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A LINEARIZED MODEL OF VIBRATIONS IN THE DNA MOLECULE IN THE QUASI-BANACH SPACES

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The system of Boussinesq equations, which simulates the vibrations in the DNA molecule, is researched in this paper. Earlier this model with a nonlinear right side was considered in a Banach space. In this paper, the model with a linear one is considered, but in quasi-Banach spaces of sequences. In the article the method of phase space and the theory of $(L,\ p)$-bounded operators, developed by G. A. Sviridyuk and T. G. Sukacheva for first order equations, are used. It is also based on the theory of the Cauchy problem for linear equations of Sobolev type of high order in quasi-Banach spaces. A mathematical model of vibrations in the DNA molecule is reduced to an equation of Sobolev type in a quasi-Sobolev space. We construct the phase space of a system of linear Boussinesq equations. Sufficient conditions for the solvability of the initial boundary value problem for the Boussinesq equations in terms of the theory of $(L,\ p)$-bounded operators are obtained.

Keywords: Sobolev type equation; quasi-Sobolev space; a mathematical model of vibrations in the DNA molecule.

Introduction

The system of the equations

$$\begin{cases} \dfrac{\rho}{a}u_{tt} = \beta u_{xx} + \dfrac{\beta}{2}(w^2)_{xx} + \dfrac{\rho}{a}\dfrac{l^2}{12}u_{xxtt}, \\ \dfrac{\rho}{a}w_{tt} = \dfrac{\beta}{2}(w^3)_{xx} + \dfrac{\rho}{a}\dfrac{l^2}{12}w_{xxtt} \end{cases}$$

simulates vibrations in DNA molecules. This model was described in [1]. The members of the third order were saved during the derivation of this system. Also it was assumed that the interaction between parts of the molecule is subjected to potential Toda $(a/b)e^{-bx}$. The functions $u(x,t)$ and $w(x,t)$ describe the longitudinal and transverse strain, respectively, and the constants ρ, β, a, l characterize properties such as linear density, strength of intermolecular interactions in the initial time, size of the molecule.

We supplement the system with initial and boundary conditions and consider a model in a more general way. Let $\Omega \subset \mathbb{R}^n$ be bounded domain with with boundary $\partial\Omega$ of class C^∞. In the cylinder $\partial\Omega \times \mathbb{R}$ we consider the Cauchy – Dirichlet problem

$$\begin{array}{ll} u(x,0) = u_0(x), & \dot{u}(x,0) = u_1(x), \\ v(x,0) = v_0(x), & \dot{v}(x,0) = v_1(x), \end{array} \quad x \in \Omega, \tag{1}$$

$$u(x,t) = v(x,t) = 0, \quad (x,t) \in \partial\Omega \times \mathbb{R} \tag{2}$$

for the system of the equations

$$\begin{cases} (b+\Delta)\ddot{u} = a\Delta u - \Delta f(u,v), \\ (b+\Delta)\ddot{v} = d\Delta v - \Delta g(u,v), \end{cases} \tag{3}$$

where $u, v \in C^\infty(\Omega \times \mathbb{R})$. Functions u and v have the same meaning, constants a, b, d characterize the properties of the molecule. The functions f and g are intermolecular interactions, the form of them depends on type of potential between adjacent base pair. The degenerate and nonlinear model given in Banach spaces was considered in [2].

Linearize this model. Suppose that the power of intermolecular interactions is sufficiently low or its potential close to zero. Then we neglect it and set $f(u, v) \equiv 0$, $g(u, v) \equiv 0$. Due to linearization, mathematical model (1)–(3) can be reduced to the Cauchy problem

$$w(0) = w_0, \quad \dot{w}(0) = w_1 \tag{4}$$

for incomplete linear Sobolev type equations of the second order

$$L\ddot{w} = Mw, \tag{5}$$

where operators $L, M \in \mathcal{L}(\mathfrak{U}; \mathfrak{F})$. The problem (4)–(5) was considered in detail by A. A. Zamyshlyaeva in [3] for the case, when $\mathfrak{U}$, $\mathfrak{F}$ are Banach spaces. The aim of this work is to study the model of vibrations in the DNA molecule in quasi-Banach spaces.

1. Advance Information

Let $\mathfrak{U}$ be lineal over field $\mathbb{R}$ and $\mathbf{0}$ be its zero. An ordered pair $({}_{\mathfrak{U}}\|\cdot\|, \mathfrak{U})$ is called quasi-normed space, if the following axioms hold:

1. $\forall u \in \mathfrak{U}\ {}_{\mathfrak{U}}\|u\| \geq 0$, ${}_{\mathfrak{U}}\|u\| = 0 \Leftrightarrow u = \mathbf{0}$;
2. $\forall u \in \mathfrak{U}$, $a \in R\ {}_{\mathfrak{U}}\|au\| = |a|{}_{\mathfrak{U}}\|u\|$;
3. $\forall\ u, w \in \mathfrak{U}\ {}_{\mathfrak{U}}\|u + w\| \leq C({}_{\mathfrak{U}}\|u\| + {}_{\mathfrak{U}}\|w\|)$, where $C \geq 1$.

Generally, the quasi-normed space is not normable, but it is metrizable. Therefore, in the quasi-normed space we can introduce a metric and define convergence, fundamental sequence and completeness. The complete quasi-normed space is called the quasi-Banach space.

Consider one of the most important examples. Suppose that the quasi-Banach spaces are spaces of sequences $u = (u_1, u_2, ...)$ of real numbers l_q, $q \in \mathbb{R}_+$ equipped with the quasi-norm

$${}_q\|u\| = \Big(\sum_{k=1}^{\infty} |u_k|^q\Big)^{(1/q)}.$$

In the future we assume the Banach spaces l_q, $q \in [1, +\infty)$ are the quasi-Banach spaces.

Next, we consider a monotonic sequence $\{\lambda_k\} \subset \mathbb{R}_+$ such as $\lim\limits_{k\to\infty} \lambda_k = +\infty$ and construct the spaces

$$l_q^m = \left\{ u = (u_1, u_2, ...) : \sum_{k=1}^{\infty} \Big(\lambda_k^{\frac{m}{2}} |u_k|\Big)^q < \infty \right\}, \quad m \in \mathbb{R},\ q \in \mathbb{R}_+.$$

The spaces l_q^m are quasi-Banach spaces with the quasi-norm

$${}_q^m\|u\| = \Big(\sum_{k=1}^{\infty} \Big(\lambda_k^{\frac{m}{2}} |u_k|\Big)^q\Big)^{\frac{1}{q}},$$

and embedding $l_q^m \hookrightarrow l_q^n$ is dense and continuous for all $m \geq n$ [4]. Therefore, the space thus obtained is called a quasi-Sobolev.

In addition, we actively use the theory of (L,p)-bounded operators developed to Banach spaces.

Let $\mathfrak{U}, \mathfrak{F}$ be quasi-Banach spaces. Denote L-resolvent set as $\rho^L(M) = \{\mu \in \mathbb{C} : (\mu L - M)^{-1} \in \mathcal{L}(\mathfrak{F}; \mathfrak{U})\}$ and L-spectrum of the operator M as $\sigma^L(M) = \mathbb{C} \setminus \rho^L(M)$. The operator M is called (L,σ)-*bounded operator*, if L-spectrum $\sigma^L(M)$ of the operator M is bounded. If the operator M is (L,σ)-bounded, then projectors P and Q exist and are defined by the formulas

$$P = \frac{1}{2\pi i} \int\limits_{\gamma} R_\mu^L(M) d\mu \in \mathcal{L}(\mathfrak{U}), \quad Q = \frac{1}{2\pi i} \int\limits_{\gamma} L_\mu^L(M) d\mu \in \mathcal{L}(\mathfrak{F}).$$

Here $R_\mu^L(M) = (\mu L - M)^{-1} L$ is *the right*, and $L_\mu^L(M) = L(\mu L - M)^{-1}$ is *the left L-resolvent* of the operator M, while the closed contour $\gamma \subset \mathbb{C}$ bounds a region including $\sigma^L(M)$. Set $\mathfrak{U}^0 = \ker P$, $\mathfrak{U}^1 = \mathrm{im} P$, $\mathfrak{F}^0 = \ker Q$, $\mathfrak{F}^1 = \mathrm{im} Q$ and denote the restrictions of L and M to the subspaces $\mathfrak{U}^k$, $k = 0, 1$ by L_k and M_k.

Theorem 1. (Splitting theorem [4]) *Let the operator M be (L,σ)-bounded. Then there exist*
(i) the operators L_k , M_k: $\mathfrak{U}^k \to \mathfrak{F}^k$, $k = 0, 1$;
(ii) the operator M_0^{-1} $\mathcal{L}(\mathfrak{F}^0, \mathfrak{U}^0)$;
(iii) the operator L_1^{-1} $\mathcal{L}(\mathfrak{F}^1, \mathfrak{U}^1)$;
(iv) the operator M_1^{-1} $\mathcal{L}(\mathfrak{F}^1, \mathfrak{U}^1)$.

We construct the operators $H = M_0^{-1} L_0 \in \mathcal{L}(\mathfrak{U}^0)$ and $S = L_1^{-1} M_1 \in \mathcal{L}(\mathfrak{U}^1)$. The operator M is called (L,p)-*bounded,* $p \in \{0\} \cup \mathbb{N}$, if ∞ is removable singular point (i.e. $H \equiv \mathbb{O}, p = 0$) or a pole of order $p \in \mathbb{N}$ (i.e. $H^p \neq \mathbb{O}$, $H^{p+1} \equiv \mathbb{O}$) of the *L-resolvent* $(\mu L - M)^{-1}$ of the operator M [5,6].

2. Reduction to an Equation of Sobolev Type

Definition 1. *A set $\mathfrak{P}$ is called the phase space of equation (5), if the following conditions hold*
(i) there exists a unique solution of the problem (4), (5) for all $(w_0, w_1) \in \mathfrak{P}$;
(ii) each solution $w = w(t)$ of equation (5) lies in the set $\mathfrak{P}$ as trajectory, i.e. $u(t) \in \mathfrak{P}$ when $t \in (-\tau, \tau)$.

Reduce the mathematical model (1)–(3) to the Cauchy problem (4) for the Sobolev type equation (5). To this end, we introduce spaces.

Let spaces $\mathfrak{U} = l_q^{m+2}$ and $\mathfrak{F} = l_q^m$ be quasi-Sobolev spaces of sequences. On $\mathfrak{U}$ define the Laplace quasi-operator $\Lambda u = (\lambda_1 u_1, \lambda_2 u_2, ...)$ $\Lambda : l_q^{m+2} \to l_q^m$ which is continuous as $\mathcal{L}(l_q^{m+2}; l_q^m)$ for all $q \in \mathbb{R}_+$, $m \in \mathbb{R}$. [4].

Define the vector function in the introduced spaces

$$w = \begin{pmatrix} u \\ v \end{pmatrix}$$

and construct the operators L, M

$$L = \begin{pmatrix} b+\Lambda & 0 \\ 0 & b+\Lambda \end{pmatrix}, \quad M = \begin{pmatrix} a\Lambda & 0 \\ 0 & d\Lambda \end{pmatrix}.$$

The following theorem was proved in paper [7].

Theorem 2. *Let an operator M be (L,p)-bounded operator, $p \in \{0\} \cup N$. Then the subspace $\mathfrak{U}^1$ is the phase space of equation (5).*

Let us prove an auxiliary lemma

Lemma 1. *The operator M is (L,p)-bounded operator for all $b \in \mathbb{R}$.*

Proof.
We must show that the operator L is Fredholm and that M-adjoint vectors of the operator L are not exist. Let a sequence $\{\lambda_k\}_{k=1}^{\infty}$ be monotonous and converges to $+\infty$, and fix b. Two cases are possible. The first case is $-b \notin \sigma(\Lambda)$, then $\ker L = \{0\}$ and the statement is obvious. The second case is $-b \in \sigma(\Lambda)$. There exists a unique eigenvalue $\lambda_l = -b$ of an operator $b + \Lambda$, because the sequence $\{\lambda_k\}_{k=1}^{infty}$ is monotonous, and the corresponding eigenfunction has the form $e_l = (0, \dots, 1, \dots, 0)$, when the unit stands at l location. Thus,

$$\ker L = \operatorname{span}\left\{ \begin{pmatrix} 0 \\ e_l \end{pmatrix}, \begin{pmatrix} e_l \\ 0 \end{pmatrix} \right\}.$$

Now we show that vector ψ_k of the kernel of the operator L has no M-adjoint vectors. By definition, M-adjoint vector of eigenvector of the operator L, is a vector φ_k, such that the equality $M\varphi_k = L\varphi_{k+1}$ is holds.

Let $-b \in \sigma(\Lambda)$, then we construct an image of the operator L

$$\mathrm{im} L = \{f \in \mathfrak{F} : (f, \psi_k)_{\mathfrak{U}} = 0\}$$

where

$$(w_1, w_2)_{\mathfrak{U}} = \langle u_1, u_2 \rangle_{l_2} + \langle v_1, v_2 \rangle_{l_2}.$$

$$M\left(\sum_{k=1}^{m} a_k \psi_k\right) = \sum_{k=1}^{m} a_k \begin{pmatrix} a\Lambda\psi_{k_1} \\ d\Lambda\psi_{k_2} \end{pmatrix} = \begin{pmatrix} -ab \sum\limits_{k=1}^{m} a_k \psi_k \\ -db \sum\limits_{k=1}^{m} a_k \psi_k \end{pmatrix} \notin \mathrm{im} L,$$

if

$$\sum_{k=1}^{m} |a_k| > 0 \text{ and } a^2 + d^2 \neq 0.$$

□

Thus the conditions of Theorem 2 and following statement is true.

Theorem 3. *Let $a^2 + d^2 \neq 0$. Then there exists a unique solution of the linearized problem (1)–(3).*

In the future, develop an algorithm of a numerical solution based on the Galerkin's modified method, which is investigated in the works of S.I. Kadchenko for example [8], is planed.

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ЛИНЕАРИЗОВАННАЯ МОДЕЛЬ КОЛЕБАНИЙ В МОЛЕКУЛЕ ДНК В КВАЗИБАНАХОВЫХ ПРОСТРАНСТВАХ

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Статья посвящена исследованию системы уравнений Буссинеска, моделирующей колебания в молекуле ДНК. Ранее данная модель рассматривалась в банаховых пространствах с нелинейной правой частью. В настоящей статье модель рассматривается с линейной правой частью, но в квазибанаховых пространствах последовательностей. В статье применяется метод фазового пространства и теория $(L,\ p)$-ограниченных операторов, разработанная Г.А. Свиридюком и Т.Г. Сукачевой, для уравнений первого порядка. Мы также опираемся на теорему о разрешимости задачи Коши для линейного уравнения соболевского типа высокого порядка в квазибанаховых пространствах. Математическая модель колебаний в молекуле ДНК редуцируется к уравнению соболевского типа в квазисоболевых пространствах. Строится фазовое пространство системы линейных уравнений Буссинеска. Получены достаточные условия разрешимости начально-краевой задачи для системы уравнений Буссинеска в терминах теории $(L,\ p)$-ограниченных операторов.

Ключевые слова: уравнения соболевского типа, квазисоболевкое пространство, математическая модель колебаний в молекуле ДНК.

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