability of the unit of semigroup is not postulated.

The semigroup $\{V^t:t\in\mathbb{R}_+\}$ is called *uniformly bounded*, if

$$\exists M>0\quad\forall t>0\quad {}_{\mathcal{L}(\mathfrak{U})}\|V^t\|\le M;$$

analytic, if it is analytically continued in some sector, containing a ray $\mathbb{R}_+$.

Let here and in the next $\mathfrak{U}{=}\ell_q^{m+2n}$, $\mathfrak{F}{=}\ell_q^m$, $m\in\mathbb{R}$, $q\in\mathbb{R}_+$, $Q_n(\lambda)=\sum_{i=0}^n c_i\lambda^i$, $R_s(\lambda)=\sum_{j=0}^s d_j\lambda^j$ be polynomials with real coefficients, with no common roots, and of degrees n и s, respectively, where $n<s$ and $d_sc_n<0$. Construct operators $L{=}Q_n(\Lambda), M{=}R_s(\Lambda): Lu=\{Q_n(\lambda_k)u_k\}, Mu=\{R_s(\lambda_k)u_k\}$.

Theorem 1. *Let operators $L,\ M$ be defined as in the above, where* $\operatorname{Re}\mu_k\le 0$. *Then there exists analytic in the sector $\{\tau\in\mathbb{C}:|\arg\tau|<\theta-\pi/2,\quad \tau\ne 0\}$, where $\theta\in(\pi/2;\pi)$, and uniformly bounded resolving semigroup $\{U^t:t\in\mathbb{R}_+\}\quad(\{F^t:t\in\mathbb{R}_+\})$ of equation (1) ((2)), and it is defined by Dunford-Taylor integrals*

$$U^t=\frac{1}{2\pi i}\int\limits_\Gamma R_\mu^L(M)e^{\mu t}d\mu \tag{6}$$

$$\left(F^t=\frac{1}{2\pi i}\int\limits_\Gamma L_\mu^L(M)e^{\mu t}d\mu\right), \tag{7}$$

where the contour Γ satisfies the condition:

$$\Gamma\subset S_\theta^L(M),\qquad \arg\mu\to\pm\theta\quad \textit{when}\quad |\mu|\to\infty. \tag{8}$$

Remark 2. If we ignore the constraint $\text{Re}\,\mu_k \le 0$ (in the case a constant $a \ne 0$ in the definition of L-sectoriality [7]), then the semigroup of equation $L\dot{u} = Mu$ will be

$$\left\{ W^t = e^{at}U^t = \frac{1}{2\pi i}\int\limits_{\Gamma} R^L_\mu(M)e^{(\mu+a)t}d\mu : t \in \mathbb{R}_+ \right\}.$$

Accordingly, for this semigroup instead of uniformly boundedness there will be exponential estimation

$$\forall t > 0 \quad {}_{\mathcal{L}(\mathfrak{U})}\|W^t\| \le Me^{bt}.$$

Proof.

$$\,^{m+2n}_{\quad q}\|W^t u\| = e^{at}(\,^{m+2n}_{\quad q}\|\frac{1}{2\pi i}\int\limits_{\Gamma} R^L_\mu(M)e^{\mu t}d\mu\|)^{1/q} \le$$

$$\le Ce^{at}(\sum{}' e^{\text{Re}\mu_k qt}| < u, e_k > |^q \cdot \,^{m+2n}_{\quad q}\|e_k\|^q)^{1/q} =$$

$$= Ce^{(a+\max\text{Re}\mu_k)t}(\sum{}' | < u, e_k > |^q \cdot (\lambda_k)^{\frac{m+2n}{2}q})^{1/q} \le,$$

$$\le Ce^{(a+\max\text{Re}\mu_k)t} \cdot \,^{m+2n}_{\quad q}\|u\|.$$

Hence,

$$\forall t > 0 \quad {}_{\mathcal{L}(\mathfrak{U})}\|W^t\| \le Ce^{bt}.$$

□

Remark 3. If the degrees of polynomials $n = s$, then an operator M (L, σ)-bounded, and ∞ is a removable singular point or a pole of order p of L-resolvent of operator M. Then a semigroup (6),(7) is continued to analytical group.

Remark 4. Under the conditions of Theorem 1, the following relations:

$$LU^t u = F^t Lu \quad \forall u \in \mathfrak{U},$$

$$MU^t u = F^t Mu \quad \forall u \in \text{dom}M \quad \forall t \in \mathbb{R}_+$$

are obvious.

From the formulas (6),(7) of resolving semigroups for equations (1), (2), seeing that the operators have nontrivial kernels, we have $\ker U^t \supset \ker R^L_\mu(M)$, $\ker F^t \supset \ker L^L_\mu(M)$ $\forall t \in \mathbb{R}_+$.

2. Kernels and Images of Analytic Semigroups

Definition 3. *The set*

$$\ker V^\bullet = \{\varphi \in \mathfrak{V} : \exists t \in \mathbb{R}_+ \quad V^t\varphi = 0\}$$

is called a kernel of the analytic semigroup $\{V^t : t \in \mathbb{R}_+\}$.

The set

$$\text{im}V^\bullet = \{v \in \mathfrak{V} : \lim_{t\to 0+} V^t v = v\}$$

is called an image of the semigroup.

Remark 5. From the analyticity of the semigroup we have $\ker V^\bullet = \ker V^t \quad \forall t \in \mathbb{R}_+$. Indeed, by the definition $\ker V^\bullet = \bigcup\limits_{t>0} \ker V^t$. Claim that $\ker V^{t_1} = \ker V^{t_2} \quad \forall t_2 > t_1 > 0$. Since $V^{t_2} = V^{t_2 - t_1} V^{t_1}$, then $\ker V^{t_1} \subset \ker V^{t_2}$. Let $u \in \ker V^{t_2}$, consider the function $v(t) = V^t u$. It is analytic in the sector, containing $\mathbb{R}_+$, and equal to zero when $t \geq t_2$. By the theorem about uniqueness of the analytic function we get $v(t) \equiv 0$ in the entire sector.

The last remark shows that the kernel of the analytic semigroup is a subspace.

Lemma 1. *For the analytic semigroup* $\{V^t : t \in \mathbb{R}_+\}$ *we have* $\ker V^\bullet \cap \mathrm{im} V^\bullet = \{0\}$.

Proof.

Let $v \in \ker V^\bullet \cap \mathrm{im} V^\bullet$. Then by remark 5 $\forall t > 0 \;\; V^t v = 0$. Therefore $v = \lim\limits_{t \to 0+} V^t v = 0$.

□

Denote $\mathfrak{U}^0 = \ker U^\bullet$, $\mathfrak{F}^0 = \ker F^\bullet$ and by $L_0 \quad (M_0)$ define restriction of the operator $L \quad (M)$ onto $\mathfrak{U}^0 \quad (\mathfrak{U}^0 \cap \mathrm{domM})$.

Lemma 2. *Under the conditions of Theorem 1 operators* $L_0 \in \mathcal{L}(\ker U^\bullet; \ker F^\bullet)$, $M_0 : \ker U^\bullet \cap \mathrm{domM} \to \ker \mathrm{F}^\bullet$.

Proof.

From Remark 4 it follows that if $U^t u = 0$ then $0 = LU^t u = F^t L u$ $(0 = MU^t u = F^t M u)$ $\forall t \in \mathbb{R}_+ \; \forall u \in \mathfrak{U}$ $(\forall u \in \mathrm{domM})$.

□

By $\sigma_0^L(M)$ define an L_0-spectrum of operator M_0.

Lemma 3. *Under the conditions of Theorem 1* $\sigma_0^L(M) = \{\infty\}$.

Proof.

Take $\lambda \in \mathbb{C}$. Consider the operator

$$(\lambda L_0 - M_0)^{-1} = \sum_{\lambda_k \in \ker Q} \frac{< \cdot, e_k >}{\lambda Q(\lambda_k) - R(\lambda_k)} e_k. \tag{9}$$

For $f \in \mathfrak{F}^0$ we have

$${}^m_q\|(\lambda L_0 - M_0)^{-1} f\| = {}^m_q\| \sum_{\lambda_k \in \ker Q} \frac{< f, e_k >}{\lambda Q(\lambda_k) - R(\lambda_k)} e_k \| \leq$$

$$\leq C(\sum_{\lambda_k \in \ker Q} \frac{| < f, e_k > |^q}{|\lambda Q(\lambda_k) - R(\lambda_k)|^q} \, {}^m_q\|e_k\|^q)^{1/q} \leq$$

$$\leq C(\sum_{\lambda_k \in \ker Q} \frac{| < f, e_k > |^q}{|\lambda Q(\lambda_k) - R(\lambda_k)|^q} \, {}^m_q\|e_k\|^q)^{1/q} \leq$$

$$\leq C \max_{\lambda_k \in \ker Q} \frac{1}{|\lambda Q(\lambda_k) - R(\lambda_k)|} (\sum_{\lambda_k \in \ker Q} |f_k|^q \; (\lambda_k)^{\frac{m}{2} q})^{1/q} \leq \mathrm{const} \; {}^m_q\|f\|.$$

Consequently, operator (9) is bounded. Let $\varphi \in \mathfrak{U}^0$. Then

$$\sum_{\lambda_k \in \ker Q} \frac{< (\lambda L_0 - M_0)\varphi, e_k >}{\lambda Q(\lambda_k) - R(\lambda_k)} e_k =$$

$$\sum_{\lambda_k \in \ker Q} \frac{< \sum\limits_{\lambda_j \in \ker Q} (\lambda Q(\lambda_j) - R(\lambda_j)) < \varphi, e_j > e_j, e_k >}{\lambda Q(\lambda_k) - R(\lambda_k)} e_k =$$

$$\sum_{\lambda_k \in \ker Q} \frac{(\lambda Q(\lambda_k) - R(\lambda_k)) < \varphi, e_k >}{\lambda Q(\lambda_k) - R(\lambda_k)} e_k =$$

$$\sum_{\lambda_k \in \ker Q} < \varphi, e_k > e_k = \varphi.$$

Similarly, for $f \in \mathfrak{F}^0$ it can be shown that

$$(\lambda L_0 - M_0) \sum_{\lambda_k \in \ker Q} \frac{< f, e_k >}{\lambda Q(\lambda_k) - R(\lambda_k)} e_k = f.$$

Consequently, operator (9) is an inverse to $\lambda L_0 - M_0$. A contour Γ, which satisfies (8), by using the analyticity of below integrand functions, can be choosen lying "to the right" of the points λ. Then for any $\varphi \in \ker U^\bullet \cap \mathrm{dom} M, \quad f \in \ker F^\bullet, \quad t \in \mathbb{R}_+$

$$\frac{1}{2\pi i} \int\limits_\Gamma \frac{(\mu L - M)^{-1} e^{(\mu - \lambda)t}}{\mu - \lambda} (\lambda L - M) \varphi d\mu =$$

$$\frac{1}{2\pi i} \int\limits_\Gamma \frac{e^{(\mu - \lambda)t} \varphi}{\mu - \lambda} d\mu - \frac{1}{2\pi i} \int\limits_\Gamma (\mu L - M)^{-1} L e^{(\mu - \lambda)t} \varphi d\mu = \varphi - e^{-\lambda t} U^t \varphi = \varphi,$$

$$(\lambda L - M) \frac{1}{2\pi i} \int\limits_\Gamma \frac{(\mu L - M)^{-1} e^{(\mu - \lambda)t}}{\mu - \lambda} f d\mu =$$

$$\frac{1}{2\pi i} \int\limits_\Gamma \frac{e^{(\mu - \lambda)t}}{\mu - \lambda} f d\mu - \frac{e^{-\lambda t}}{2\pi i} \int\limits_\Gamma L(\mu L - M)^{-1} e^{\mu t} f d\mu = f - e^{-\lambda t} F^t f = f$$

by the Cauchy theorem.

Thus, $\forall \lambda \in \mathbb{C}$ there exists an operator

$$(\lambda L_0 - M_0)^{-1} \in \mathcal{L}(\mathfrak{F}^0; \mathfrak{U}^0).$$

□

Corollary 1. *Under the conditions of Theorem 1 there exists an operator* $M_0^{-1} \in \mathcal{L}(\mathfrak{F}^0; \mathfrak{U}^0)$.

Proof.

It is sufficient to take in the previous lemma $\lambda = 0$. In this case

$$(M_0)^{-1} = \sum_{\lambda_k \in \ker Q} \frac{< \cdot, e_k >}{R(\lambda_k)} e_k. \tag{10}$$

□

Theorem 2. *Under the conditions of Theorem 1* $\ker R^L_\mu(M) = \mathfrak{U}^0, \quad \ker L^L_\mu(M) = \mathfrak{F}^0$.

Proof.

Take $\varphi \in \ker R^L_\mu(M) \setminus \{0\}$, i.e. φ is an eigenfunction of operator L. It is clear that the eigenfunction, according to (6), belongs to $\ker U^\bullet$. So, $\ker R^L_\mu(M) \subset \ker U^\bullet$. Let us prove the inverse inclusion. Consider the vector $\psi = R^L_\mu(M)\varphi$, where $\varphi \in \ker U^\bullet$. By using Lemmas 2 and 3, $\psi = R^{L_0}_\mu(M_0)\varphi \in \ker U^\bullet$, therefore by Lemma 1

$$0 = \lim_{t\to 0+} U^t\psi = \psi = R^L_\mu(M)\varphi.$$

Thus, the vector $\varphi \in \mathfrak{U}^0$. Consequently, $\ker R^L_\mu(M) = \mathfrak{U}^0$.

Now take $f \in \ker L^L_\mu(M)$. Then $f = M\varphi$, where $\varphi \in \ker L^L_\mu(M) \cap \mathrm{domM}$. By Remark 5 we get $\forall t \in \mathbb{R}_+$

$$F^t f = F^t M\varphi = MU^t\varphi = M0 = 0.$$

Thus, $f \in \ker F^\bullet, \quad \mathfrak{F}^0 \subset \ker F^\bullet$.

Now let $f \in \ker F^\bullet$, then $M_0^{-1} f = \varphi \in \ker U^\bullet = \ker L^L_\mu(M)$. Therefore

$$L^L_\mu(M)f = L^L_\mu(M)M_0\varphi = MR^L_\mu(M)\varphi = 0.$$

□

Remark 6. Under the conditions of Theorem 1 operators $H = M_0^{-1}L_0$ and $J = L_0 M_0^{-1}$ are equal to $\mathbb{O}$.

Lemma 4. *Under the conditions of Theorem 1* $\forall u \in \mathrm{im} R^L_\mu(M)$ $(\forall f \in \mathrm{im} L^L_\mu(M))$ $\lim\limits_{t\to 0+} U^t u = u$ $(\lim\limits_{t\to 0+} F^t f = f)$.

Proof.

Take an arbitrary vector $u = R^L_\mu(M)v$, $\mu \in S^L_\theta$. By using the L-resolvent identities

$$\begin{aligned}(\mu-\lambda)R^L_\lambda(M)R^L_\mu(M) = R^L_\lambda(M) - R^L_\mu(M),\\ (\mu-\lambda)L^L_\lambda(M)L^L_\mu(M) = L^L_\lambda(M) - L^L_\mu(M),\end{aligned} \tag{11}$$

then

$$U^t u = \frac{1}{2\pi i}\int_\Gamma R^L_\lambda(M)e^{\lambda t}d\lambda R^L_\mu(M)v =$$

$$= \frac{1}{2\pi i}\int_\Gamma \frac{R^L_\lambda(M)v}{\mu-\lambda}e^{\lambda t}d\lambda + R^L_\mu(M)\frac{1}{2\pi i}\int_\Gamma \frac{e^{\lambda t}v}{\lambda-\mu}d\lambda,$$

while the second integral on the right is equal to zero by the Cauchy theorem. Letting $t \to 0+$, we get

$$\lim_{t\to 0+} U^t u = \frac{1}{2\pi i}\int_\Gamma \frac{R^L_\lambda(M)v}{\mu-\lambda}d\lambda = R^L_\mu(M)v = u,$$

as the latter integral is equal to a deduction function $R^L_\lambda(M)v$ (around the contour in the negative direction).

(The claim about semigroup $\{F^t : t \in \mathbb{R}_+\}$ is proved similarly.)

□

The closure of image $\mathrm{im}R^L_\mu(M)$ $(\mathrm{im}L^L_\mu(M))$ right (left) L-resolvent in the normed space $\mathfrak{U}$ $(\mathfrak{F})$ is denoted by $\mathfrak{U}^1$ $(\mathfrak{F}^1)$.

Theorem 3. *Under the conditions of Theorem 1* $\mathrm{im}U^\bullet = \mathfrak{U}^1$, $\mathrm{im}F^\bullet = \mathfrak{F}^1$.

Proof.
According to Lemma 4, $\mathrm{im}R^L_\mu(M) \subset \mathrm{im}U^\bullet$. And as the limit $\lim\limits_{t\to 0+} U^t u = u$ exists on the dense in $\mathfrak{U}^1$ lineal $\mathrm{im}R^L_\mu(M)$, then using Banach – Steinhaus theorem for uniformly bounded semigroup implies the existence of this limit in a whole $\mathfrak{U}^1$, i.e. $\mathfrak{U}^1 \subset \mathrm{im}U^\bullet$.
According to the Cauchy theorem and the L-resolvent identity

$$U^t u = \frac{1}{2\pi i}\int\limits_\Gamma R^L_\lambda(M)e^{\lambda t}ud\lambda = \frac{1}{2\pi i}\int\limits_\Gamma R^L_\lambda(M)e^{\lambda t}ud\lambda$$

$$\frac{1}{2\pi i}\int\limits_\Gamma R^L_{\mu_0}(M)e^{\lambda t}ud\lambda = \frac{1}{2\pi i}R^L_{\mu_0}(M)\int\limits_\Gamma (\mu_0-\lambda)R^L_\lambda(M)e^{\lambda t}ud\lambda$$

$\forall \mu_0 \in \rho^L(M)$. Thus, $\forall t > 0$ $\mathrm{im}U^t \subset \mathrm{im}R^L_{(\mu,p)}(M)$. This means, $\mathrm{im}U^\bullet \subset \mathfrak{U}^1$.
Affirmation about the image of semigroup $\mathrm{im}F^\bullet$ is proved similarly.

□

Consequently, the images of the semigroups are subspaces, and we can define operators

$$L_1 = L\Big|_{\mathfrak{U}^1}, \quad M_1 = M\Big|_{\mathfrak{U}^1\cap \mathrm{domM}}.$$

Lemma 5. *Under the conditions of Theorem 1* $L_1 \in \mathcal{L}(\mathfrak{U}^1;\mathfrak{F}^1)$.

Proof.
Let $u = \lim\limits_{t\to 0+} U^t u$. Then, by the continuity of the operator L and Remark 5

$$Lu = L\lim_{t\to 0+} U^t u = \lim_{t\to 0+} LU^t u = \lim_{t\to 0+} F^t Lu,$$

that was required.

□

Remark 7. By Lemma 1 and Theorems 2 and 3

$$\mathfrak{U}^0 \cap \mathfrak{U}^1 = \{0\}, \quad \mathfrak{F}^0 \cap \mathfrak{F}^1 = \{0\}.$$

Introduce the notation:

$$\tilde{\mathfrak{U}} = \overline{\mathfrak{U}^0 \oplus \mathfrak{U}^1}, \quad \tilde{\mathfrak{F}} = \overline{\mathfrak{F}^0 \oplus \mathfrak{F}^1},$$

$$\forall t > 0 \quad \tilde{U}^t = U^t\Big|_{\tilde{\mathfrak{U}}}, \quad \tilde{F}^t = F^t\Big|_{\tilde{\mathfrak{F}}}.$$

Lemma 6. *Under the conditions of Theorem 1*

$$\tilde{\mathfrak{U}} = \mathfrak{U}^0 \oplus \mathfrak{U}^1, \quad \tilde{\mathfrak{F}} = \mathfrak{F}^0 \oplus \mathfrak{F}^1.$$

Proof.

We must show that the operator $\tilde{P} = s - \lim\limits_{t\to 0+} \tilde{U}^t$ is a projector. It is continuous due to the Banach-Steinhaus theorem as semigroup is uniformly bounded, and the set $\mathfrak{U}^0 \oplus \mathfrak{U}^1$, which obviously defined by $\tilde{P}$ by Theorems 1 and 2, is dense in the space $\tilde{\mathfrak{U}}$.

Further, from the continuity of the operator $\tilde{P}$ we get

$$\forall u \in \tilde{\mathfrak{U}} \quad \tilde{P}^2 u = \tilde{P} \lim_{k\to\infty} \tilde{P}(u_k^0 + u_k^1) = \tilde{P} \lim_{k\to\infty} u_k^1$$

$$= \lim_{k\to\infty} \tilde{P} u_k^1 + \lim_{k\to\infty} \tilde{P} u_k^0 = Pu.$$

Here $u_k^l \in \mathfrak{U}^l, \quad l = 0, 1.$

For $u \in \mathfrak{U}^0 \quad \tilde{P}u = 0$. Let $u \in \ker \tilde{P}$, i.e.

$$0 = \tilde{P}u = \lim_{k\to\infty} \tilde{P}(u_k^0 + u_k^1) = \lim_{k\to\infty} u_k^1.$$

Therefore $u = \lim\limits_{k\to\infty} u_k^0 \in \mathfrak{U}^0$, as $\mathfrak{U}^0$ is closed.

By Theorem 2 $\mathfrak{U}^1 \subset \mathrm{im}\tilde{P}$. Take a vector $u \in \mathrm{im}\tilde{P}$. Then, for some $v \in \tilde{\mathfrak{U}} \quad u = \tilde{P}v$. Taking into account the idempotency of the operator $\tilde{P}$ we get

$$\tilde{P}u = \tilde{P}^2 v = \tilde{P}v = u.$$

Hence $\mathrm{im}P \subset \mathfrak{U}^1$. For the space $\tilde{\mathfrak{F}}$ lemma is proved similarly using a projector

$$\tilde{Q} = s - \lim_{t\to 0+} \tilde{F}^t.$$

□

By Theorems 2 and 3, Remark 5 and Lemma 6 it follows that for the semigroup $\{\tilde{U}^t : t \in \mathbb{R}_+\} \quad (\{\tilde{F}^t : t \in \mathbb{R}_+\})$ we can define the unit

$$\tilde{U}^0 = \tilde{P} = s - \lim_{t\to 0+} U^t \in \mathcal{L}(\tilde{\mathfrak{U}}) \qquad (\tilde{F}^0 = \tilde{Q} = s - \lim_{t\to 0+} F^t \in \mathcal{L}(\tilde{\mathfrak{F}}),$$

which will be a projector on $\tilde{\mathfrak{U}}^1$ $(\tilde{\mathfrak{F}}^1)$ along $\tilde{\mathfrak{U}}^0$ $(\tilde{\mathfrak{F}}^0)$.

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О ЯДРАХ И ОБРАЗАХ ВЫРОЖДЕННЫХ АНАЛИТИЧЕСКИХ РАЗРЕШАЮЩИХ ПОЛУГРУПП В КВАЗИСОБОЛЕВЫХ ПРОСТРАНСТВАХ

Дж.К.Т. Аль Исави

Теория голоморфных вырожденных полугрупп операторов была построена ранее в банаховых пространствах и пространствах Фреше. Однако, изучение математических объектов как правило требует максимального обобщения теории. В статье теория голоморфных вырожденных полугрупп операторов переносится в квазисоболевы пространства последовательностей, которые являются квазинормированными и более того, квазибанаховыми пространствами.

Статья кроме введения и списка литературы содержит два параграфа. В первом, строятся аналитические вырожденные полугруппы, содержащая несколько определений, замечаний и теоремы со своими доказательствами. Второй параграф содержит изучение ядер и образов аналитических полугрупп.

Ключевые слова: ядра и образы полугрупп, аналитические вырожденные полугруппы, квазибанаховы пространства, квазисоболевы пространства.

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