MSC 60H10, 60H30, 60K30 DOI: 10.14529/jcem1602004

ON EXISTENCE OF SOLUTIONS TO STOCHASTIC DIFFERENTIAL EQUATIONS WITH OSMOTIC VELOCITIES

Yu. E. Gliklikh, yeg@math.vsu.ru,
K. A. Samsonova, kristina-samsonova92@rambler.ru.
[1] Voronezh State University, Voronezh, Russian Federation.

The notion of mean derivatives was introduced by E. Nelson in 60-th years of XX century and at the moment there are a lot of mathematical models of physical processes constructed in terms of those derivatives. The paper is devoted to investigation of stochastic differential equations with osmotic velocities, i.e., with Nelson's antisymmetric mean derivatives. Since the osmotic velocities of stochastic processes shows "how fast the randomness grows up", such research is important for investigation of models of physical processes that take into account stochastic properties. An existence of solution theorem for those equations is obtained.

Keywords: mean derivatives, equations with osmotic velocities, existence of solutions.

Introduction

The notion of mean derivatives was introduced by E. Nelson [1, 2, 3] for the needs of the so-called Nelson's stochastic mechanics (a version of quantum mechanics). Later a lot of applications of mean derivatives to some other branches of science were found. It should be pointed out that among Nelson's mean derivatives (forward, backward, symmetric and antisymmetric, etc.) the symmetric derivatives called current velocities, play the role of natural analogues of physical velocity of deterministic processes. That is why inestigation of equations with with current velocities is very important for stochastic models for many physical processes.

In this paper we investigate those equations and obtain an existence and uniqueness theorem for their solutions.

Some remarks on notations. In this paper we deal with equations and inclusions in the linear space $\mathbb{R}^n$, for which we always use coordinate presentation of vectors and linear operators. Vectors in $\mathbb{R}^n$ are considered as columns. If X is such a vector, the transposed row vector is denoted by X^*. Linear operators from $\mathbb{R}^n$ to $\mathbb{R}^n$ are represented as $n \times n$ matrices, the symbol $*$ means transposition of a matrix (pass to the matrix of conjugate operator). The space of $n \times n$ matrices is denoted by $L(\mathbb{R}^n, \mathbb{R}^n)$.

By $\mathrm{S}(n)$ we denote the linear space of symmetric $n \times n$ matrices that is a subspace in $L(\mathbb{R}^n, \mathbb{R}^n)$. The symbol $\mathrm{S}_+(n)$ denotes the set of positive definite symmetric $n \times n$ matrices that is a convex open set in $\mathrm{S}(n)$. Its closure, i.e., the set of positive semi-definite symmetric $n \times n$ matrices, is denoted by $\bar{\mathrm{S}}_+(n)$.

For the sake of simplicity we consider equations, their solutions and other objects on a finite time interval $t \in [0, T]$.

We use Einstein's summation convention with respect to shared upper and lower indices. By the symbol $\frac{\partial}{\partial x^i}$ we denote both the partial derivative and the vector of basis in the tangent space.

1. Preliminaries on the mean derivatives

Consider a stochastic process $\xi(t)$ in $\mathbb{R}^n$, $t \in [0, l]$, given on a certain probability space $(\Omega, \mathcal{F}, \mathsf{P})$ and such that $\xi(t)$ is L_1-random variable for all t.

Every stochastic process $\xi(t)$ in $\mathbb{R}^n$, $t \in [0, l]$, determines three families of σ-subalgebras of σ-algebra $\mathcal{F}$:
(i) the "past" $\mathcal{P}_t^\xi$ generated by pre-images of Borel sets in $\mathbb{R}^n$ by all mappings $\xi(s) : \Omega \to \mathbb{R}^n$ for $0 \le s \le t$;
(ii) the "future" $\mathcal{F}_t^\xi$ generated by pre-images of Borel sets in $\mathbb{R}^n$ by all mappings $\xi(s) : \Omega \to \mathbb{R}^n$ for $t \le s \le l$;
(iii) the "present" ("now") $\mathcal{N}_t^\xi$ generated by pre-images of Borel sets in $\mathbb{R}^n$ by the mapping $\xi(t)$.
All families are supposed to be complete, i.e., containing all sets of probability 0.

For convenience we denote the conditional expectation of $\xi(t)$ with respect to $\mathcal{N}_t^\xi$ by $E_t^\xi(\cdot)$.

Ordinary ("unconditional") expectation is denoted by E.

Strictly speaking, almost surely (a.s.) the sample paths of $\xi(t)$ are not differentiable for almost all t. Thus its "classical" derivatives exist only in the sense of generalized functions. To avoid using the generalized functions, following Nelson (see, e.g., [1, 2, 3]) we give a definition.

Definition 1. *(i) Forward mean derivative $D\xi(t)$ of $\xi(t)$ at time t is an L_1-random variable of the form*

$$D\xi(t) = \lim_{\Delta t \to +0} E_t^\xi \left(\frac{\xi(t + \Delta t) - \xi(t)}{\Delta t} \right), \tag{1}$$

where the limit is supposed to exists in $L_1(\Omega, \mathcal{F}, \mathsf{P})$ and $\Delta t \to +0$ means that Δt tends to 0 and $\Delta t > 0$.

(ii) Backward mean derivative $D_*\xi(t)$ of $\xi(t)$ at t is an L_1-random variable

$$D_*\xi(t) = \lim_{\Delta t \to +0} E_t^\xi \left(\frac{\xi(t) - \xi(t - \Delta t)}{\Delta t} \right), \tag{2}$$

where the conditions and the notation are the same as in (i).

Note that mainly $D\xi(t) \neq D_*\xi(t)$, but if, say, $\xi(t)$ a.s. has smooth sample paths, these derivatives evidently coinside.

From the properties of conditional expectation (see [4]) it follows that $D\xi(t)$ and $D_*\xi(t)$ can be represented as compositions of $\xi(t)$ and Borel measurable vector fields (regressions)

$$Y^0(t, x) = \lim_{\Delta t \to +0} E \left(\frac{\xi(t + \Delta t) - \xi(t)}{\Delta t} | \xi(t) = x \right),$$

$$Y^0_*(t, x) = \lim_{\Delta t \to +0} E \left(\frac{\xi(t) - \xi(t - \Delta t)}{\Delta t} | \xi(t) = x \right) \tag{3}$$

on $\mathbb{R}^n$. This means that $D\xi(t) = Y^0(t, \xi(t))$ and $D_*\xi(t) = Y^0_*(t, \xi(t))$.

Definition 2. *The derivative $D_S = \frac{1}{2}(D + D_*)$ is called symmetric mean derivative. The derivative $D_A = \frac{1}{2}(D - D_*)$ is called anti-symmetric mean derivative .*

Consider the vector fields

$$v^{\xi}(t,x) = \frac{1}{2}(Y^0(t,x) + Y^0_*(t,x))$$

and

$$u^{\xi}(t,x) = \frac{1}{2}(Y^0(t,x) - Y^0_*(t,x)).$$

Definition 3. *$v^{\xi}(t) = v^{\xi}(t,\xi(t)) = D_S\xi(t)$ is called current velocity of $\xi(t)$;*
$u^{\xi}(t) = u^{\xi}(t,\xi(t)) = D_A\xi(t)$ is called osmotic velocity of $\xi(t)$.

For stochastic processes the current velocity is a direct analogue of ordinary physical velocity of deterministic processes (see, e.g., [1, 2, 3, 5]). The osmotic velocity measures how fast the "randomness" grows up.

Recall that Ito process is a process $\xi(t)$ of the form

$$\xi(t) = \xi_0 + \int_0^t a(s)ds + \int_0^t A(s)dw(s), \tag{4}$$

where $a(t)$ is a process in $\mathbb{R}^n$ whose sample paths a.s. have bounded variation; $A(t)$ is a process in $L(\mathbb{R}^n, \mathbb{R}^n)$ such that for any element $A^j_i(t)$ of matrix $A(t)$ the condition $\mathsf{P}(\omega| \int_0^T (A^j_i)^2 dt < \infty) = 1$ holds; $w(t)$ is a Wiener process in $\mathbb{R}^n$; the first integral is Lebesgue integral, the second one is Itô integral and all integrals are well-posed.

Definition 4. *An Itô process $\xi(t)$ is called a process of diffusion type if $a(t)$ and $A(t)$ are not anticipating with respect to $\mathcal{P}^{\xi}_t$ and the Wiener process $w(t)$ is adapted to $\mathcal{P}^{\xi}_t$. If $a(t) = a(t,\xi(t))$ and $A(t) = A(t,\xi(t))$, where $a(t,x)$ and $A(t,x)$ are Borel measurable mappings from $[0,T]\times\mathbb{R}^n$ to $\mathbb{R}^n$ and to $L(\mathbb{R}^n,\mathbb{R}^n)$, respectively, the Itô process is called a diffusion process.*

Below we are dealing with smooth fields of non-degenerate linear operators $A(x) : \mathbb{R}^n \to \mathbb{R}^n$, $x \in \mathbb{R}^n$ (i.e., (1,1)-tensor field on $\mathbb{R}^n$). Let $\xi(t)$ be a diffusion process in which the integrand under Itô integral is of the form $A(\xi(t))$. Then its diffusion coefficient $A(x)A^*(x)$ is a smooth field of symmetric positive definite matrices $\alpha(x) = (\alpha^{ij}(x))$ ((2,0)-tensor field on $\mathbb{R}^n$). Since all these matrices are non-degenerate and smooth, there exist the smooth field of converse symmetric and positive definite matrices (α_{ij}). Hence this field can be used as a new Riemannian $\alpha(\cdot,\cdot) = \alpha_{ij}dx^i \otimes dx^j$ on $\mathbb{R}^n$. The volume form of this metric has the form $\Lambda_\alpha = \sqrt{\det(\alpha_{ij})}dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n$.

Denote by $\rho^{\xi}(t,x)$ the probability density of random element $\xi(t)$ with respect to the volume form $dt \wedge \Lambda_\alpha = \sqrt{\det(\alpha_{ij})}dt \wedge dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n$ on $[0,T]\times\mathbb{R}^n$, i.e., for every continuous bounded function $f : [0,T]\times\mathbb{R}^n \to \mathbb{R}$ the relation

$$\int_0^T E(f(t,\xi(t)))dt = \int_0^T \left(\int_\Omega f(t,\xi(t))d\mathsf{P}\right)dt = \int_{[0,T]} \left(\int_{\mathbb{R}^n} f(t,x)\rho^{\xi}(t,x)\Lambda_\alpha\right)dt.$$

holds.

Lemma 1. *[[6, 7]] Let $\xi(t)$ satisfy the Ito equation*

$$\xi(t) = \xi_0 + \int_0^t a(s, \xi(s))ds + \int_0^t A(s, \xi(s))dw(s).$$

Then

$$u^\xi(t,x) = \frac{1}{2}\frac{\frac{\partial}{\partial x^j}(\alpha^{ij}\rho^\xi(t,x))}{\rho^\xi(t,x)}\frac{\partial}{\partial x^i}, \tag{5}$$

where (α^{ij}) is the matrix of operator AA^*.

Remark 1. Denote by $\Xi(x)$ the vector field whose coordinate presentation is $\frac{\partial \alpha^{ij}}{\partial x^j}\frac{\partial}{\partial x^i}$. One can easily derive from (5) that $u^\xi(t,x) = \frac{1}{2}Grad \log \rho^\xi(t,x) + \frac{1}{2}\Xi(x)$ where Grad denotes the gradient with respect to metric $\alpha(\cdot,\cdot)$. Indeed,

$$\frac{\frac{\partial}{\partial x^j}(\alpha^{ij}\rho^\xi(t,x))}{\rho^\xi(t,x)}\frac{\partial}{\partial x^i} = \frac{1}{2}\alpha^{ij}\frac{\frac{\partial \rho^\xi}{\partial x^j}}{\rho^\xi}\frac{\partial}{\partial x^i} + \frac{1}{2}\frac{\partial \alpha^{ij}}{\partial x^j}\frac{\partial}{\partial x^i},$$

where $\alpha^{ij}\frac{\frac{\partial \rho^\xi}{\partial x^j}}{\rho^\xi}\frac{\partial}{\partial x^i} = Grad \log \rho^\xi$ and $\frac{\partial \alpha^{ij}}{\partial x^j}\frac{\partial}{\partial x^i} = \Xi$.

Lemma 2. *[[3, 5]] For $v^\xi(t,x)$ and $\rho^\xi(t,x)$ the following interrelation*

$$\frac{\partial \rho^\xi(t,x)}{\partial t} = -Div(v^\xi(t,x)\rho^\xi(t,x)), \tag{6}$$

(known as the equation of continuity) takes place where Div denotes the divergence with respect to Riemannian metric $\alpha(\cdot,\cdot)$.

Following [8, 5] we introduce the differential operator D_2 that differentiates an L_1 random process $\xi(t)$, $t \in [0,T]$ according to the rule

$$D_2\xi(t) = \lim_{\triangle t \to +0} E_t^\xi\left(\frac{(\xi(t+\triangle t) - \xi(t))(\xi(t+\triangle t) - \xi(t))^*}{\triangle t}\right), \tag{7}$$

where $(\xi(t+\triangle t) - \xi(t))$ is considered as a column vector (vector in $\mathbb{R}^n$), $(\xi(t+\triangle t) - \xi(t))^*$ is a row vector (transposed, or conjugate vector) and the limit is supposed to exists in $L_1(\Omega, \mathcal{F}, \mathsf{P})$. We emphasize that the matrix product of a column on the left and a row on the right is a matrix so that $D_2\xi(t)$ is a symmetric positive semi-definite matrix function on $[0,T]\times\mathbb{R}^n$. We call D_2 the quadratic mean derivative.

Theorem 1. *([5, 8]) For an Itô diffusion type process $\xi(t)$ of (4) form the forward mean derivative $D\xi(t)$ exists and equals $E_t^\xi(a(t))$. In particular, if $\xi(t)$ a diffusion process, $D\xi(t) = a(t, \xi(t))$.*

Theorem 2. *([5, 8]) Let $\xi(t)$ be a diffusion type process of (4) form. Then $D_2\xi(t) = E_t^\xi[\alpha(t)]$ where $\alpha(t) = AA^*$ is the diffusion coefficient. In particular, if $\xi(t)$ is a diffusion process, $D_2\xi(t) = \alpha(t, \xi(t))$ where $\alpha = AA^*$ is the diffusion coefficient.*

Lemma 3. *([5, 8]) Let $\alpha(t,x)$ be a jointly continuous (measurable, smooth) mapping from $[0,T]\times\mathbb{R}^n$ to $\mathrm{S}_+(n)$. Then there exists a jointly continuous (measurable, smooth, respectively) mapping $A(t,x)$ from $[0,T]\times\mathbb{R}^n$ to $L(\mathbb{R}^n,\mathbb{R}^n)$ such that for all $t \in R$, $x \in \mathbb{R}^n$ the equality $A(t,x)A^*(t,x) = \alpha(t,x)$ holds.*

2. Main results

Consider C^∞-vector field $u(t,x)$ and $(2,0)$ symmetric autonomous positive definite C^∞-tensor field $\alpha(x)$ on $\mathbb{R}^n$. Recall that the field $\alpha(x)$ is described by the field of symmetric positive definite matrices $(\alpha^{ij})(x)$. Since that field is C^∞-smooth and positive definite, the field of converse matrices (α_{ij}) exists and can be considered as a Riemannian metric (see Section 1). To avoid some technical difficulties, we suppose that $u(t,x)$, $\alpha(x)$ and $\Xi(x)$ (see Remark 1) are uniformly bounded with respect to the corresponding norms.

Definition 5. *The system*

$$\begin{cases} D_A\xi(t) = u(t,\xi(t)), \\ D_2\xi(t) = \alpha(t,\xi(t)) \end{cases} \tag{8}$$

is called the first order differential equation with osmotic velocity.

Definition 6. *We say that (8) has a solution on the interval* $[0,T]$ *if there exists a probability space* $(\Omega,\mathcal{F},\mathsf{P})$ *and a process* $\xi(t)$ *given on* $(\Omega,\mathcal{F},\mathsf{P})$ *for* $t\in[0,T]$*, such that it satisfies (8).*

Taking into account formula (5) and Remark 1, one can easily see that (8) can have a solution not for all right-hand sides since u and α should be connected by the formula $u(t,x) = \frac{1}{2}(\Xi(x) + Grad\log\rho(t,x))$ where $\rho(t,x)$ for every t is a probabilistic density with respect to the Lebesgue measure on $\mathbb{R}^n$. Introduce $p_{(t)}(x) = \log\rho(t,x)$ (hence $\rho(t,x) = e^{p_{(t)}(x)}$) and so we can transform the above expression for u to the form $u(t,x) = \frac{1}{2}Grad\,p_{(t)}(x) + \frac{1}{2}\Xi(x)$ where $\Xi(x)$ is known since its coordinates consist of partial derivatives of known and smooth field $\alpha(x)$. Note that $u(t,x)$ is also known by the hypothesis. Hence, $Grad\,p(t,x) = 2u(t,x) - \Xi(x)$. Denote by $(2u(t,x)-\Xi(x))^i$ the i-th coordinate of vector $2u(t,x)-\Xi(x)$ and by $(dp_{(t)})_j$ the j-th coordinate of 1-form $dp_{(t)}$ (total differential of $p_{(t)}$, here d is external differential). Then $(dp_{(t)})_j = \alpha_{ij}(2u(t,x)-\Xi(x))^i$ and so $dp_{(t)}$ is known.

It is a well-known fact that one can recover $p_{(t)}$ from $dp_{(t)}$ up to an additive constant. Indeed, specify a certain value of $p_{(t)}$ (for simplicity we take this value equal to 0) at the origin in $\mathbb{R}^n$. For any point $x\in\mathbb{R}^n$ construct the value $p_{(t)}(x)$ for an arbitrary specified t as follows. Let σ be a smooth curve connecting 0 and x. Then define $p_{(t)}(x) = \int_\sigma dp_{(t)}$. Take another smooth curve σ_1 connecting 0 and x, and consider the union of σ and σ_1 as the boundary $\partial\Omega$ of an arbitrary specified 2-dimensional sub-manifold Ω in $\mathbb{R}^n$. Then by the Stokes formula (see, e.g., [10]) $\oint_{\partial\Omega} dp_{(t)} = \iint_\Omega ddp_{(t)}$. Since $d^2=0$, $\oint_{\partial\Omega} dp(t) = 0$ and so the value $p_{(t)}(x)$ does not depend on the choice of σ.

Assumption 1. *We suppose that for every* $t\in[0,T]$ *the integral* $\int_{\mathbb{R}^n} e^{p_{(t)}(x)}dx$ *is finite, i.e., is equal to a certain finite constant* $C_{(t)}$ *that is* C^∞*-smooth in* t*.*

Theorem 3. *If Assumption 1 is satisfied, equation* (8) *has a solution.*

Proof.

It follows from Assumtion 1 that $\rho(t,x) = e^{-C_{(t)}}e^{p_{(t)}(x)}$ is a probabilistic density. This allows us to find the current velocity of the solution. Here we use formula (6). Note that $\rho(0,x) = e^{-C_{(0)}}e^{p_{(0)}(x)}$ does not equal zero everywhere in $\mathbb{R}^n$. It is shown in [11] that in this case $p_{(t)}(x) = p_{(0)}(g_{-t}(x)) - \int_0^t (Div\ \ v)(s, g_s(g_{-t}(x))ds,\ \ p_0 = \log\rho_0$ where g_t is the flow of vector field v, the current velocity of solution. From this formula one can easily see that the densities of this sort and C^∞ vector fields with complete flows are in one-to-one correspondence. Thus v is uniquely determined.

Thus we can find the forward mean derivative of the solution by the formula $a(t,x) = v(t,x) + u(t,x)$.

From Lemma 3 and from the hypothesis of Theorem it follows that there exists smooth and uniformly bounded $A(x)$ such that $A(x)A^*(x) = \alpha(x)$. Then from the general theory of equations with forward mean derivatives it follows that $\xi(t)$ having the density $\rho(t,x)$ as above, must satisfy the stochastic differential equation

$$\xi(t) = \xi_0 + \int_0^t a(s,\xi(s))ds + \int_0^t A(s,\xi(s))dw(s). \tag{9}$$

From the hypothesis and from results of [9] it follows that (9) for initial condition with density ρ_0 has a unique strong solution $\xi(t)$ well-posed for $t \in [0,T]$. Thus, by Theorem 2 $D_2\xi(t) = \alpha(\xi(t))$. The fact that $D_A\xi(t) = u(t,\xi(t))$ follows from the construction.

$\square$

This research is supported by Russian Science Foundation (RSF) Grant 14-21-00066, being carried out in Voronezh State University.

References

1. Nelson E. Derivation of the Schrödinger Equation from Newtonian Mechanics. *Physical Review*, 1966, vol. 150, no. 4, pp. 1079–1085. doi: 10.1103/PhysRev.150.1079
2. Nelson E. *Dynamical Theory of Brownian Motion.* Princeton, Princeton University Press, 1967.
3. Nelson E. *Quantum fluctuations.* Princeton, Princeton University Press, 1985.
4. Parthasarathy K.R. *Introduction to Probability and Measure.* New York, Springer–Verlag, 1978.
5. Gliklikh Yu.E. *Global and Stochastic Analysis with Applications to Mathematical Physics.* London, Springer–Verlag, 2011. doi: 10.1007/978-0-85729-163-9
6. Cresson J., Darses S. Stochastic Embedding of Dynamical Systems. *Journal of Mathematical Physics*, 2007, vol. 48, no. 7, pp. 072703-1–072303-54. doi: 10.1063/1.2736519
7. Gliklikh Yu.E., Mashkov E.Yu. Stochastic Leontieff Type Equations and Mean Derivatives of Stochastic Processes. *Bulletin of the South Ural State University. Series: Mathematical Modelling, Programming and Computer Software*, 2013, vol. 6, no. 2, pp. 25–39.
8. Azarina S.V., Gliklikh Yu.E. Differential Inclusions with Mean Derivatives. *Dynamic Systems and Applications*, 2007, vol. 16, no. 1, pp. 49–71.

9. Gihman I.I., Skorohod A.V. *Theory of Stochastic Processes. V. 3.* New York, Springer–Verlag, 1979. doi: 10.1007/978-1-4615-8065-2

10. Sternberg S. *Lectures on Differential Geometry.* N.J., Englewood Cliffs, Prentice Hall, 1964.

11. Azarina S.V., Gliklikh Yu.E. On Existence of Solutions to Stochastic Differential Equations with Current Velocities. *Bulletin of the South Ural State University. Series: Mathematical Modelling, Programming and Computer Software*, 2015, vol. 8, no. 4, pp. 100–106. doi: 10.14529/mmp150408

Yuri E. Gliklikh, Doctor of Physico-Mathematical Sciences, Full Professor, Department of Algebra and Topological Analysis Methods, Voronezh State University (Voronezh, Russian Federation), yeg@math.vsu.ru.

Kristina A. Samsonova, Undergraduate, Department of Algebra and Topological Analysis Methods, Voronezh State University (Voronezh, Russian Federation), kristina-samsonova92@rambler.ru.

Received May 13, 2016

УДК 517.9+519.216.2 **DOI: 10.14529/jcem1602004**

О СУЩЕСТВОВАНИИ РЕШЕНИЙ СТОХАСТИЧЕСКИХ ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ С ОСМОТИЧЕСКИМИ СКОРОСТЯМИ

Ю. Е. Гликлих, К. А. Самсонова

Понятие производных в среднем было введено Э. Нельсоном в 60-х годах XX века и к настоящему моменту имеется много математических моделей физических процессов, построенных в терминах этих производных. Настоящая статья посвящена изучению стохастических дифференциальных уравнений с осмотическими скоростями, т.е. с антисимметрическими производными в среднем по Нельсону. Поскольку осмотические скорости стохастических процессов показывают «насколько быстро нарастает случайность», это исследование важно для изучения моделей физических процессов, которые учитывают стохастические свойства. Получена теорема существования решений для указанных уравнений.

Ключевые слова: производные в среднем, уравнения с осмотическими скоростями, существование решений.

Литература

1. Nelson, E. Derivation of the Schrödinger Equation from Newtonian Mechanics / E. Nelson // Physical Review. – 1966. – V. 150, № 4. – P. 1079–1085.

2. Nelson, E. Dynamical Theory of Brownian Motion / E. Nelson. – Princeton: Princeton University Press, 1967.

3. Nelson, E. Quantum Fluctuations / E. Nelson. – Princeton: Princeton University Press, 1985.

4. Партасарати, К.Р. Введение в теорию вероятностей и теорию меры / К.Р. Партасарати. – М.: Мир, 1988.

5. Gliklikh, Yu.E. Global and Stochastic Analysis with Applications to Mathematical Physics / Yu.E. Gliklikh. – London: Springer–Verlag, 2011.

6. Cresson, J. Stochastic Embedding of Dynamical Systems / J. Cresson, S. Darses // Journal of Mathematical Physics. – 2007. – V. 48, № 7. – P. 072703-1–072303-54.

7. Гликлих, Ю.Е. Стохастические уравнения леонтьевского типа и производные в среднем случайных процессов / Ю.Е. Гликлих, Е.Ю. Машков // Вестник Южно-Уральского государственного университета. Серия: Математическое моделирование и программирование. – 2013. – Т. 7, № 2. – С. 25–39.

8. Azarina, S.V. Differential inclusions with mean derivatives / S.V. Azarina, Yu.E. Gliklikh // Dynamic Systems and Applications. – 2007. – V. 16, №. 1. – P. 49–71.

9. Гихман, И.И. Теория случайных процессов / И.И. Гихман, А.В. Скороход. – М.: Наука, 1975. – Т. 3.

10. Стернберг, С. Лекции по дифференциальной геометрии / С. Стернберг. – М.: Мир, 1970.

11. Азарина, С.В. О существовании решений стохастических дифференциальных уравнений с текущими скоростями / С.В. Азарина, Ю.Е. Гликлих // Вестник Южно-Уральского государственного университета. Серия: Математическое моделирование и программирование. – 2015. – Т. 8, № 4. – С. 100–106.

Гликлих Юрий Евгеньевич, доктор физико-математических наук, профессор, кафедра алгебры и топологических методов анализа, Воронежский государственный университет (г. Воронеж, Российская Федерация), yeg@math.vsu.ru.

Самсонова Кристина Александровна, магистрантка, кафедра алгебры и топологических методов анализа, Воронежский государственный университет (г. Воронеж, Российская Федерация), kristina-samsonova92@rambler.ru.

Поступила в редакцию 13 мая 2016 г.