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NUMERICAL MODELING OF QUASI-STEADY PROCESS IN CONDUCTING NONDISPERSIVE MEDIUM WITH RELAXATION

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Sufficient conditions of existence and uniqueness of weak generalized solution to the Dirichlet–Cauchy problem for equation modeling a quasi-steady process in conducting nondispersive medium with relaxation are obtained. The main equation of the model is considered as a representative of the class of quasi-linear equations of Sobolev type. It enables to prove a solvability of the Dirichlet–Cauchy problem in a weak generalized meaning by methods developed for this class of equations. In suitable functional spaces the Dirichlet–Cauchy problem is reduced to the Cauchy problem for abstract quasi-linear operator differential equation of the special form. Algorithm of numerical solution to the Dirichlet–Cauchy problem based on the Galerkin method is developed. Results of computational experiment are provided.

Keywords: Galerkin method, quasi-linear Sobolev type equation, weak generalized solution, numerical modeling.

Introduction

Assume that $\Omega \subset \mathbb{R}^n, n \geq 2$ is a bounded region with boundary of class C^∞. Consider the Dirichlet–Cauchy problem

$$u(x,t) = 0, (x,t) \in \partial\Omega \times (0,\tau), \tag{1}$$

$$u(x,0) = u_0(x), x \in \Omega, \tag{2}$$

for the equation

$$(\Delta u - \Phi(u))_t = \Phi(u). \tag{3}$$

in cylinder $\Omega \times T$, $T \in \mathbb{R}$.

This problem arises during a research of quasi-steady processes in conducting nondispersive media [1]. Unknown function u corresponds to the electric field potential. Function $\Phi(u) \equiv |u|^{p-2}u, p > 2$ is monotonely increasing and smooth. Problem (1) – (3) was considered earlier in the [2], global solvability in strong generalized meaning was established under some conditions. We consider the equation (3) as a representative of the class of quasi-linear equations of Sobolev type. It enables us to prove a solvability of problem (1) – (3) in a weak generalized meaning by methods developed for this class of equations.

In suitable functional spaces we reduce problem (1) – (3) to the Cauchy problem

$$u(0) = u_0 \tag{4}$$

for abstract operator differential equation of the form

$$\frac{d}{dt}(L(u)) + M(u) = 0, \tag{5}$$

where $L(u) = Au + \lambda M(u)$, $\lambda \in \mathbb{R}_+$. Equation (5) is a quasi-linear Sobolev type equation. Nonsolvable in relation to high derivative equations attract the attention of many researchers [3] – [7]. Problem (4), (5) was considered in the [8], conditions of existence and uniqueness of the weak generalized solution were developed.

The article contains two parts. Reduction of problem (1) – (3) to the abstract problem is developed, and the theorem of existence and uniqueness of weak generalized solution to problem (1) – (3) is provided in the first part. Results of the computational experiment based on the theoretical results are provided in the second part.

1. Solvability

Introduce some definitions and assumptions necessary for further consideration.

Assume that $\mathfrak{H} = (\mathfrak{H}, \langle \cdot, \cdot \rangle)$ is a real Hilbert space identified with its dual and equipped with dual pairs of reflexive Banach spaces $\mathfrak{U} \equiv (\mathfrak{U}, \| \cdot \|)$, $\mathfrak{U}^* \equiv (\mathfrak{U}, \| \cdot \|_*)$, $\mathfrak{F} \equiv (\mathfrak{F}, \| \cdot \|)$ and $\mathfrak{F}^* \equiv (\mathfrak{F}, \| \cdot \|_*)$ such that we have a continuous dense embedding

$$\mathfrak{U} \hookrightarrow \mathfrak{F} \hookrightarrow \mathfrak{H} \hookrightarrow \mathfrak{F}^* \hookrightarrow \mathfrak{U}^*. \tag{6}$$

Definition 1. *Refer as a weak generalized solution to the Cauchy problem* (4), (5) *to a function* $u(t) \in L_\infty(0, \tau, \mathfrak{U})$, *with* $\frac{du}{dt} \in L_2(0, \tau, \mathfrak{U})$, *satisfying*

$$\int_0^\tau \left(\frac{d}{dt} \langle L(u), w \rangle + \langle M(u), w \rangle\right)\varphi(t)dt = 0,$$

$$u(0) = u_0, \forall w \in \mathfrak{U}, \forall \varphi \in L_2(0, \tau).$$

Condition 1. $\exists F(s) \geq 0$ *for almost all* $s \in [0, \infty)$, *such that* $F \in C[0, \infty)$ *possibly after a change on a negligible set, and for almost all* $s_0 \in [0, \infty)$, *for any* $u = u(s_0), v = v(s_0) \in \mathfrak{U}$ *condition*

$$\|M(u) - M(v)\|_* \leq F(s_0)\|u - v\|.$$

is satisfied.

Condition 2. $\exists\, C^M > 0$, *and* $\exists\, p \geq 2$ *such that* $\|M(u)\|_* \leq C^M \|u\|^{p-1}$ $\forall u \in \mathfrak{U}$ *and* $\langle M(u), u \rangle \geq 0$.

Assume that $M \in C^{r+1}(\mathfrak{F}; \mathfrak{F}^*)$, $r \in \mathbb{N}$, is s-monotonous, homogeneous of degree k and satisfies to conditions 1 and 2, and, furthermore, Fréchet derivative of the operator M is symmetric, and the operator $A \in \mathfrak{L}(\mathfrak{U}; \mathfrak{U}^*)$ is symmetric and positive definite.

Theorem 1. *Suppose that the unique local solution to problem* (4), (5) *exists for some interval* $(-\tau_0, \tau_0)$, $\tau_0 \in \mathbb{R}_+$. *Then there exists a unique weak generalized solution to problem* (4), (5).

Proof.

The proof is completely similar to one provided in [8], except the requirement of p-coercivity of operator M is replaced by the weaker condition 2.

□

To reduce problem (1) – (3) to problem (4), (5) assume $\mathfrak{H} = L_2(\Omega)$, $\mathfrak{U} = \overset{0}{W}{}_2^1(\Omega)$ and $\mathfrak{F} = L_p(\Omega)$. Note that we have continuous dense embeddings (6) because of the Sobolev embedding theorem [9, p. 53].

Define operators A and M as follows:

$$\langle Au, v\rangle = \int\limits_\Omega \nabla u \nabla v \, dx, u, v \in \mathfrak{U},$$

$$\langle M(u), v\rangle = \int\limits_\Omega |u|^{p-2} uv dx, \quad u, \, v \in \mathfrak{F}.$$

Lemma 1. *Operator $A : \mathfrak{U} \to \mathfrak{U}^*$ is linear, positive definite, symmetric and continuous.*

Lemma 2. *Operator $M \in C^2(\mathfrak{F}; \mathfrak{F}^*)$, is s-monotonous, homogeneous of degree k and satisfies to conditions 1 and 2, and Fréchet derivative of the operator M is symmetric.*

Proof.

First show the effect of the operator $M : \mathfrak{F} \to \mathfrak{F}^*$. Because of the Hölder's inequality and embeddings (6) we have

$$|\langle M(u), v\rangle| \leq \int\limits_\Omega |u|^{p-1}|v| \, dx \leq \|u\|_{L_p}^{p-1} \|v\|_{L_p}.$$

therefore

$$\|M(u)\|_* = \sup_{\|v\|=1} |\langle M(u), v\rangle| \leq C \|u\|_{L_p}^{p-1}, \tag{7}$$

i.e, operator $M : \mathfrak{F} \to \mathfrak{F}^*$ actually. Moreover, operator M satisfies to condition 2 because of (7) and

$$\langle M(u), u\rangle = \int\limits_\Omega |u|^p dx \geq 0.$$

It is evident that operator M is homogeneous of degree $p - 1$.

Further, develop the Fréchet derivative M'_u of the operator M. At the point $u \in \mathfrak{F}$ it is defined by formula

$$\langle M'_u v, w\rangle = (p-1) \int\limits_\Omega |u|^{p-2} vw \, dx, u, v, w \in \mathfrak{F}$$

and is symmetric. Because of the Hölder's inequality and embeddings (6) we have

$$|\langle M'_u v, w\rangle| = (p-1) \int\limits_\Omega |u|^{p-2}|vw| \, dx \leq (p-1)\|u\|_{L_p}^{p-2} \|v\|_{L_p} \|w\|_{L_p},$$

operator $M'_u \in \mathfrak{L}(\mathfrak{F}; \mathfrak{F}^*)$ for all $u \in \mathfrak{F}$. Prove the s-monotonity of operator M:

$$\langle M'_u v, v\rangle = (p-1) \int\limits_\Omega |u|^{p-2} v^2 \, dx > 0, u, v \in \mathfrak{F} \setminus \{0\}.$$

Demonstrate the inclusion $M \in C^2(\mathfrak{F};\mathfrak{F}^*)$:

$$|\langle M''_u(v,w),z\rangle| \le (p-1)(p-2)\alpha\|u\|^{p-3}_{L_p}\|v\|_{L_p}\|w\|_{L_p}\|z\|_{L_p}.$$

Finally, prove that operator M satisfies to condition 1.

There exists a nonnegative continuous function $g:\mathbb{R}^2\to\mathbb{R}$ such that

$$||u|^{p-2}u-|v|^{p-2}v| \le g(u,v)|u-v|. \tag{8}$$

for all $u,v\in\mathbb{R}$. It is easy to show that, for instance, the function

$$g(u,v)=\begin{cases}\dfrac{||u|^{p-2}u-|v|^{p-2}v|}{|u-v|} & \text{for } u\ne v,\\ (p-2)|u|^{p-1} & \text{for } u=v\end{cases}$$

satisfies these conditions. By (8), for all real-valued functions $u=u(x,s)$, $v=v(x,s)\in\mathfrak{U}$

$$||u|^{p-2}u-|v|^{p-2}v| \le f(x,s)|u-v| \tag{9}$$

almost everywhere, where $f(x,s)=g(u(x,s),v(x,s))$. It follows from (9) that

$$\sup_{\|w\|=1}\alpha\int\limits_\Omega ||u|^{p-2}u-|v|^{p-2}v||w|dx \le \sup_{\|w\|=1}\alpha\int\limits_\Omega f(x,s)|u-v||w|dx.$$

In the left-hand side we have

$$\sup_{\|w\|=1}\alpha\int\limits_\Omega ||u|^{p-2}u-|v|^{p-2}v||w|dx \ge \sup_{\|w\|=1}\alpha\left|\int\limits_\Omega |u|^{p-2}uwdx-\int\limits_\Omega |v|^{p-2}vwdx\right| =$$

$$=\sup_{\|w\|=1}|\langle M(u)-M(v),w\rangle| = \|M(u)-M(v)\|_*.$$

In the right-hand side we have

$$\sup_{\|w\|=1}\int\limits_\Omega f(x,s)|u-v||w|dx = \sup_{\|w\|=1}\langle f(x,s)|u-v|,|w|\rangle \le \sup_{\|w\|=1}\|f(x,s)|u-v|\|_{L_2}\|w\|_{L_2} \le$$

$$\le A\|f(x,s)|u-v|\|_{L_2} \le C\|f(x,s)\|_{L_2}\|u-v\|_{L_2} \le F(s)\|u-v\|_{L_p}.$$

Hence,

$$\|M(u)-M(v)\|_* \le F(s)\|u-v\|.$$

□

Existence of the unique local solution to the problem (1) – (3) was proved in [9] for any initial conditions assuming that $p>2$.

Hence, the next theorem holds.

Theorem 2. *Assume that $2<p\le frac2nn-2$, then for any $u_0\in \overset{0}{W}{}^1_2(\Omega)$ and for any $\tau\in\mathbb{R}_+$ there exists a unique weak generalized solution to problem (1) – (3).*

2. Computational Experiment

Algorithm of numerical solution to problem (1) – (3) and modeling a quasi-steady process in conducting medium with relaxation was developed and implemented in Maple 15.0 environment basing on the theoretical results. The developed program allows us:

1. To specify initial condition $u_0(r, \phi)$, radius R of circle in which the problem is solved, number N of Galerkin approximations.

2. To find an approximate solution to the Dirichlet–Cauchy in the circle with initial conditions specified.

3. To show the graph of the approximate solution on the display.

For example, let us find a numerical solution to problem (1) – (3) in the circle of radius $R = 1$ with conditions: $u_0 = 1 - r^2$, $\Phi(u) = u^3(r, \phi, t)$. Initial and boundary conditions are symmetric (independent of the variable ϕ). Provide the problem (1) – (3) with formulated conditions:

$$\begin{cases} \left(\frac{1}{r}\left(r(u(r,t))_r\right)_r\right)_t - (u^3(r,t))_t = u^3(r,t), \\ u(r,0) = 1 - r^2, \\ u(1,t) = 0, \text{ при } \phi \in [0; 2\pi]. \end{cases} \tag{10}$$

Define the set of eigenfunctions of homogeneous Dirichlet problem for Laplace operator in the circle of radius $R = 1$ orthonormal with scalar product in space $L_2(\Omega)$ as $\{\Phi_k\}$. We represent an unknown function in the form of Galerkin summ:

$$u(r,t) = \sum_{i=k}^{\infty} u_k(t)\Phi_k(r).$$

Let us find the approximate solution with 2 Galerkin approximations in the summ:

$$u(r,t) = \frac{\sqrt{2}}{J_1(\mu_1^0)} J_0(r\mu_1^0)u_1(t) + \frac{\sqrt{2}}{J_1(\mu_2^0)} J_0(r\mu_2^0)u_2(t),$$

where $\mu_i^{(k)}$ is a i-th zero of J_k function. Substitute this representation to the equation. Taking the scalar product with eigenfunctions of Laplace operator, we get differential system for the coefficients $u_1(t)$ and $u_2(t)$. Solving this system numerically, we get the approximate solution to the problem (10). Graphs of the approximate solution at various time points ($t = 0$, $t = 5$, $t = 10$, $t = 50$) are shown in Figure 1.

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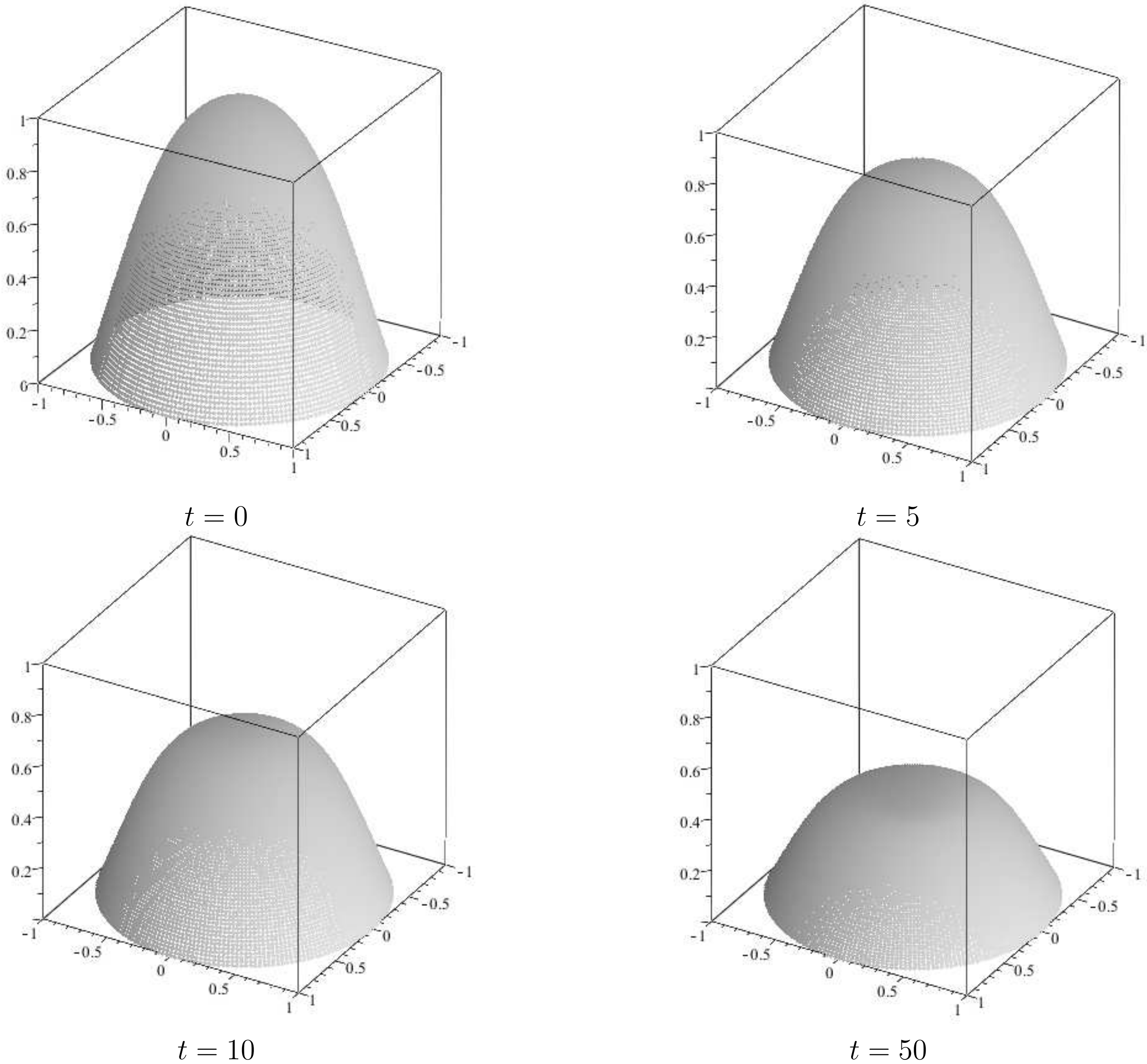

Fig. 1. The electric field potential at various time points

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