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THE CONVERGENCE OF APPROXIMATE SOLUTIONS OF THE CAUCHY PROBLEM FOR THE MODEL OF QUASI-STEADY PROCESS IN CONDUCTING NONDISPERSIVE MEDIUM WITH RELAXATION

E. A. Bogatyreva, South Ural State University, Chelyabinsk, Russian Federation, bogatyrevaea@susu.ru.

This article deals with numerical method for solving of the Dirichlet–Cauchy problem for equation modeling the quasi-steady process in conducting nondispersive medium with relaxation. This problem describes a complex electrodynamic process, allows to consider and predict its flow in time. The study of electrodynamic models is necessary for the development of electrical engineering and new energy saving technologies. The main equation of the model is considered as a quasi-linear Sobolev type equation. The convergence of approximate solutions obtained from the use of the method of straight lines with ε-embedding method and complex Rosenbrock method is proven in the article. The lemmas on the local error and on the distribution of error are proven. Estimates of a global error of the method are obtained.

Keywords: Rosenbrock method, quasi-linear Sobolev type equation, weak generalized solution, numerical solution.

Introduction

During a research of quasi-steady processes in conducting nondispersive media [1], the Dirichlet–Cauchy problem

$$u(x,t) = 0, (x,t) \in \partial\Omega \times (0,\tau), \tag{1}$$

$$u(x,0) = u_0(x), x \in \Omega, \tag{2}$$

arises for the equation modeling the quasi-steady process in conducting nondispersive medium with relaxation

$$(\Delta u - \Phi(u))_t = \Phi(u). \tag{3}$$

Here $\Omega \subset \mathbb{R}^n$ is a bounded domain with boundary of class C^∞ representing an ideal conductivity domain, $\tau \in \mathbb{R}_+$, and an unknown function u represents a potential of the electric field. Function $\Phi(u) \equiv |u|^{p-2}u, p > 2$ is monotonely increasing and smooth. Problem (1) – (3) was considered earlier in the [2], global solvability in strong generalized meaning was established under some conditions. A number of similar, both in physical interpretation and nature, mathematical models was considered in [3]. We consider the equation (3) as a quasi-linear Sobolev type equation. It allows us to prove a solvability of problem (1) – (3) in a weak generalized meaning by methods developed for this class of equations [4].

Applied nature of the problem causes the necessity of numerical modeling of the process described by the problem (1) – (3). However the model is nonlinear; nonlinearity creates significant difficulties in its consideration. But authors point out that such models, in some cases, describe the physical process more qualitatively than simple linear analogues. Since finding the analytical solutions and use of known numerical methods for such models usually is not possible, development of new numerical methods and proof of their convergence becomes important. In the article the method of straight lines with ε-embedding method and complex Rosenbrock method are used to solve the problem (1) – (3) numerically. The lemmas on the local error and on the distribution of error are proven. Estimates of a global error of the method are obtained.

1. Convergence

Consider the case of one space variable

$$\begin{gathered}(x_{ss}-|x|^{p-2}x)_t=|x|^{p-2}x,\\ x(0,t)=0,\\ x(l,t)=0,\\ x(s,0)=x_0(s),s\in(0,l).\end{gathered}\tag{4}$$

Perform a decomposition of the problem (4). Suppose $w=x_{ss}-|x|^{p-2}\,x$. We obtain the problem

$$\begin{gathered}w_t=-w+x_{ss},\\ 0=w-x_{ss}+|x|^{p-2}x,\\ x(s,t)=0,(s,t)\in\partial\Omega\times(0,T),\\ x(s,0)=x_0(s),s\in\Omega,\\ w(s,t)=0,(s,t)\in\partial\Omega\times(0,T),\\ w(s,0)=x_{0ss}(s)-|x_0|^{p-2}x_0,s\in\Omega,\end{gathered}\tag{5}$$

which is equivalent to (4).

The finite difference method is suitable for this problem. Proof of convergence of the method will be performed in a grid norm C_{w_h} ($\|U\|_{C_{w_h}}=\max\limits_{1\le i\le N-1}|u_i|$). The idea of the proof is similar to the idea of the proof of method convergence in [3]. We turn to the differential-algebraic system:

$$\begin{cases}\frac{d}{dt}W=-W+MX,\\ 0=W-MX+|X|^{p-2}\,X.\end{cases}\tag{6}$$

The resulting system will be solved by one-step Rosenbrock method with coefficient $\alpha=\frac{1}{2}+\frac{1}{2}i$. This method and its application to differential-algebraic systems by means of the ε-embeddings method was examined in detail in [5]. The reason of a choice of complex coefficient was also given there.

We prove a theorem on the local error.

Theorem 1. *There exist τ_0,h_0 such that for $\forall\tau\in(0,\tau_0)$, $\forall h\in(0,h_0)$ a local error of the method for the problem* (5) *satisfies the estimates:*

$$\left\|\hat{W}-w(t+\tau)\right\|_{C_{w_h}}\le C\tau(\tau^2+h^2),$$

$$\left\|\hat{X} - x(t+\tau)\right\|_{C_{w_h}} \leq C(\tau^2 + h^2).$$

Proof.

Let X, W, W_t, X_t be values of the exact solution (5), taken in points of the spatial grid at time t_n. On the next time step in accordance with the Rosenbrock method we have

$$\begin{cases} \hat{W} = W + \mathrm{Re}\vec{k}, \\ \hat{X} = X + \mathrm{Re}\vec{l}. \end{cases} \tag{7}$$

Vectors $\vec{k}, \vec{l}$ are determined from the system of linear algebraic equations

$$\begin{cases} \vec{k} + \alpha\tau\vec{k} - \alpha\tau M\vec{l} = \tau(-W + MX), \\ -\alpha\vec{k} + \alpha(M - (p-1)\left|X\right|^{p-2})\vec{l} = W - MX + \left|X\right|^{p-2} X, \end{cases}$$

τ is a step of the time grid.

Suppose that

$$\vec{k} = \tau W_t + \alpha\tau^2(-W_t + MX_t) + \vec{k_1},$$
$$\vec{l} = \tau X_t + \vec{l_1}.$$

Taking into account the order of the local error of differential and algebraic component of ε-embeddings method [5], we obtain the system for $\vec{k_1}$, $\vec{l_1}$:

$$\tau W_t + \alpha\tau^2(-W_t + MX_t) + \vec{k_1} + \alpha\tau(\tau W_t + \alpha\tau^2(-W_t + MX_t) + \vec{k_1}) - \alpha\tau M(\tau X_t + \vec{l_1}) =$$
$$= \tau(-W + MX),$$
$$-\alpha\tau(\tau W_t + \alpha\tau^2(-W_t + MX_t) + \vec{k_1}) + \alpha\tau(M - (p-1)\left|X\right|^{p-2})(\tau X_t + \vec{l_1}) =$$
$$= \tau(W - MX + \left|X\right|^{p-2} X).$$

Further, considering

$$W_t = -W + MX + O(h^2),$$
$$(M - (p-1)\left|X\right|^{p-2})X_t = W_t + O(h^2),$$

we get

$$\vec{k_1} + \alpha\tau\vec{k_1} - \alpha\tau M\vec{l_1} = O(\tau^3),$$
$$-\alpha(\vec{k_1} - (M - (p-1)\left|X\right|^{p-2})\vec{l_1}) = O(h^2 + \tau^2).$$

Let $(M - (p-1)\left|X\right|^{p-2})^{-1} = A$. The operator A according to results in [3] is uniformly bounded in the norm of C_{w_h}. Then

$$\vec{k_1} + \alpha\tau\vec{k_1} - \alpha\tau MA\vec{k_1} = O(\tau h^2 + \tau^3).$$

The operator $S\vec{k_1} = -\alpha\vec{k_1} + \alpha MA\vec{k_1}$ is uniformly bounded. Hence, there exists τ_0 such that the operator $E + \tau S$ with $0 < \tau \leq \tau_0$ has an uniformly bounded inverse $\vec{k_1} = O(\tau^3 + \tau h^2)$. Therefore, $\vec{l_1} = O(\tau^2 + h^2)$, and finally

$$\begin{aligned} \vec{k} &= \tau W_t + \alpha\tau^2(-W_t + MX_t) + O(\tau^3 + \tau h^2), \\ \vec{l} &= \tau X_t + O(\tau^2 + h^2). \end{aligned} \tag{8}$$

Substitute (8) in (6) and take into account that

$$W(t+\tau) = W(t) + \tau W_t + \frac{\tau^2}{2} W_{tt} + O(\tau^3),$$

$$X(t+\tau) = X(t) + \tau X_t + O(\tau^2),$$

$$W_{tt} = -W_t + MX_t,$$

$$\mathrm{Re}\alpha = \frac{1}{2}.$$

This completes the proof.

□

Definition 1. *Let $x(s,t)$ and $w(s,t)$ be classical solution of the problem (6). We say that a point $(t,s,\tilde{w},\tilde{x})$ belongs to δ - neighborhood of a classical solution if $|\tilde{w} - w(s,t)| < \delta$ and $|\tilde{x} - x(s,t)| < \delta$.*

Let (W_0, X_0) and $(\tilde{W}_0, \tilde{X}_0)$ be two pairs of initial data, (W_1, X_1) and $(\tilde{W}_1, \tilde{X}_1)$ be values at the next time layer obtained by the method (7) by ε-embeddings with Rosenbrock scheme with complex coefficients. We have

Lemma 1. *Let $p > 2$. There exists τ_0, h_0 such that for $0 < \tau < \tau_0$, $0 < h < h_0$ following estimates are satisfied*

$$\left\|W_1 - \tilde{W}_1\right\|_{C_w} \leq (1 + \tau L) \left\|W_0 - \tilde{W}_0\right\|_{C_w} + \tau P \left\|X_0 - \tilde{X}_0\right\|_{C_w},$$

$$\left\|X_1 - \tilde{X}_1\right\|_{C_w} \leq Q \left\|W_0 - \tilde{W}_0\right\|_{C_w} + q \left\|X_0 - \tilde{X}_0\right\|_{C_w}.$$

Moreover, constants Q, P, L, q do not depend on the initial data, and we can achieve $q < 1$ by decreasing δ and h.

Proof.

Vectors $\vec{a}, \vec{l}$ are defined by the system

$$\begin{cases} \vec{k} + \alpha\tau\vec{k} - \alpha\tau M\vec{l} = \tau(-W + MX), \\ -\alpha\vec{k} + \alpha(M - (p-1)\,|X|^{p-2})\vec{l} = W - MX + |X|^{p-2}\,X. \end{cases}$$

Expressing $\vec{l}$ from the second equation and substituting in the first equation, we find that there exist τ_0, h_0 such that for $0 < \tau < \tau_0$, $0 < h < h_0$ vector $\vec{k}$ is uniformly bounded in the norm of C_w. It is following from the second equation that $\vec{l}$ is also uniformly bounded. Then, from the first equation it is following that $k = O(\tau)$. Substituting in the second one, we obtain $\vec{l} = O(\tau + \delta + h^2)$. Differentiating the first and the second equations by W, we get

$$\left\|\frac{\partial\vec{k}}{\partial W}\right\|_{C_w} = O(\tau),$$

$$\frac{\partial\vec{l}}{\partial W} = \frac{1}{\alpha}(M - (p-1)\,|X|^{p-2})^{-1} + O(\tau),$$

$$\left\| \frac{\partial \vec{l}}{\partial W} \right\|_{C_w} \leq Q.$$

By differentiating the first and the second equations by X, we get

$$\frac{\partial \vec{k}}{\partial X} = O(\tau),$$

$$\frac{\partial \vec{l}}{\partial X} = -\frac{1}{\alpha}E + (M - (p-1)\left|X\right|^{p-2})^{-1}(p-1)(p-2)\left|X\right|^{p-1} \times$$
$$\times \text{sign}X(M - (p-1)\left|X\right|^{p-2})^{-1} \times (MX - \left|X\right|^{p-2}X) + O(\tau) =$$
$$= -\frac{1}{\alpha} + O(\tau + \delta + h^2).$$

Note that for the Rosenbrock scheme with $\alpha = \frac{1+i}{2}$, the value $(1 - \text{Re}\frac{1}{\alpha}) = 0$.

From (6) it follows that

$$\frac{\partial W_1}{\partial W_0} = E + O(\tau),\ \frac{\partial X_1}{\partial W_0} = O(1),\ \frac{\partial W_1}{\partial X_0} = O(\tau),$$

$$\frac{\partial X_1}{\partial X_0} = E(1 - \text{Re}\frac{1}{\alpha}) + O(\tau + \delta + h^2).$$

Hence, we obtain the lemma, because by the decreasing of τ, δ, h we can achieve

$$\left\| \frac{\partial X_1}{\partial X_0} \right\| \leq q < 1.$$

□

The following Lemma about the distribution of error is proven.

Lemma 2. *If for the initial data (W_0, X_0) and $(\bar{W}_0, \bar{X}_0)$ for $\forall k \in [1, N]$, $N\tau \leq const$, (W_k, X_k) and $(\bar{W}_k, \bar{X}_k)$ are in δ-neighborhood, then*

$$\left\|W_N - \bar{W}_N\right\|_{C_w} \leq C\left(\left\|W_0 - \bar{W}_0\right\|_{C_w} + \tau\left\|X_0 - \bar{X}_0\right\|_{C_w}\right),$$

$$\left\|X_N - \bar{X}_N\right\|_{C_w} \leq C\left(\left\|W_0 - \bar{W}_0\right\|_{C_w} + (\tau + q^N)\left\|X_0 - \bar{X}_0\right\|_{C_w}\right),\ q < 1.$$

Proof.

The proof is similar to the proof given in [3].

□

The global error theorem is proven.

Theorem 2. *There exist τ_0, h_0 such that for $\forall \tau \in (0, \tau_0)$, $\forall h \in (0, h_0)$, for the problem (5), global error of the method satisfies the estimates:*

$$\left\|\bar{W}_N - w(N\tau)\right\|_{C_w} \leq C(\tau^2 + h^2),$$

$$\left\|\bar{X}_N - x(N\tau)\right\|_{C_w} \leq C(\tau^2 + h^2).$$

Proof.

The proof is analogous to the proof given in [3].

□

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Ekaterina A. Bogatyreva, Candidate of Physico-Mathematical Sciences, Department of Equations of Mathematical Phisics, South Ural State University (Chelyabinsk, Russian Federation), bogatyrevaea@susu.ru.

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СХОДИМОСТЬ ЧИСЛЕННЫХ РЕШЕНИЙ ЗАДАЧИ КОШИ ДЛЯ МОДЕЛИ КВАЗИСТАЦИОНАРНОГО ПРОЦЕССА В ПРОВОДЯЩЕЙ СРЕДЕ БЕЗ ДИСПЕРСИИ С УЧЕТОМ РЕЛАКСАЦИИ

Е. А. Богатырева

В работе рассмотрен численный метод решения задачи Коши – Дирихле для уравнения, моделирующего квазистационарный процесс в проводящей среде без дисперсии с учетом релаксации. Данная задача описывает сложный электродинамический процесс, позволяет рассматривать и прогнозировать его течение во времени. Изучение электродинамических моделей необходимо для развития электротехники и разработки новых энергосберегающих технологий. Основное уравнение модели рассматривается как квазилинейное уравнение соболевского типа. Доказана сходимость численных решений, полученных с использованием метода прямых в сочетании с методами ε-вложений и методом Розенброка с комплексным коэффициентом. Получены оценки глобальной ошибки метода.

Ключевые слова: метод Розенброка, квазилинейное уравнение соболевского типа, слабое обобщенное решение, численное решение.

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Богатырева Екатерина Александровна, кандидат физико-математических наук, кафедра уравнений математической физики, Южно-Уральский государственный университет (г. Челябинск, Российская Федерация), bogatyrevaea@susu.ru

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