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MATHEMATICAL BASES OF OPTIMAL MEASUREMENTS THEORY IN NONSTATIONARY CASE

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Recently, the use of mathematical results is becoming increasingly vast field of study for solving technical problems. An example of such approach is the recently developed optimal measurement theory. In the article the mathematical reasoning for solution of the measurement problem of dynamically distorted signal, taking into account the multiplier effect on the measuring transducer (MT). Making such a change can improve the adequacy of the mathematical model of the MT, namely, the problem is considered under the assumption that the MT are subject to change over time, which allows us to describe a decrease in sensitivity of elements of the MT.

Keywords: nonstationary Sobolev type equations, relatively bounded operator, degenerate flow of operators, optimal control problem, Showalter–Sidorov problem.

Introduction

The methods for solving ill-posed problems originally used in solving problems of the dynamical measurements theory (see [1]). On the basis of the methods of automatic control theory, A.L. Shestakov and his students proposed and justified technical hypotheses to solving the reconstruction problem of dynamically distorted signal [2]. However, increasing precision requirements has led to creation other methods for solving such problems. By virtue of what to solve this problem A.L. Shestakov and G.A. Sviridyuk proposed to use the methods of optimal control theory [3]. The problem resulting from this, it was suggested to call the problem of optimal measurement [4]. Based on the results of the numerical solution for Leontiev type system [5, 6] A.V. Keller brought this mathematical model "to the number" [7]. The first review of this approach was published in 2014 [8]. The resulting theory will be called the theory of optimal measurement of Shestakov – Sviridyuk – Keller.

Today this theory is rapidly developing. For example, the optimal measurement problem is investigated in spaces of "noise" [9, 10]. In addition, based on the results [11, 12], the problem of optimal measurement began to be studied in the nonstationary case, i.e. when the parameters, witch describing the measuring transducer, may change over time. The main purpose of this article is the full mathematical justification for the optimal measurement theory of Shestakov – Sviridyuk – Keller in the nonstationary case.

The basis of the theory of optimal measurement in the nonstationary case is a presentation the model of the measuring trasducer (MT) via a nonstationary Leontief type system

$$L\dot{x}(t) = a(t)Mx(t) + Bu(t), \quad \ker L \neq \{0\}, \tag{1}$$

$$y(t) = Nx(t), \tag{2}$$

where $x = (x_1, x_2, \dots, x_n)$ and $\dot{x} = (\dot{x}_1, \dot{x}_2, \dots, \dot{x}_n)$ are the vector-functions of a state and a rate of state change for the MT respectively; n is a dimension of the state vector-function. Square matrices L and M of order n, representing the mutual velocity of state changes and the actual state of the MT. Note that the matrix L is degenerate. The scalar function a :

$(0,T) \to \mathbb{R}_+$ describes the time variation of the parameters of the MT. B and N are square matrices of order n, characterizing the interference of the measurement parameters and the relationship between system state and observation correspondingly. It should be noted that systems (1), (2) are also called descriptor system (see, for example,[15]). To obtain a mathematical model of the measurement transducer the system (1), (2) supplement the initial condition Showalter – Sidorov

$$[R^L_\alpha(M)]^{p+1}(x(0)-x_0)=0. \tag{3}$$

Is required to find the optimal measurement $v \in \mathfrak{U}_{ad}$ almost everywhere on $(0,T)$, satisfying the problem (1)–(3) and the condition

$$J(v)=\min_{(u,x(u))\in\mathfrak{U}_{ad}\times\mathfrak{X}} J(u) \tag{4}$$

for functional

$$J(u)=\sum_{q=0}^{1}\int_0^T \|y^{(q)}(t)-y_0^{(q)}(t)\|^2_{\mathfrak{Y}}dt+\sum_{q=0}^{k}\int_0^T \left\langle N_q u^{(q)}(t),u^{(q)}(t)\right\rangle_{\mathfrak{U}} dt, \tag{5}$$

where $0 \le k \le p+1$, N_q $(q=0,1,\ldots,p+1)$ are square positive definite matrices of order n. Here $u=(u_1,u_2,\ldots,u_n)$ and $y=(y_1,y_2,\ldots,y_m)$ are the vector-functions of measurements and observations for the MT correspondingly; $y_0=(y_{01},y_{02},\ldots,y_{0m})$ are the observations obtained in the field experiment results. Problem (1)–(5) is called the optimal measurement problem. Solution of this problem allows reconstructing the signal $v \in \mathfrak{U}_{ad}$, corresponding to the results of observations y_0.

The theory of stationary Sobolev type equations based on the phase space method [16, 17]. This theory became the basis of the optimal measurement theory and is developing very rapidly, as can be seen by a large number of monographs, in whole or in part devoted to their study [18–25]. Moreover, Sobolev type equation began to be considered in quasi-Banach spaces [26–28], as well as in spaces of "noises" [29, 30].

Article except for the introduction and bibliography contains four parts. In the first part the relatively p-bounded operators are described. Also in this part degenerate groups and flows of operators are considered. The second part consists the solutions of the initial value problems for non-stationary Sobolev type equations obtained using degenerate flows of operators. In the third part of the existence and uniqueness of solutions of optimal control problem of solutions for non-stationary equation is proved. In the fourth part consists the statement of the optimal measurement problem and the existence of solutions for this problem, based on the results of the third part. Bibliography is not exhaustive and represents only the tastes and preferences of the author.

It is a pleasant duty to express my sincere gratitude to my scientific advisor Professor A.L. Shestakov for attention to this work, Professor G.A. Sviridyuk for strict, but constructive criticism, as well as the staff of the department of Mathematical Physics equations for useful discussions and interest in this work.

1. Relatively p-Bounded Operators and Degenerate Groups and Flows of Operators

Let $\mathfrak{X}$ and $\mathfrak{Y}$ are Banach spaces. Operator $L \in \mathcal{L}(\mathfrak{X};\mathfrak{Y})$ is linear and continuous and operator $M \in \mathcal{C}l(\mathfrak{X};\mathfrak{Y})$ is linear, closed and densely defined in $\mathfrak{X}$. Besides the operator L

is degenerate ($\ker L \neq \{0\}$). By [20, 31] sets $\rho^L(M) = \{\mu \in \mathbb{C} : (\mu L - M)^{-1} \in \mathcal{L}(\mathfrak{Y};\mathfrak{X})\}$ and $\sigma^L(M) = \mathbb{C} \setminus \rho^L(M)$ are called *L-resolvent set* and *L-spectrum* of operator M. By the results of [20, 31] set $\rho^L(M)$ is open and that's why L-spectrum of operator M is always closed.

L-resolvent set of operator M can be an empty set, for example, if $\ker L \cap \ker M \neq \{0\}$. We suppouse that $\rho^L(M) \neq \oslash$ and consider the operator-valued functions $(\mu L - M)^{-1}$, $R^L_\mu(M) = (\mu L - M)^{-1}L$, $L^L_\mu(M) = L(\mu L - M)^{-1}$ of a complex variable $\mu \in \mathbb{C}$ with domain $\rho^L(M)$, which will be called *L-resolvent, right* and *left L-resolvent* of operator M respectively. Also in view of the results [20, 31] the L resolvent, right and left L-resolvents of operator M are holomorphic in $\rho^L(M)$.

Definition 1. Operator M is called *spectrally bounded with respect to* operator L (or shortly (L,σ)*-bounded*), if

$$\exists r > 0 \quad \forall \mu \in \mathbb{C} \quad (|\mu| > r) \Rightarrow (\mu \in \rho^L(M)).$$

Let operatop M be a (L,σ)-bounded. We take in complex plane $\mathbb{C}$ closed circuit $\gamma = \{\mu \in \mathbb{C} : |\mu| = h > r\}$. Then the integrals of holomorphic operator-functions in a closed circuit are worthwhile

$$P = \frac{1}{2\pi i}\int_\gamma R^L_\mu(M)d\mu, \quad Q = \frac{1}{2\pi i}\int_\gamma L^L_\mu(M)d\mu. \tag{6}$$

If operator M is (L,σ)-bounded, then operators $P \in \mathcal{L}(\mathfrak{X})$ and $Q \in \mathcal{L}(\mathfrak{Y})$ are projectors [20, 31].

Denote $\mathfrak{X}^0 = \ker P$, $\mathfrak{Y}^0 = \ker Q$, $\mathfrak{X}^1 = \text{im } P$, $\mathfrak{Y}^1 = \text{im } Q$. So we have $\mathfrak{X} = \mathfrak{X}^0 \oplus \mathfrak{X}^1$, $\mathfrak{Y} = \mathfrak{Y}^0 \oplus \mathfrak{Y}^1$. By L_k (M_k) denote restriction of operator L (M) on $\mathfrak{X}^k$ (dom $M_k =$ dom $M \cap \mathfrak{X}^k$), $k = 0, 1$.

We formulate the splitting theorem.

Theorem 1. (G.A. Sviridyuk) [20, 31] *Let operator M be a (L,σ)-bounded. Then*
(i) $L_k \in \mathcal{L}(\mathfrak{X}^k;\mathfrak{Y}^k)$, $k = 0, 1$;
(ii) $M_0 \in \mathcal{C}l(\mathfrak{X}^0;\mathfrak{Y}^0)$, $M_1 \in \mathcal{L}(\mathfrak{X}^1;\mathfrak{Y}^1)$;
(iii) *operators* $L_1^{-1} \in \mathcal{L}(\mathfrak{Y}^1;\mathfrak{X}^1)$ *and* $M_0^{-1} \in \mathcal{L}(\mathfrak{Y}^0;\mathfrak{X}^0)$ *are exits.*

Infinity is called *a pole of order* $p \in \mathbb{N}_0$ ($\equiv \{0\} \cup \mathbb{N}$), if $H = \mathbb{O}$ ($p = 0$) or $H^p \neq \mathbb{O}$ and $H^{p+1} = \mathbb{O}$ with $p \in \mathbb{N}$.

Definition 2. Let ∞ is pole of order $p \in \mathbb{N}_0$ for L-resolvent of operator M. Then (L,σ)-bounded operator M is called (L,p)*-bounded.*

Corollary 1. [32] *Let operator M be a (L,p)-bounded ($p \in \mathbb{N}_0$). Then*

$$P = \lim_{\mu\to\infty}\left(\mu R^L_\mu(M)\right)^{p+1}, \qquad Q = \lim_{\mu\to\infty}\left(\mu L^L_\mu(M)\right)^{p+1}.$$

Definition 3. A one-parameter family $X^\bullet : \mathbb{R} \to \mathcal{L}(\mathfrak{X})$ is called *degenerate group of operators*, if the following conditions are met
(i) $X^0 = P$;
(ii) $X^tX^s = X^{t+s}$ for all $t, s \in \mathbb{R}$.

Degenerate group of operators called *of analytical* if it has an analytic continuation to the whole complex plane $\mathbb{C}$ retaining properties (i) and (ii) of the definition of 3.

Theorem 2. [20, 31] *Let operator M be a (L,σ)-bounded. Then there exists analytical group $\{X^t \in \mathcal{L}(\mathfrak{X}) : t \in \mathbb{R}\}$ ($\{Y^t \in \mathcal{L}(\mathfrak{Y}) : t \in \mathbb{R}\}$) and its operators can be represent by Danford–Taylor type integrals*

$$X^t = \frac{1}{2\pi i}\int\limits_\gamma R^L_\mu(M)e^{\mu t}d\mu \quad \left(Y^t = \frac{1}{2\pi i}\int\limits_\gamma L^L_\mu(M)e^{\mu t}d\mu\right), \tag{7}$$

where close circuit $\gamma = \{\mu \in \mathbb{C} : |\mu| = h > r\}$.

Corollary 2. [32] *Let operator M be a (L,p)-bounded ($p \in \mathbb{N}_0$). Then operators of group $\{X^t \in \mathcal{L}(\mathfrak{X}) : t \in \mathbb{R}\}$ ($\{Y^t \in \mathcal{L}(\mathfrak{Y}) : t \in \mathbb{R}\}$) can be represent by the Hille–Widder–Post approximations*

$$X^t = \lim_{k\to\infty}\left(\frac{k}{t}R^L_{\frac{k}{t}}(M)\right)^k \qquad \left(Y^t = \lim_{k\to\infty}\left(\frac{k}{t}L^L_{\frac{k}{t}}(M)\right)^k\right). \tag{8}$$

Definition 4. A two-parameter family $X(\cdot,\cdot) : \mathbb{R}\times\mathbb{R} \to \mathcal{L}(\mathfrak{X})$ is called *degenerate flows of operators*, if the following conditions are met
(i) $X(t,t) = P$;
(ii) $X(t,s)X(s,\tau) = X(t,\tau)$.

Degenerate flows of operators called the *analytical* if its operators can be analytically continued to the whole complex plane $\mathbb{C}$ retaining properties (i), (ii) of the definition 4.

Let operator M be a (L,p)-bounded ($p \in \mathbb{N}_0$) and function $a \in C(\mathbb{R};\mathbb{R})$. By the analogy with (7) consider with $s,t \in \mathbb{R}$ and close circuit $\gamma = \{\mu \in \mathbb{C} : |\mu| = h > r\}$

$$X(t,s) = \frac{1}{2\pi i}\int\limits_\gamma R^L_\mu(M)\exp\left(\mu\int\limits_s^t a(\zeta)d\zeta\right)d\mu, \qquad s < t. \tag{9}$$

Theorem 3. [32] *Let operator M be a (L,p)-bounded ($p \in \mathbb{N}_0$) and function $a \in C(\mathbb{R};\mathbb{R})$. Then family $\{X(t,s) \in \mathcal{L}(\mathfrak{X}) :\ t,s \in \mathbb{R}\}$ defined by (9) is an analitical degenerate flows of operators.*

Similarly, corollary 2 we have the following

Corollary 3. [32] *Let operator M be a (L,p)-bounded ($p\in\mathbb{N}_0$) and function $a\in C(\mathbb{R};\mathbb{R}_+)$. Then operators of the flow $\{X(t,s) \in \mathcal{L}(\mathfrak{X}) : t,s \in \mathbb{R}\}$ u $\{Y(t,s) \in \mathcal{L}(\mathfrak{Y}) : t,s \in \mathbb{R}\}$ can be represented by Hille–Widder–Post approximations*

$$X(t,s) = \lim_{k\to\infty}\left(\left(L - \frac{1}{k}M\int\limits_s^t a(\zeta)d\zeta\right)^{-1}L\right)^k,\ Y(t,s) = \lim_{k\to\infty}\left(L\left(L - \frac{1}{k}M\int\limits_s^t a(\zeta)d\zeta\right)^{-1}\right)^k. \tag{10}$$

2. Solvability of Initial Problems for Non-stationary Sobolev Type Equations

Let $\mathfrak{X}$ and $\mathfrak{Y}$ are Banach spaces. Operators $L \in \mathcal{L}(\mathfrak{X};\mathfrak{Y})$ and $M \in \mathcal{C}l(\mathfrak{X};\mathfrak{Y})$. On the interval $\mathfrak{J} \subset \mathbb{R}$ consider the Cauchy problem ($t_0 \in \mathfrak{J}$)

$$x(t_0) = x_0, \tag{11}$$

for homogeneous non-stationary equation

$$L\dot{x}(t) = a(t)Mx(t), \tag{12}$$

where function $a : \mathfrak{J} \to \mathbb{R}_+$ to be further defined.

Definition 5. Vector-function $x \in C^1(\mathfrak{J};\mathfrak{X})$ is called *solution of equation* (12), if it satisfies this equation on $\mathfrak{J}$. Solution of equation (12) is called *solution of Cauchy problem* (11), (12), if it also satisfies to (11).

Closed set $\mathfrak{P} \subset \mathfrak{X}$ is called *phase space* of equation (12), if
(i) any solution $x(t)$ of equation (12) lies in $\mathfrak{P}$ (pointwise);
(ii) for any x_0 from $\mathfrak{P}$ there exists unique solution of the Cauchy problem (11) for equation (12).

In addition to equation (12) consider the precise equivalent equation

$$L(\nu L - M)^{-1}\dot{y} = aM(\nu L - M)^{-1}y, \quad \nu \in \rho^L(M). \tag{13}$$

Theorem 4. [32] *Let operator M be a (L,p)-bounded ($p \in \mathbb{N}_0$) and function $a \in C(\mathbb{R},\mathbb{R}_+)$. Then the set $\mathfrak{X}^1$ ($\mathfrak{Y}^1$) is a phase space of equation* (12) ((13)).

The flows of operators $X(\cdot,\cdot) : \mathbb{R} \times \mathbb{R} \to \mathcal{L}(\mathfrak{X})$ is called *flows of solving operators* for equation (12), if for any $x_0 \in \mathfrak{X}$ the vector-function $x(t) = X(t,t_0)x_0$ is a solution of equation (12) by the definition 5.

Consider the Showalter–Sidorov problem

$$P(x(0) - x_0) = 0 \tag{14}$$

for non-homogeneous equation

$$L\dot{x}(t) = a(t)Mx(t) + g(t) \tag{15}$$

with function $g : \mathfrak{J} \to \mathfrak{Y}$. Denote $(\mathbb{I}_{\mathfrak{Y}} - Q)g(t) = g^0(t)$.

Definition 6. Solution of equation (15) is called *solution of Showalter–Sidorov problem* (14), (15), if it satisfies (14).

Theorem 5. [32] *Let $0, T \in \mathfrak{J}$, operator M be a (L,p)-bounded ($p \in \mathbb{N}_0$) and function $a \in C^{p+1}([0,T];\mathbb{R}_+)$. Then for all $x_0 \in \mathfrak{X}$ and vector-function $g : [0,T] \to \mathfrak{Y}$, such that $Qg \in C^1([0,T];\mathfrak{Y}^1)$ and $g^0 \in C^{p+1}([0,T];\mathfrak{Y}^0)$ there exists a unique solution $x \in C^1([0,T];\mathfrak{X})$ of Showalter–Sidorov problem* (14) *for equation* (15), *given by the next formula*

$$x(t) = X(t,0)Px_0 + \int_0^t X(t,s)L_1^{-1}Qg(s)ds - \sum_{k=0}^{p} H^k M_0^{-1} \left(\frac{1}{a(t)}\frac{d}{dt}\right)^k \frac{g^0(t)}{a(t)}. \tag{16}$$

If in addition initial data x_0 *satisfies*

$$(\mathbb{I}_{\mathfrak{X}} - P)x_0 = -\sum_{k=0}^{p} H^k M_0^{-1} \left(\frac{1}{a(0)}\frac{d}{dt}\right)^k \frac{g^0(0)}{a(0)},$$

then solution (16) *is a unique solution of the Cauchy problem* (11), (15).

3. Solutions for Optimal Control Problem

Let $\mathfrak{X}$, $\mathfrak{Y}$ and $\mathfrak{U}$ are Hilbert spaces. On the interval $[0,T) \subset \mathbb{R}_+$ $(T < +\infty)$ consider Showalter–Sidorov problem

$$P(x(0) - x_0) = 0, \tag{17}$$

for equation

$$L\dot{x}(t) = a(t)Mx(t) + g(t) + Bu(t). \tag{18}$$

Here operators $L \in \mathcal{L}(\mathfrak{X};\mathfrak{Y})$, $M \in \mathcal{C}l(\mathfrak{X};\mathfrak{Y})$ and operator $B \in \mathcal{L}(\mathfrak{U};\mathfrak{Y})$. Function $a : [0,T) \to \mathbb{R}_+$ and vector-functions $u : [0,T) \to \mathfrak{U}$ and $g : [0,T) \to \mathfrak{Y}$ would be defined in future.

Definition 7. The vector-function $x \in H^1(\mathfrak{X}) = \{x \in L_2(0,T;\mathfrak{X}) : \dot{x} \in L_2(0,T;\mathfrak{X})\}$ is called *strong solution* of equation (18), if it almost everywhere in $(0,T)$ transform equation (18) to right equality. The strong solution $x = x(t)$ of equation (18) is called *strong solution for Showalter–Sidorov problem* (17), (18), if it is satisfied (17).

Construct the space $H^{p+1}(\mathfrak{Y}) = \{\xi \in L_2(0,T;\mathfrak{Y}) : \xi^{(p+1)} \in L_2(0,T;\mathfrak{Y}),\ p \in \mathbb{N}_0\}$ which is a Hilbert space with a inner product $[\xi,\eta] = \sum\limits_{q=0}^{p+1} \int\limits_0^T \langle \xi^{(q)}, \eta^{(q)} \rangle_{\mathfrak{Y}}\, dt.$

By the theorem 5 we have the following

Theorem 6. *Let the operator* M *be a* (L,p)*-bounded* $(p \in \mathbb{N}_0)$ *and function* $a \in C^{p+1}([0,T);\mathbb{R}_+)$. *Then for any* $x_0 \in \mathfrak{X}$, $g \in H^{p+1}(\mathfrak{Y})$ *and* $u \in H^{p+1}(\mathfrak{U})$ *there is exist a unique solution* $x \in H^1(\mathfrak{X})$ *for Showalter–Sidorov problem* (17), (18) *and it has the form*

$$x(t) = X(t,0)Px_0 + \int_0^t X(t,s)L_1^{-1}Q(g(s)+Bu(s))ds - \sum_{k=0}^{p} H^k M_0^{-1}(I-Q)\left(\frac{1}{a(t)}\frac{d}{dt}\right)^k \frac{g(t)+Bu(t)}{a(t)}, \tag{19}$$

where the symbol $\left(\frac{1}{a(t)}\frac{d}{dt}\right)^k$ *means the continuous application this operator* k *times.*

Let $\mathfrak{Z}$ is a Hilbert space and operator $C \in \mathcal{L}(\mathfrak{X};\mathfrak{Z})$. Consider the quality functional in the form

$$J(x,u) = \sum_{q=0}^{1} \int_0^T \|z^{(q)} - z_d^{(q)}\|_{\mathfrak{Z}}^2 dt + \sum_{q=0}^{k} \int_0^T \langle N_q u^{(q)}, u^{(q)} \rangle_{\mathfrak{U}}\, dt, \qquad z = Cx, \tag{20}$$

where $0 \leq k \leq p+1$. Operators $N_q \in \mathcal{L}(\mathfrak{U})$, $q = 0, 1, \ldots, p+1$ are self adjoin and positively defined ones, $z_d = z_d(t, s)$ is a observations from some space of observations $\mathfrak{Z}$. Remark that if $x \in H^1(\mathfrak{X})$ then $z \in H^1(\mathfrak{Z})$. By analogy with $H^{p+1}(\mathfrak{Y})$ denote the space $H^{p+1}(\mathfrak{U})$, which is a Hilbert one although $\mathfrak{U}$ is a such space. We distinguish a convex and closed subset $H^{p+1}_\partial(\mathfrak{U}) = \mathfrak{U}_{ad}$ in the space $H^{p+1}(\mathfrak{U})$ which we call a *set of admissible controls*.

Definition 8. The vector-function $\hat{u} \in H^{p+1}_\partial(\mathfrak{U})$ is called an *optimal control* over solutions of problem (17), (18), if

$$J(\hat{x}, \hat{u}) = \min_{(x,u)\in\mathfrak{X}\times\mathfrak{U}_{ad}} J(x, u), \tag{21}$$

where pairs $(x, u) \in \mathfrak{X} \times \mathfrak{U}_{ad}$ are satisfied (17), (18).

Let fix $x_0 \in \mathfrak{X}$, $g \in H^{p+1}(\mathfrak{Y})$ and consider (19) as a map $D : u \to x(u)$.

Lemma 1. *Let spaces $\mathfrak{X}$, $\mathfrak{Y}$ and $\mathfrak{U}$ are Hilbert ones, an operator M be a (L, p)-bounded, $p \in \mathbb{N}_0$, function $a \in C^{p+1}(\overline{\mathbb{R}}_+; \mathbb{R}_+)$ and elements $x_0 \in \mathfrak{X}$, $g \in H^{p+1}(\mathfrak{Y})$ are fix. Than map $D : H^{p+1}(\mathfrak{U}) \to H^1(\mathfrak{X})$ defined by formula* (19) *is a continuous.*

Proof. Thus operator $B \in \mathcal{L}(H^{p+1}(\mathfrak{U}); H^{p+1}(\mathfrak{Y}))$ and solution has the form (19) then Lemma is true due to properties of flows $X(t, s)$ and continuously of $a(t)$ with $t \in \overline{\mathbb{R}}_+$. □

At last, we proof the main result of this part.

Theorem 7. *Let an operator M be a (L, p)-bounded, $p \in \mathbb{N}_0$ and function $a \in C^{p+1}([0, T); \mathbb{R}_+)$. Then for any $x_0 \in \mathfrak{X}$ and $g \in H^{p+1}(\mathfrak{Y})$ the unique solution $\hat{u} \in \mathfrak{U}_{ad}$ of optimal control problem* (17), (18), (20), (21) *is exist.*

Proof. Thus set $\mathfrak{U}_{ad}$ is a nonempty, convex, closed and bouded one, then minimizung sequence $u_m \in \mathfrak{U}_{ad}$ is exists

$$\lim_{m\to\infty} J(x_m, u_m) = \inf_{(x,u)\in\mathfrak{X}\times\mathfrak{U}_{ad}} J(x, u).$$

Here sequence x_m construct by the element $u_m \in \mathfrak{U}_{ad}$ using the map D from Lemma 1. Thus the set $\mathfrak{U}_{ad}$ is convex and closed, then by the Mazur theorem (see, for example [33]) this set is sequensed closed. And we have that

$$J(\hat{x}, \hat{u}) = \lim_{m\to\infty} J(x_m, u_m) = \min_{(x,u)\in\mathfrak{X}\times\mathfrak{U}_{ad}} J(x, u).$$

Thus the quality functional (20) is a quadratical, precisely, strict quadratical, then the optimal control $\hat{u} \in \mathfrak{U}_{ad}$ is unique. □

4. Mathematical Model of Optimal Measurement in Nonstationary Case

For posing of the optimal measurement problem in non-stationary case for the model of the measuring transducer (MT) we introduce a *state space* $\mathfrak{X} = \{x \in L_2((0, T); \mathbb{R}^n) : \dot{x} \in L_2((0, T); \mathbb{R}^n)\}$, a *space of measurements* $\mathfrak{U} = \{u \in L_2((0, T); \mathbb{R}^n) : u^{p+1} \in L_2((0, T); \mathbb{R}^n)\}$ and a *space of observations* $\mathfrak{Y} = N[\mathfrak{X}]$ with some fixed $T \in \mathbb{R}_+$. We distinguish a convex and closed subset $\mathfrak{U}_{ad}$ in the space $\mathfrak{U}$ which we call a *set of admissible measurements*. Our aim is to find the *optimal measurement* $v \in \mathfrak{U}_{ad}$ which almost everywhere in $(0, T)$ satisfies the Leontiev type system

$$L\dot{x}(t) = a(t)Mx(t) + Bu(t), \quad \ker L \neq \{0\}, \tag{22}$$

$$y(t) = Nx(t). \tag{23}$$

Also this function satisfies Showalter–Sidorov initial condition

$$[R^L_\alpha(M)]^{p+1}(x(0) - x_0) = 0, \quad \alpha \in \rho^L(M), \tag{24}$$

and such that

$$J(v) = \min_{(u,x(u)) \in \mathfrak{U}_{ad} \times \mathfrak{X}} J(u) \tag{25}$$

with *penalty functional* in the form

$$J(u) = \sum_{q=0}^{1} \int_0^T \|y^{(q)}(t) - y_0^{(q)}(t)\|^2_{\mathfrak{Y}} dt + \sum_{q=0}^{k} \int_0^T \left\langle N_q u^{(q)}(t), u^{(q)}(t) \right\rangle_{\mathfrak{U}} dt, \tag{26}$$

where $0 \leq k \leq p+1$, matrices N_q $(q = 0, 1, \ldots, p+1)$ of order n are positively defined. Here $x = (x_1, x_2, \ldots, x_n)$ and $\dot{x} = (\dot{x}_1, \dot{x}_2, \ldots, \dot{x}_n)$ are the vector-functions of a state and a rate of state change for the MT respectively; $u = (u_1, u_2, \ldots, u_n)$ and $y = (y_1, y_2, \ldots, y_n)$ are the vector-functions of measurements and observations for the MT correspondingly; $y_0 = (y_{01}, y_{02}, \ldots, y_{0n})$ are the observations obtained in the field experiment results; n is a number of variables of the system states. Square matrices L and M of order n are representing the mutual velocity of state changes and the actual state of the MT ($\det L \neq \{0\}$). The scalar function $a : (0, T) \to \mathbb{R}_+$ describes the time variation of the parameters of the MT. B and N are square matrices of order n, characterizing the interference of the measurement parameters and the relationship between system state and observation correspondingly. For getting the mathematical model of MT the system (22), (23) are supplemented by the Showalter–Sidorov condition (24). Problem (22)–(26) is called the *optimal measurement problem.* A solution $v \in \mathfrak{U}_{ad}$ of problem (22)–(26) is a reconstructed signal which is consistent with the results of observations y_0.

Definition 9. *A pair* $(x, v) \in \mathfrak{X} \times \mathfrak{U}_{ad}$ *is named the exact solution of problem* (22)–(26) *if* $x = x(t)$ *satisfies the system* (22), (23) *almost everywhere in* $[0, T]$ *(where* $u = v$*) , the conditions* (24) *(for some vector* $x_0 \in \mathbb{R}^n$*) and* (25) *with* (26)*.*

The matrix M is named the (L, p)*-regular* ($p \in \{0\} \cup \mathbb{N}$) if the set $\rho^L(M) \neq \oslash$ and ∞ is a removable singularity ($p = 0$) or a pole of order $p \in \mathbb{N}$ for function $\det(\mu L - M)^{-1}$.

Due to the fact that problem (22)–(24) is a finite-dimensional version of problem (17), (18), by the Theorem 7 there is exists the solution of the optimal measurement problem.

Theorem 8. *Let the matrix* M *be a* (L, p)*-regular* ($p \in \{0\} \cup \mathbb{N}$) *and* $\det M \neq 0$*. Then for any vector* $x_0 \in \mathfrak{X}$ *u* $y_0 \in \mathfrak{Y}$ *and function* $a \in C^{p+1}([0, T]; \mathbb{R}_+)$ *there exists a unique optimal measurement* $v \in \mathfrak{U}_{ad}$ *for the problem* (22)–(26)*.*

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МАТЕМАТИЧЕСКИЕ ОСНОВЫ ТЕОРИИ ОПТИМАЛЬНЫХ ИЗМЕРЕНИЙ В НЕСТАЦИОНАРНОМ СЛУЧАЕ

М. А. Сагадеева

В последнее время применение математических результатов при решении технических задач становится все более обширной областью изучения. Примером такого подхода является развиваемая в последнее время теория оптимальных измерений. В статье проведено математическое обоснование решения задачи измерений динамически искаженного сигнала с учетом мультипликативного воздействия на измерительное устройство (ИУ). Внесение такого изменения позволяет повысить адекватность математической модели ИУ, а именно, задача рассматривается при допущении, что параметры ИУ могут меняться во времени, что позволяет описать снижение чувствительности элементов ИУ.

Keywords: нестационарные уравнения соболевского типа; относительно ограниченный оператор; вырожденный поток операторов; задача оптимального управления; задача Шоуолтера–Сидорова.

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